CHAPTER - I

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1.1 Integral Transforms :

Many functions in analysis can be expressed as Lebesgue integrals or improper Riemann integrals of the form

$$F(s) = \int_{o \text{ or } -\infty}^{\infty} K(s,x)f(x)\,dx \cdot \tag{1.1-1}$$

A function F defined by an equation of this type (in which s may be real or complex) is called an integral transform of f . The function K which appears in the integrand is called the kernel of the transformation. It is assumed that the infinite integral in equation (1.1-1) is convergent. When the range of integration (o or $-\infty$, ∞) is replaced by a finite range (a,b), F(s) is called the finite integral transform of f(x).

Integral transformations are employed very extensively in both pure and applied mathematics. With the help of the different form of kernel K(s,x) and the range of integration, certain boundary value problems and certain types of integral equations can be solved. The important aspect of integral transformation is its inversion theorem.

The problems involving several variables can be solved by applying integral transformations successively with regard to several variables.

There are several problems which can be solved by the repeated application of Laplace and Hankel transforms.

The important integral transform, the Hankel transform, arises as a result of separation of variables in the problems posed in the cylindrical co-ordinates, involving Bessel functions. The Hankel type transform of a suitably restricted function f(x) is defined by the integral

$$F(y) = h_\lambda(f) = \int_0^\infty (y/x)^{\lambda/2} J_\lambda (2\sqrt{xy}) f(x)dx \qquad (1.1\text{-}2)$$

where $J_\lambda (2\sqrt{xy})$ is the Bessel function of first kind and of order λ .

If we construct an integral transform for which the kernel is the product of two Hankel type kernels, we may term this integral transform as the two-dimensional Hankel transform. If f(x,y) is a suitably restricted function on $o < x < \infty$, $o < y < \infty$, then its Hankel type transform F(u,v) is defined by the integral

$$F(u,v) = h_{\lambda,\mu}(f) = \int_0^\infty \int_0^\infty f(x,y)(u/x)^{\lambda/2}(v/y)^{\mu/2} J_\lambda(2\sqrt{ux})J_\mu(2\sqrt{vy}) \cdot dxdy \qquad (1.1\text{-}3)$$

where J_α is the Bessel function of first kind and of order α with α real. The inversion formula for the above Hankel type transform is given by

$$f(x,y) = \int_0^\infty \int_0^\infty F(u,v)(x/u)^{\lambda/2}(y/v)^{\mu/2} J_\lambda(2\sqrt{ux})J_\mu(2\sqrt{vy})dudv \ . \qquad (1.1\text{-}4)$$

1.2 Generalized Functions :

A collection V of elements $\phi, \psi, \theta \ldots$ is called a Linear-Space if the following axioms are satisfied :

(1) There is an operation $+$, called "addition" by which pair of elements ϕ and ψ can be combined to yield an element $\phi + \psi$ in such a way that the following properties are satisfied:

(1.a) $\phi + \psi = \psi + \phi$ (Commutativity)

(1.b) $(\phi + \psi) + \theta = \phi + (\psi + \theta)$ (Associativity)

(1.c) There exists a unique element 0, called zero in V, such that $\phi + 0 = \phi$ for every $\phi \in V$.

(1.d) For every $\phi \in V$, there exists a unique element $-\phi$ in V such that $\phi + (-\phi) = 0$.

(2) There is an operation, called "multiplication by a complex number", by which any complex number α and any $\phi \in V$ can be combined to yield an element $\alpha\phi \in V$ in such a way that the following conditions are fulfilled :

(2.a) $\alpha(\beta\phi) = (\alpha\beta)\phi$, for all complex numbers α and β.

(2.b) $1.\phi = \phi$ (1 denotes the number one)

(3) The following distributive laws hold :

(3.a) $\alpha(\phi + \psi) = \alpha\phi + \alpha\psi$

$(\alpha + \beta)\phi = \alpha\phi + \beta\phi$.

Let V be a linear space. A <u>seminorm</u> on V is a rule γ that assigns a real number $\gamma(\phi)$ to each $\phi \in V$ and that satisfies the following axioms :

(1) $\gamma(\alpha\phi) = |\alpha|\,\gamma(\phi)$ for every $\phi \in V$, $\alpha \in C^1$

(2) $\gamma(\phi + \psi) \leqslant \gamma(\phi) + \gamma(\psi)$, $\phi, \psi \in V$

The collection $S = \{\gamma_\nu\}_{\nu \in A}$ of seminorms on a linear space V is called <u>multinorm</u> on V if for every $\phi \in V$ with $\phi \neq 0$, there is some $\gamma \in S$ such that $\gamma(\phi) \neq 0$. Here, A denotes any finite or infinite index set. The sufficient condition for S to be a multinorm is that one of the seminorms in S is a norm.

A <u>multinormed space</u> V is a linear space having a topology generated by a multinorm S. If S is countable, V is called a <u>countably multinormed space</u>.

Let V be a multinormed space with the multinorm S . A sequence $\{\phi_\nu\}_{\nu=1}^{\infty}$ converges in V to the limit ϕ if and only if, for each $\gamma \in S$, $\gamma(\phi - \phi_\nu) \to 0$ as $\nu \to \infty$. The limit ϕ is unique.

$\{\phi_\nu\}$ is a Cauchy sequence in V if and only if all ϕ_ν are in V, and for each $\gamma \in S$, $\gamma(\phi_\nu - \phi_\mu) \to 0$ as ν and μ tend to infinity independently.

Every convergent sequence in V is a Cauchy sequence. When every Cauchy sequence in V is convergent, V is said to be

<u>complete</u>. A complete countably multinormed space is called a <u>Fréchet space</u>.

Let $\{V_m\}_{m=1}^{\infty}$ be a sequence of countably multinormed spaces such that $V_1 \subset V_2 \subset \ldots$. Assume that the topology of each V_m is stronger than the topology induced on it by V_{m+1} . Let $V = \bigcup_{m=1}^{\infty} V_m$. V is a linear space. A sequence $\{\phi_\nu\}_{\nu=1}^{\infty}$ is said to converge in V to ϕ if all the ϕ_ν and ϕ belong to some particular V_m and $\{\phi_\nu\}_{\nu=1}^{\infty}$ converges to ϕ in V_m. Under these circumstances V is called a <u>countable-union space</u>.

A sequence $\{\phi_\nu\}_{\nu=1}^{\infty}$ is called a Cauchy sequence in the countable-union space V if it is a Cauchy sequence in one of the spaces V_m . The countable-union space V is complete, whenever all the V_m are complete countably multinormed spaces.

Let V be a countably multinormed space. A rule that assigns a unique complex number to each $\phi \in V$ is called a <u>functional</u> on V. This complex number is denoted by $\langle f, \phi \rangle$. The functional f is said to be linear if for any $\phi, \psi \in V$ and any complex numbers α and β ,

$$\langle f, \alpha \phi + \beta \psi \rangle = \alpha \langle f, \phi \rangle + \beta \langle f, \psi \rangle .$$

Generalized functions were first introduced into science as a result of Dirac's research into quantum mechanics, where he systematically used the δ-function. Delta function, $\delta(x)$, is equal to zero everywhere except at origin where it is infinite and its integral over the infinite interval is one. It is obvious that $\delta(x)$ is not a function in the sense of classical analysis. This led to the introduction of the concept of a "generalized function."

Let I be an open subset of R^n or C^n. The set $V(I)$ is said to be a testing function space if the following conditions are satisfied :

(i) $V(I)$ consists entirely of smooth functions defined on I.

(ii) $V(I)$ is either complete countably multinormed space or complete countable-union space.

(iii) If $\{\phi_\nu\}_{\nu=1}^{\infty}$ converges in $V(I)$ to zero, then for every nonnegative integer $k \in R^n$, $\{D^k\phi_\nu\}_{\nu=1}^{\infty}$ converges to the zero function uniformly on every compact subset of I.

The collection of all continuous linear functionals on a countably multi-normed space or a countable-union space V is called the dual space of V and is denoted by V'.

A generalized function on I is any continuous linear functional on any testing function space $V(I)$ on I. In other words, f is called generalized function if it is a member of the dual space $V'(I)$ of some testing function space $V(I)$.

Let K be any compact subset of $I \subset R^n$. $D_K(I)$ is a set of all complex-valued smooth functions defined on I which vanish outside of K. $D_K(I)$ is a linear space under the usual definitions of addition and multiplication by scalars. The zero element of $D_K(I)$ is the identically zero function on I. The topology of $D_K(I)$ is generated by the multinorms $\{\gamma_k\}_{k=0}^{\infty}$ where γ_k is a seminorm on $D_K(I)$ defined by

$$\gamma_k(\emptyset) = \sup_{t \in I} \left| D^k \emptyset(t) \right| , \quad \emptyset \in D_K(I) ,$$

$k = 0, 1, 2, \ldots$.

The space $D_K(I)$ is a testing function space on I. If $\{K_m\}_{m=1}^{\infty}$ is sequence of compact subsets of I with the following properties : (i) $K_m \subset K_{m+1}$; $m = 1, 2, \ldots$. (ii) Each compact subset of I is contained in one of K_m, then $D_{K_m}(I) \subset D_{K_{m+1}}(I)$ and the topology of $D_{K_m}(I)$ is stronger than the topology induced on it by $D_{K_{m+1}}(I)$. Therefore, the countable-union space $D(I)$ is the space $D(I) = \bigcup_{m=1}^{\infty} D_{K_m}(I)$. $D'(I)$ is its dual.

A continuous linear functional on the space $D(I)$ is called a <u>distribution</u> on I. Thus the members of $D'(I)$, the dual space of $D(I)$ are the distributions.

$E(I)$ is the space of all complex-valued smooth functions on I. For each compact subset K of I and nonnegative integer $k \in R^n$, the seminorm $\gamma_{K,k}(\emptyset)$ on $E(I)$ is defined by

$$\gamma_{K,k}(\emptyset) = \sup_{t \in K} \left| D^k \emptyset(t) \right| , \quad \emptyset \in E(I)$$

$E(I)$ is a multinormed space with topology generated by the multinorm $\{\gamma_{K,k}\}$ where K traverses through the set of all compact subsets of I and $k = 0,1,2, \ldots$ in R^n.

$E(I)$ is also testing function space. The members of the dual space $E'(I)$ of $E(I)$ are called distributions with compact supports on I.

The overall advantage of generalized functions and distributions is that by widening the class of functions, many theorems and operations are freed from tedious restrictions. Number of treatises are available on various aspects of generalized functions and notably among those are Gelfand and Shilov [3], Zemanian [8] and Vladimirov [6].

1.3 Generalized Integral Transformations :

The topic of the Generalized Integral Transformations has been evolved as the confluence of two mathematical disciplines, "The Theory of Integral Transforms" and "The Theory of Generalized Functions" about which a brief discussion is given in the last two articles.

To extend the classical integral transformation, one has to construct a testing function space which satisfies certain properties of the kernel function of the transformation. The testing function spaces are different for different transforms.

There are mainly three ways in which a classical integral transform say

$$T(f)(x) = F(s) = \int_I K(s,x)\, f(x)dx \qquad (1.3\text{-}1)$$

can be extended to generalized functions. In the first , we construct a testing function space $V(I)$ containing the kernel $K(s,x)$, a dual space $V'(I)$ - the space of all continuous linear functionals defined on $V(I)$ and then we define an integral transform $F(s)$ of generalized functions directly as the application of a generalized function to the kernel function. Therefore, if $f \in V'(I)$ and $K(s,x) \in V(I)$, then

$$F(s) = \langle f(x), K(s,x) \rangle \;.$$

We will call this method as direct method or M_1 - method.

If the kernel function does not belong to $V(I)$ then one has to extend integral transform (1.3-1) by another way. First, construct a testing function space $V(I)$ on which integral transformation T is defined. Again, construct another testing function space

$$\overline{V(I)} = \{ T(\phi) = \Phi \;/\; \phi \in V(I) \}$$

such that T is an isomorphism from $V(I)$ to $\overline{V(I)}$. The inverse mapping T^{-1} is also an isomorphism from $\overline{V(I)}$ to $V(I)$. Then the generalized integral transformation T' on $V'(I)$ can be defined as the adjoint of T^{-1} on $\overline{V(I)}$. More specifically, for arbitrary $\Phi = T(\phi) \in \overline{V(I)}$ and $f \in V'(I)$ we define $F = T'(f)$ by

$$\langle F, \Phi \rangle = \langle f, \phi \rangle \qquad (1.3\text{-}2)$$

or, we write,

$$\langle T'f, \Phi \rangle = \langle f, T^{-1}(\Phi) \rangle .$$

This method will be called as M_2-method.

Another method which we may call M_3-method is to generalize an integral transform with kernel K_2 by first reducing it to another transform with kernel K_1 by a suitable change of variable, which can be generalized by the method-M_1 and then studying its properties with the help of the corresponding study of the K_1 transform for generalized functions.

An important and the first achievement to the theory of Generalized Integral Transformation is the extension of Fourier Transformation to the generalized functions.

Several distinct approaches have been proposed for an extension of an integral transform to generalized functions.

In 1952, Schwartz extended Laplace transform to generalized functions.

The first one to extend the Hankel transformation to generalized functions is J.L. Lions [5], Zemanian extended the Hankel transformation to generalized functions in 1966 [7].

For a real number μ and a positive real number a, Koh and Zemanian [4] have defined $H_{\mu,a}$ as a testing function space which contains the kernel, $\sqrt{xy}\, J_\mu(xy)$, as a function on $o < x < \infty$ for each fixed complex y in the strip

$$\Omega = \{ y : |I_m\, y| < a,\ y \neq o \text{ or a negative number} \}.$$

The Hankel transformation h'_μ is defined on the dual space $H'_{\mu,a}$ as follows : For $f \in H'_{\mu,a}$, $\mu \geq -\frac{1}{2}$,

$$(h'_\mu f)(y) \triangleq \langle f(x), \sqrt{xy}\ J_\mu\,(xy) \rangle , \qquad (1.3\text{-}3)$$

where y is a complex parameter belonging to the strip Ω.

Choudhary [1,2] has constructed a testing function space $H_{a,\lambda}$ for a real number λ and a positive number a which contains the kernel $(y/x)^{\lambda/2}\ J_\lambda\ (2\sqrt{xy})$ as a function on $o < x < \infty$ for each fixed y. The Hankel type transform F(y) of a distribution f in the dual space $H'_{a,\lambda}$ is defined by

$$F(y) = h'_\lambda f = \langle f(x),\ (y/x)^{\lambda/2}\ J_\lambda(2\sqrt{xy}) \rangle \qquad (1\cdot 3\cdot 4)$$

for suitably restricted y .

In the present work, the attempt has been made to extend the Hankel transformation defined by the equation (1.1-3) to a certain class of generalized functions.

1.4 Notations and Terminology :

The notations and terminology of this work follows that of [9].

R^n and C^n denote the real and complex n-dimensional euclidean spaces respectively.

By a compact set in R^n we mean a closed nd bounded set in R^n. If I is an open set in R^n and K is compact set in R^n such that $K \subset I$, then K is called compact subset of I .

If k is a nonnegative integer in R^1 the partial differential with respect to x is denoted y $D^k_x = \frac{\partial^k}{\partial x^k}$. We shall use the notation $\Delta_{\lambda,x}$ for $D_x x^{-2\lambda+1} D_x x^{\lambda}$.

A function whose domain is contained in either R^n or C^n and whose range is in either R^1 or C^1 is called conventional function. By a smooth function we mean a function that possesses continuous derivatives of all orders everywhere on R^n.

Let I be an open set in R^n. By a locally integrable function on I we mean a conventional function that is Lebesgue integrable on every open set J in R^n whose closure $\bar{J}$ is a compact subset of I .

A function of rapid descent is a conventional function $f(t)$ on R or C such that $|f(t)| = o(|t|^{-m})$ as $|t| \to \infty$ for every integer $m \in R$. A function $f(t)$ is said to be a function of slow growth if it is a conventional function on R or C such that there exists an integer $k \in R$ for which
$|f(t)| = O(|t|^k)$ as $|t| \to \infty$.

The support of a continuous function $f(t)$ defined on some open set Ω in R^n is the closure with respect to Ω of a set of points t where $f(t) \neq 0$. It is denoted by supp f .

If f is a generalized function on R^2, the notation $f(x,y)$, where $(x,y) \in R^2$, is used merely to irdicate that the testing functions, on which f is defined, hae (x,y) as their independent variable; it does not mean hat f is a function of (x,y). $\langle f, \emptyset \rangle$ denotes the numer assigned to element $\emptyset$ in a testing function space by a mmber of the dual space.

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