α-ideals in 0-distributive Semilattices

Section II

α-ideals in O-distributive semilattices

For convenience we repeat the definition of a O-distributive semilattice.

A meet semilattice $< S, \wedge >$ with 0 is said to be O-distributive if $a \wedge x_1 = 0$, $a \wedge x_2 = 0$, ... $a \wedge x_n = 0$; $a, x_1, \ldots x_n \in S$ (n finite) and $x_1 \vee x_2 \vee \ldots \vee x_n$ exists in S then $a \wedge (x_1 \vee x_2 \vee \ldots \vee x_n) = 0$.

Throughout this section we denote a O-distributive semilattice by S.

We define

Definition 2.1 α-ideal in S : An ideal I in S is said to be an α-ideal if $x \in I$ implies that $(x]^{**} \subseteq I$

Note that ~~4,~~ Jayaram [6] has also defined α-ideals in O-distributive semilattices. But the definitions of ideals and O-distributivity used in [6] are different

Example 2.2:

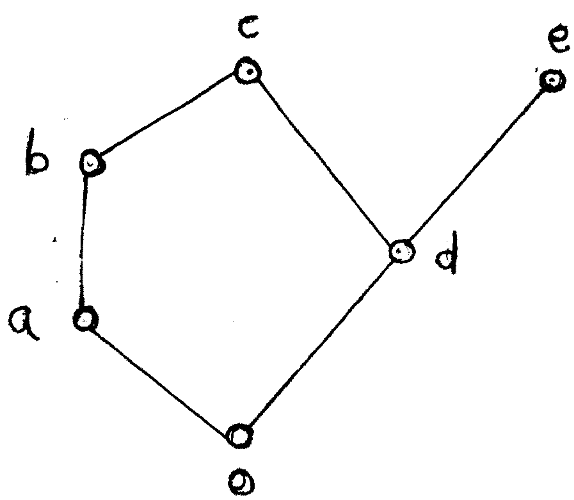

Figure-I

In the O-distributive semilattice S represented by Figure-I, the following are α-ideals.

$$I_1 = \{0, a, b\}, \quad I_2 = \{0, d, e\}, \quad I_3 = \{0\},$$

$$I_4 = \{0, a, b, c, d, e\}$$

Remark 2.3 : Obviously every α-ideal is an ideal but every ideal need not be an α-ideal. This we illustrate in the following example :

<u>Example 2.4</u> :

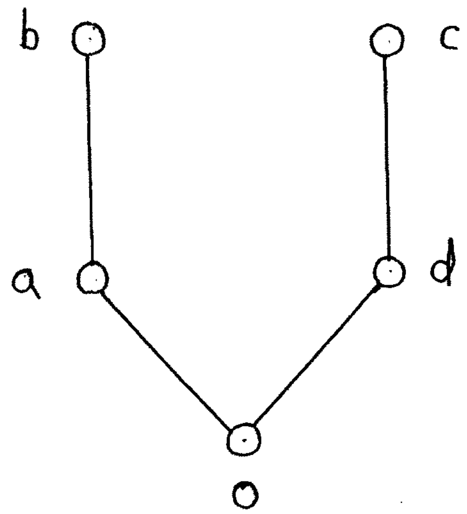

Figure-II

In the 0-distributive semilattice sketched in the Figure-II, $\{0, a\}$ is an ideal but not an α-ideal.

Consider the ideal, $I = \{0, a, c, d, e, f\}$ in the following 0-distributive semilattice whose diagramatic representation is as shown in Figure-III.

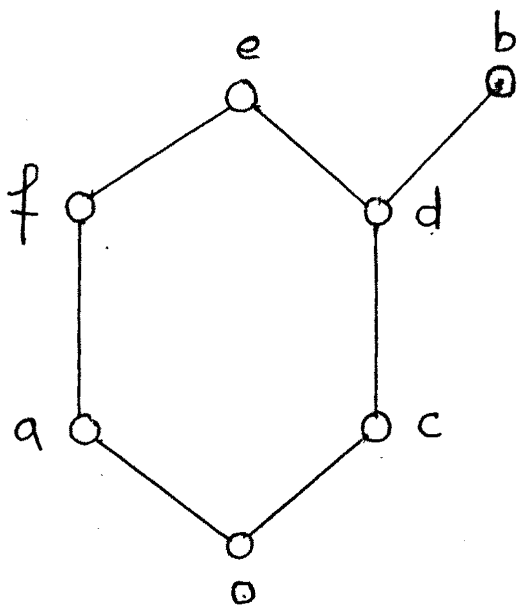

Figure-III

Here I is a prime ideal but not an α-ideal.

For minimal prime ideal we have the following :

Theorem 2.5 : Every minimal prime ideal in S is an α-ideal.

Proof - Let M be any minimal prime ideal in S and $x \in M$. As M is minimal prime ideal, S-M is maximal filter (See Result 1.2.2). Since S-M is maximal filter and $x \notin S-M$ ~~therefore~~ there exists $y \in S-M$ such that $x \wedge y = 0$ (See Result 1.2.3).

Now if $z \in (x)^{**}$ then $z \wedge y = 0$ (since $y \in (x)^{*}$). But then $z \wedge y \in M$ and M is prime ideal will imply $z \in M$. Thus $(x)^{**} \subseteq M$ for every x in M proving that M is an α-ideal.//

For any prime ideal P in S define,

$$O(P) = \{ x \in S / \ x \wedge y = 0 \text{ for some } y \notin P \}$$

We now prove that O(P) is an α-ideal

<u>Theorem 2.6</u> : For any prime ideal P in S, O(P) is an α-ideal.

<u>Proof</u> - <u>Claim-I</u> O(P) is an ideal in S

(i) Let, $a \leqslant b$ and $b \in O(P)$. $(a, b \in S)$

$b \in O(P)$ implies that $b \wedge y = 0$ for some $y \notin P$.

As $a \leqslant b$ we get $a \wedge y = 0$: Hence $a \in O(P)$.

Thus $a \leqslant b$ and $b \in O(P) \Longrightarrow a \in O(P)$.

(ii) Let $a_1, a_2, \ldots a_n$ be the elements of O(P) such that $a_1 \vee a_2 \vee \ldots \vee a_n$ exists in S. Then by the definition of O(P) we have $a_1 \wedge p_1 = 0$, $a_2 \wedge p_2 = 0 \ldots$ $\ldots a_n \wedge p_n = 0$ where $p_1, p_2, \ldots p_n$ are not in P.

As P is prime, $p_1 \ p_2 \ \ldots \ p_n \notin P$.

$a_i \wedge p_i = 0$ $(1 \leqslant i \leqslant n)$ will imply

$a_1 \wedge (p_1 \wedge p_2 \ \ldots \wedge p_n) = 0$

$a_2 \wedge (p_1 \wedge p_2 \wedge \dots \wedge p_n) = 0$

$a_n \wedge (p_1 \wedge p_2 \wedge \dots \wedge p_n) = 0.$

As S is 0-distributive we get,

$(a_1 \vee a_2 \vee \dots \vee a_n) \wedge (p_1 \wedge p_2 \wedge \dots \wedge p_n) = 0$. But $p_1 \wedge p_2 \wedge \dots \wedge p_n \notin P$ proving that $a_1 \vee a_2 \vee \dots \vee a_n \in 0(p)$. Thus by (i) and (ii) we get $0(P)$ is an ideal in S.

<u>Claim-II</u> $0(P)$ is an α-ideal in S.

Let $x \in 0(P)$. Then $x \wedge y = 0$ for some $y \notin P$. Hence $y \in (x)^*$. Therefore for any $z \in (x)^{**}$, $z \wedge y = 0$. But then $z \in 0(P)$. This proves that $(x)^{**} \subseteq 0(P)$ for $x \in 0(P)$. Hence $0(P)$ is an α-ideal in S.//

As for any prime ideal P in S, $0(P) \subseteq P$ we have the following corollary.

<u>Corollary 2.7</u> : Every prime ideal in S contains an α-ideal.

For an ideal I in S define,

$$I' = \{x \in S \mid x \in (a)^{**} \text{ for some } a \in I\}$$

It is clear that I' not necessarily be an ideal in S. But when I' is an ideal we have

<u>Theorem 2.8</u> : For any ideal I in S if I' is an ideal in S then I' is the smallest α-ideal containing I.

<u>Proof</u> (I) I' is an α-ideal in S.

Let $x \in I'$. Then by the definition of I' there exists $a \in I$ such that $x \in (a)^{**}$. Therefore $(x)^{**} \subseteq (a)^{**}$. Hence for any $t \in (x)^{**}$ we have $t \in (a)^{**}$. Hence $(x)^{**} \subseteq I'$. Thus $x \in I'$ implies that $(x)^{**} \subseteq I'$ proving that I' is an α-ideal in S.

(II) I' is the smallest α-ideal containing I.

As $a \in (a)^{**}$ for any a in S we get $I \subseteq I'$. Let J be any α-ideal in S such that $I \subseteq J$. If $x \in I'$ then $x \in (a)^{**}$ for some $a \in I$. Hence $x \in (a)^{**}$ for some $a \in J$. But since J is an α-ideal, $a \in J$ implies that $(a)^{**} \subseteq J$. Therefore $x \in J$. Thus $x \in I' \implies x \in J$ proving that $I' \subseteq J$. Hence from (I) and (II) we get I' is the smallest α-ideal containing I. //

For any filter F in S define,

$$O(F) = \left\{ x \in S \;/\; x \wedge y = 0 \text{ for some } y \in F \right\}$$

We now prove that O(F) is an α-ideal in S.

<u>Theorem 2.9</u> : For any filter F in S, O(F) is an α-ideal

<u>Proof</u> (I) O(F) is an ideal in S

(i) Let $x_1 \leq x_2$ and $x_2 \in O(F)$ $(x_1, x_2 \in S)$. Then $x_2 \wedge y = 0$ for some $y \in F$ implies that $x_1 \wedge y = 0$. Hence $x_1 \in O(F)$

(ii) Let $x_1, x_2, \ldots, x_n \in O(F)$ such that $x_1 \vee x_2 \vee \ldots \ldots \vee x_n$ exists in S. Then $x_i \wedge y_i = 0$ for some $y_i \in F$ $(1 \leq i \leq n)$. As $y_1, y_2, \ldots, y_n \in F$, $y_1 \wedge y_2 \wedge \ldots \wedge y_n \in F$. Therefore

$x_1 \wedge (y_1 \wedge y_2 \wedge \ldots \wedge y_n) = 0$

$x_2 \wedge (y_1 \wedge y_2 \wedge \ldots \wedge y_n) = 0$

$x_n \wedge (y_1 \wedge y_2 \wedge \ldots \wedge y_n) = 0$

But S being a O-distributive, we get

$(x_1 \vee x_2 \vee \ldots \vee x_n) \wedge (y_1 \wedge y_2 \wedge \ldots \wedge y_n) = 0$. Hence $x_1 \vee x_2 \vee \ldots \vee x_n \in O(F)$.

Thus from (i) and (ii), O(F) is an ideal in S.

(II) O(F) is an α-ideal in S.

Let $x \in O(F)$. Then $x \wedge f = 0$ for some $f \in F$. Hence $f \in (x)^*$. If $y \in (x)^{**}$ then $y \wedge f = 0$.

This in turn implies that $y \in O(F)$. Thus $y \in (x)^{**}$ implies that $y \in O(F)$ i.e. $(x)^{**} \subseteq O(F)$ for $x \in O(F)$ proving that $O(F)$ is an α-ideal in S. //

For any proper α-ideal I in S we have $I \cap D = \emptyset$ where D denotes the set of all dense elements in S i.e. $D = \{d \in S \,/\, (d)^* = \{0\}\}$. This is proved in the following

<u>Theorem 2.10</u> : Any proper α-ideal in S does not contain a dense element.

<u>Proof</u> Let I be proper α-ideal in S and let d be dense element in S. If possible suppose that $d \in I$. Then I being an α-ideal $(d)^{**} \subseteq I$. Hence $S \subseteq I$ (since $(d)^* = \{0\}$ $(d)^{**} = S$). This contradicts that I is proper. Hence $d \notin I$. Thus proper α-ideal does not contain a dense element. //

Now we state crucial result about the prime α-ideals.

<u>Theorem 2.11</u> : Let I be an annihilator ideal and F be a filter in S such that $I \cap F = \emptyset$. Then there exists a prime α-ideal P containing I and disjoint with F.

<u>Proof</u> : As I is an annihilator ideal in S, $I = A^*$ for some $A \subseteq S$. Further $I \cap F = \emptyset$ implies that $A^* \cap F =$

$$\bigcap_{a \in A} (a)^* \cap F = \emptyset$$

(since $A^* = \bigcap_{a \in A} (a)^*$) Hence $\bigcap_{a \in A} [(a)^* \cap F] = \emptyset$.

This implies that $(a)^* \cap F = \emptyset$ for some $a \in A$.

Consider the family $\mathfrak{F}$ of all filters in S containing F and disjoint with $(a)^*$. Then obviously $F \in \mathfrak{F}$. Hence by Zone's lemma there exists maximal element M in such that $F \subseteq M$ and $(a)^* \cap M = \emptyset$

Claim-I $a \in M$

If $a \notin M$ then the filter generated by $M \cup \{a\}$ intersects $(a)^*$. Hence there exists an element b in S such that $b \geq c \wedge a$ for some $c \in M$ and $b \wedge a = 0$. But this gives $c \wedge a = 0$ i.e. $c \in (a)^*$ which is a contradiction since $M \cap (a)^* = \emptyset$. Hence $a \in M$.

Claim-II M is maximal filter.

Let $z \in S$ such that $z \notin M$. As the filter generated by $M \cup \{z\}$ intersects $(a)^*$ there exists an element b in $(a)^*$ such that $b \geq f \wedge z$ for some $f \in F$. Now $0 = b \wedge a \geq f \wedge z \wedge a$ gives $f \wedge z \wedge a = 0$. But as $f \in M$, $a \in M$ we have $f \wedge a \in M$. Thus for $z \notin M$ there exists $f \wedge a$ in M such that $(f \wedge a) \wedge z = 0$. Hence M is maximal filter (See Result 1.2.3).

Thus we have shown that there exists maximal filter M in S such that $F \subseteq M$ and $(a)^* \cap M = \emptyset$. But then $A^* \cap M = \emptyset$.

Since M is a maximal filter, S-M is a minimal prime ideal (See Result 1.2.2). Thus we have $A^* \subseteq S - M$ and $F \cap (S-M) = \emptyset$. By Theorem 2.5, S-M is a prime α-ideal. This completes the proof. //

A relation between α-ideal and annihilator ideal is an follows.

<u>Theorem 2.12</u> : Every annihilator ideal in S is an α-ideal.

<u>Proof</u> Let I be any annihilator ideal in S. Then $I = I^{**}$. If $x \in I$ then $x \in I^{**}$. This implies that $y \in (x)^*$ for all $y \in I^*$. Hence $I^* \subseteq (x)^*$ gives $(x)^{**} \subseteq I^{**} = I$. Thus $(x)^{**} \subseteq I$ for $x \in I$, proving that I is α-ideal. //

<u>Remark 2.13</u> : Every α-ideal need not be an annihilator ideal. For example if I is the proper dense α-ideal then I will not be an annihilator ideal.

There are some 0-distributive semilattices in which every α-ideal is an annihilator ideal. For this consider the following example.

Example 2.14 :

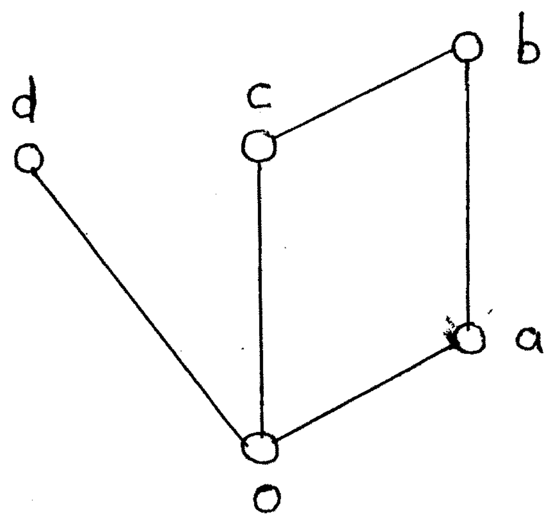

Figure-IV

In the semilattice represented by Figure-IV, α-ideals are, I_1, I_2, I_3, I_4, I_5, I_6 and I_7

where,

$I_1 = \{o\}$

$I_2 = \{o, a\}$, $I_3 = \{o, c\}$, $I_4 = \{o, d\}$

$I_5 = \{o, c, d\}$,

$I_6 = \{o, a, b, c\}$

and $I_7 = \{o, a, b, c, d, e\}$

Consider an α-ideal,

$$I_3 = \{0, c\}$$

$$\begin{aligned} I_3^{**} &= (I_3^*)^* \\ &= (\{0, a, d\})^* \\ &= \{0, c\} \\ &= I_3 \end{aligned}$$

Hence I_3 is an annihilator ideal. Similarly it can be verified that the remaining α-ideals are also annihilator ideals.

In the following theorem we study 0-distributive semilattices in which every α-ideal is an annihilator ideal.

Theorem 2.15 : If each α-ideal is an annihilator ideal in S then every minimal prime ideal is nondense.

Proof By Theorem 2.5, every minimal prime ideal M is an α-ideal. Hence by assumption M is an annihilator ideal and hence $M = M^{**}$. Let if possible suppose that M is a dense ideal. Then $M^* = \{0\} \implies M^{**} = \{0\}^* = S$. Hence M = S which contradicts that M is proper. Hence M is nondense. //

A property of a dense ideal is established in the following theorem :

<u>Theorem 2.16</u> : A dense ideal I contains a dense element if $I' = \{x \in S \,/\, x \in (a)^{**}$ for some $a \in I\}$ is an ideal in S and each α-ideal is an annihilator ideal.

<u>Proof</u> - If possible assume that $I \cap D = \emptyset$; where D is the set of all dense elements in S. Claim that $I' \cap D = \emptyset$. If $I' \cap D \neq \emptyset$ then there exists an element d in $I' \cap D$. But then $d \in (a)^{**}$ for some $a \in I$. Hence $(a)^{***} = (a)^{*} \subseteq (d)^{*} = \{0\}$; since $d \in D$. Hence $(a)^{*} = \{0\}$ i.e. $a \in D$. This in turn implies that $a \in I \cap D = \emptyset$, a contradiction. Hence the claim.

As I' is an ideal in S, by Theorem 2.8 I' is an α-ideal. By data I' is an innihilator ideal. Since D is filter in S (See Result 1.2.5), there exists maximal filter M in S such that $D \subseteq M$ and $I' \cap D = \emptyset$ (See the proof of Theorem 2.11). Denote by $P = S - M$. Then P is a minimal prime ideal in S (See Result 1.2.2). Hence P is a nondense (See Theorem 2.15).

Now $I \subseteq P \implies P^{*} \subseteq I^{*} = \{0\}$, since I is a dense ideal by data. This given $P^{*} = \{0\}$. i.e. P is a dense ideal. This contradicts the fact that P is a nondense ideal. Therefore $I \cap D \neq \emptyset$ ~~i.e.~~ proved that I contains a dense element.

<u>Remark 2.17</u> : When S becomes lattice, I' is an α-ideal [6] . Hence the converse of Theorem 2.16 is always

true for a 0-distributive lattice.

S is said to be quasicomplemented if for any $x \in S$ there exists $y \in S$ such that $(x)^{**} = (y)^{*}$ [8].

We characterize quasicomplemented semilattice in the following :

<u>Theorem 2.18</u> : Following statements are equivalent :

1) S is quasicomplemented

2) For any α-ideal J in S,

$$J = \cup \left\{ (f)^{*} \,/\, f \in F(J) \right\} ;$$

where,

$$F(J) = \left\{ x \in S \,/\, (z)^{*} \subseteq (x)^{**} \text{ for some } z \in J \right\}$$

3) For any α-ideal J in S there exists a semifilter F in S such that $J = \cup \left\{ (f)^{*} \,/\, f \in F \right\}$

<u>Proof</u> - (1) ===> (2)

Let $w \in \cup \left\{ (f)^{*} / f \in F(J) \right\}$. Then $w \wedge x = 0$ for some $x \in F(J)$. This implies that $w \wedge x = 0$ and $(z)^{*} \subseteq (x)^{**}$ for some $z \in J$. Hence $w \in (x)^{*} \subseteq (z)^{**}$. As J is an α-ideal and $z \in J$ we get that $(z)^{**} \subseteq J$. Thus $w \in (z)^{**} \subseteq J$ proving that $\cup \left\{ (f)^{*} / f \in F(I) \right\} \subseteq J$.

Now let $x \in J$. Then S being quasicomplemented there exists $y \in S$ such that $(x)^{**} = (y)^{*}$. But as $x \in (x)^{**}$, we get $x \in (y)^{*}$. Hence $x \wedge y = 0$. As $(x)^{*} \subseteq (y)^{**}$ and $x \in J$, we get $y \in F(I)$. Hence $x \in \cup \left\{ (f)^{*} / f \in F(J) \right\}$ proving that $J \subseteq \cup \{(f)^* / f \in F(J)\}$. By combining both the inclusions we get, $J = \cup \{(f)^* / f \in F(J)\}$ for any α-ideal J in S.

(2) ===> (3)

Let J be any α-ideal. Then by assumption we have, $J = \cup \{(f)^* / f \in F(J)\}$. Only it remains to prove that, $F(J) = \{x \in S / (z)^* \subseteq (x)^{**}, z \in J\}$ is a semifilter. Let $x_1 \leq x_2$ and $x_1 \in F(J)$. Then $(z)^* \subseteq (x_1)^{**}$ for some $z \in J$. But $x_1 \leq x_2$ implies that $(x_1)^{**} \subseteq (x_2)^{**}$. Therefore $(z)^* \subseteq (x_2)^{**}$ for $z \in J$. Hence $x_2 \in F(J)$ proving that $F(J)$ is a semifilter.

(3) ===> (1)

Let $x \in S$. Then $(x)^{**}$ is an α-ideal. Hence by assumption there exists semifilter F such that $(x)^{**} = \cup \{(f)^* / f \in F\}$. As $x \in (x)^{**}$, we get $x \wedge f = 0$ for some $f \in F$. Let $y \in (x)^{**}$. Then $y \wedge z = 0$ for all $z \in (x)^*$. As $y \in (x)^*$, we get $y \wedge f = 0$ proving that $y \in (f)^*$. Hence $(x)^{**} \subseteq (f)^*$.

Now obviously $(x)^{**} = \cup \{(f)^* / f \in F\}$ implies that $(x)^{**} \supseteq (f)^*$. Hence $(x)^* = (f)^*$. But this in turn implies that S is quasicomplemented. //

We know that,

$$O(F) = \{x \in S \,/\, x \wedge f = 0, \text{ for some } f \in F\}$$

where, F is any filter in S. But then by the definition of $O(F)$ we get,

$$O(F) = \cup \{(f)^* \,/\, f \in F\}$$

Using above characterization as every filter is also semi-filter we have the following :

in S we have $I = O(F)$

<u>Corollary 2.19</u> : If any α-ideal $I \wedge O(F)$ for some filter F in S then S is quasicomplemented.

••