C H A P T E R - I I

HANKEL TYPE TRANSFORM OF DISTRIBUTIONS AND ITS PROPERTIES

CHAPTER - II

2.1 Introduction :

In this chapter we construct a testing function space $H_{a,b,\lambda,\mu}$, its dual space $H'_{a,b,\lambda,\mu}$ and extend the Hankel type transformation defined by the equation (1.1.3) to a certain class of generalized functions and study some properties of it.

For real numbers λ, μ and positive numbers a, b; we construct a testing function space $H_{a,b,\lambda,\mu}$ which contains the kernel $(u/x)^{\lambda/2}\ (v/y)^{\mu/2}\ J_\lambda(2\sqrt{ux})\ J_\mu(2\sqrt{vy})$ as a function on $o < x < \infty$, $o < y < \infty$ for each fixed u and v.

The Hankel type transform F(u,v) of a distribution f in the dual space $H'_{a,b,\lambda,\mu}$ is defined by

$$F(u,v) = h'_{\lambda,\mu}(f) = \left\langle f(x,y),\ (u/x)^{\lambda/2}(v/y)^{\mu/2}J_\lambda(2\sqrt{ux})J_\mu(2\sqrt{vy})\right\rangle$$

for suitably restricted u and v .

Let I denotes the open set ~~rectangle~~ $(o < x < \infty,\ o < y < \infty)$. D(I) is the space of all smooth functions on I having compact support on I and D'(I) is the dual space of Schwartz distributions on I.

2.2 The Testing Function Spaces $H_{a,b,\lambda,\mu}$ and Their Duals

Let λ, μ be any real numbers and a, b are positive real numbers. Then $H_{a,b,\lambda,\mu}$ can be defined as the space of all complex-valued smooth functions $\phi(x,y)$ defined on I such that for each k_1, $k_2 = 0, 1, 2, 3, \ldots$

$$T_{k_1,k_2}(\phi) = T^{\lambda,\mu,a,b}_{k_1,k_2}(\phi) = \sup_{\substack{0 < x < \infty \\ 0 < y < \infty}} \left| e^{-ax-by} \Delta^{k_1,k_2}_{\lambda,\mu,x,y} \phi(x,y) \right| < \infty \qquad \ldots(2.2\text{-}1)$$

where $\Delta^{k_1,k_2}_{\lambda,\mu,x,y} = \Delta^{k_1}_{\lambda,x} \Delta^{k_2}_{\mu,y}$ and

$$\Delta_{\lambda,x} = D_x x^{-\lambda+1} D_x x^{\lambda} \quad , \quad \Delta_{\mu,y} = D_y y^{-\mu+1} D_y y^{\mu}$$

where $D_x = \frac{\partial}{\partial x}$, $D_y = \frac{\partial}{\partial y}$.

Clearly $H_{a,b,\lambda,\mu}$ is a linear space over the field of complex numbers. Moreover, $\left\{ T^{\lambda,\mu,a,b}_{k_1,k_2} \right\}_{k_1,k_2=0}^{\infty}$ is a multinorm on $H_{a,b,\lambda,\mu}$.

Indeed, for any complex number β and $\phi \in H_{a,b,\lambda,\mu}$

$$T^{\lambda,\mu,a,b}_{k_1,k_2}(\beta\phi) = |\beta| \, T^{\lambda,\mu,a,b}_{k_1,k_2}(\phi) .$$

Also, for each ϕ_1, $\phi_2 \in H_{a,b,\lambda,\mu}$,

$$T^{\lambda,\mu,a,b}_{k_1,k_2}(\phi_1 + \phi_2) \leq T^{\lambda,\mu,a,b}_{k_1,k_2}(\phi_1) + T^{\lambda,\mu,a,b}_{k_1,k_2}(\phi_2)$$

Hence, each $T^{\lambda,\mu,a,b}_{k_1,k_2}(\emptyset)$ is a seminorm, and in addition $T^{\lambda,\mu,a,b}_{o,o}(\emptyset)$ is a norm on $H_{a,b,\lambda,\mu}$. We assign to $H_{a,b,\lambda,\mu}$, the topology generated by the multinorm $\left\{T^{\lambda,\mu,a,b}_{k_1,k_2}\right\}^{\infty}_{k_1,k_2=o}$ and this makes $H_{a,b,\lambda,\mu}$ a countably multinormed space. Moreover, $H_{a,b,\lambda,\mu}$ is a Hausdorff locally convex topological vector space that satisfies the first axiom of countability. The dual space $H'_{a,b,\lambda,\mu}$ consists of all continuous linear functionals on $H_{a,b,\lambda,\mu}$. The dual is a linear space to which we assign the weak topology generated by the multinorm $\left\{\xi_{\emptyset}(f)\right\}_{\emptyset}$, where $\xi_{\emptyset}(f) = |\langle f, \emptyset\rangle|$ and $\emptyset$ varies through $H_{a,b,\lambda,\mu}$.

A sequence $\{\emptyset_m\}^{\infty}_{m=1}$ converges in $H_{a,b,\lambda,\mu}$ to $\emptyset$ if and only if for each pair of nonnegative integers k_1 and k_2,

$$T^{\lambda,\mu,a,b}_{k_1,k_2}(\emptyset_m - \emptyset) \to 0 \quad \text{as} \quad m \to \infty .$$

A sequence $\{\emptyset_m\}^{\infty}_{m=1}$ is a Cauchy sequence in $H_{a,b,\lambda,\mu}$ if and only if

$T^{\lambda,\mu,a,b}_{k_1,k_2}(\emptyset_m - \emptyset_n) \to 0$ for every k_1, k_2 as m and n tend to infinity independently.

Lemma 2.2-1

$H_{a,b,\lambda,\mu}$ is complete and therefore a Fre'chet space.

Proof :

Let $\{\emptyset_m\}^{\infty}_{m=1}$ be a Cauchy sequence in $H_{a,b,\lambda,\mu}$. Then by equation (2.2-1), we have a uniform Cauchy sequence $\{\Psi_m\}^{\infty}_{m=1}$ on I for each k_1, k_2 where

$$\psi_m(x,y) = e^{-ax-by} \Delta^{k_1,k_2}_{\lambda,\mu,x,y} (\phi_m(x,y)) . \tag{2.2-2}$$

By Cauchy criterion, $\{\Delta^{k_1,k_2}_{\lambda,\mu,x,y} \psi_m\}$ converges uniformly to $\{\Delta^{k_1,k_2}_{\lambda,\mu,x,y} \psi\}$ on I, for all k_1,k_2. Hence, by standard theorem [1, p.402] there is a smooth function $\phi(x,y)$ on I such that $\psi_m(x,y) \to \psi(x,y)$ uniformly on I and $\Delta^{k_1,k_2}_{\lambda,\mu,x,y} \psi_m(x,y) \to \Delta^{k_1,k_2}_{\lambda,\mu,x,y} \psi(x,y)$ where

$$\psi(x,y) = e^{-ax-by} \Delta^{k_1,k_2}_{\lambda,\mu,x,y} (\phi(x,y)) \tag{2.2-3}$$

Since $\psi_m(x,y)$ is a uniform Cauchy sequence then for each $\epsilon > 0$, there is an integer N_{k_1,k_2} such that

$$\sup_{\substack{0<x<\infty \\ 0<y<\infty}} \left| \psi_m(x,y) - \psi_n(x,y) \right| < \epsilon$$

for every $m,n > N_{k_1,k_2}$.

Taking the limit as $n \to \infty$, we have

$$\sup_{\substack{0<x<\infty \\ 0<y<\infty}} \left| \psi_m(x,y) - \psi(x,y) \right| \leq \epsilon, \; m > N_{k_1,k_2} . \tag{2.2-4}$$

Thus, for each k_1,k_2

$$T^{\lambda,\mu,a,b}_{k_1,k_2} (\phi_m - \phi) \to 0 \text{ as } m \to \infty .$$

Finally, because of the uniformity of the convergence and the fact that each $\psi_m(x,y)$ is bounded on I, there exists a constant C_{k_1,k_2} not depending on m such that

$$|\psi_m(x,y)| \leq C_{k_1,k_2} \quad \text{for all} \quad (x,y).$$

Therefore, from (2.2-4), we get

$$\sup_{\substack{0<x<\infty \\ 0<y<\infty}} |\psi(x,y)| \leq \epsilon + C_{k_1,k_2}$$

which shows that $\psi(x,y)$ is bounded on I. Hence, a function $\phi(x,y)$, which is the limit of a given sequence $\{\phi_m\}$ is a member of $H_{a,b,\lambda,\mu}$. Thus the sequence $\{\phi_m\}$ converges in $H_{a,b,\lambda,\mu}$ to the unique limit ϕ. Hence $H_{a,b,\lambda,\mu}$ is complete. Since $H_{a,b,\lambda,\mu}$ is countably multinormed space which is complete, then $H_{a,b,\lambda,\mu}$ is a Fréchet space. $H_{a,b,\lambda,\mu}$ is complete and hence $H'_{a,b,\lambda,\mu}$ is complete.

<u>Theorem 2.2-1</u> : $H_{a,b,\lambda,\mu}$ is a testing function space.

<u>Proof</u> : Clearly, $H_{a,b,\lambda,\mu}$ satisfies the first two conditions of testing function space. Now we shall prove the third.

Let $\{\phi_m\}_{m=1}^{\infty}$ converge in $H_{a,b,\lambda,\mu}$ to zero. In view of (2.2-2) and the seminorms defined in (2.2-1), it follows by induction on k_1, k_2 that, for each pair of k_1, k_2, $\{D_x^{k_1} D_y^{k_2} (\phi_m)\}_{m=1}^{\infty}$ converges uniformly to zero function on every compact subset of I.

This completes the proof of the Theorem 2.2-1.

Now we list some properties of these spaces.

(i) Let $\lambda, \mu \geq -\frac{1}{2}$. For a fixed complex number (u,v) belonging the strip

$$\Omega = \{(u,v) \in C^2 \ / \ u, \ v \neq 0 \ \text{ or a negative number}\},$$

$$(u/x)^{\lambda/2} (v/y)^{\mu/2} J_\lambda (2\sqrt{ux}) J_\mu (2\sqrt{vy}) \in H_{a,b,\lambda,\mu} .$$

Indeed, by the analyticity of
$w^{-\lambda} z^{-\mu} J_\lambda (w) J_\mu (z)$, $w, z \neq 0$ it follows that
$(u/x)^{\lambda/2} (v/y)^{\mu/2} J_\lambda (2\sqrt{ux}) J_\mu (2\sqrt{vy})$ is smooth on I.

Also, in view of the property

$$\Delta^{k_1,k_2}_{\lambda,\mu,x,y} [(u/x)^{\lambda/2}(v/y)^{\mu/2} J_\lambda (2\sqrt{ux}) J_\mu(2\sqrt{vy})]$$
$$= (-1)^{k_1+k_2} u^{k_1} v^{k_2} (u/x)^{\lambda/2}(v/y)^{\mu/2} J_\lambda(2\sqrt{ux})J_\mu(2\sqrt{vy}) \quad (2.2\text{-}5)$$

and the fact $\left| e^{-ax-by}(2\sqrt{ux})^{-\lambda}(2\sqrt{vy})^{-\mu} J_\lambda(2\sqrt{ux})J_\mu(2\sqrt{vy})\right|$ is bounded for $0 < x < \infty$, $0 < y < \infty$, $(u,v) \in \Omega$ [4], the quantities

$$T^{\lambda,\mu,a,b}_{k_1,k_2} [(u/x)^{\lambda/2}(y/y)^{\mu/2} J_\lambda (2\sqrt{ux}) J_\mu(2\sqrt{vy})]$$

are finite for all $k_1, k_2 = 0, 1, 2, \ldots$. Thus, our assertion is verified.

(ii) Let $\lambda \geq -\frac{1}{2}$, $\mu \geq -\frac{1}{2}$. For a fixed complex number (u,v) belonging to the strip $\Omega = \{(u,v) \in C^2 \ / \ u,v \neq 0$ or a negative number$\}$,

$$D\left[\left(\frac{u}{x}\right)^{\lambda/2} \left(\frac{v}{y}\right)^{\mu/2} J_{\lambda}(2\sqrt{ux})\, J_{\mu}(2\sqrt{vy}) \right] \in H_{a,b,\lambda,\mu}$$

where $D = \frac{\partial}{\partial u}$ or $\frac{\partial}{\partial v}$.

Hence, $T^{\lambda,\mu,a,b}_{k_1,k_2} \left[D \left(\frac{u}{x}\right)^{\lambda/2} \left(\frac{v}{y}\right)^{\mu/2} J_{\lambda}(2\sqrt{ux})\, J_{\mu}(2\sqrt{vy}) \right] <$

$< \infty$ for any fixed (u,v) in Ω .

(iii) Let $o < c < a$, $o < d < b$. Then $H_{c,d,\lambda,\mu} \subset H_{a,b,\lambda,\mu}$, and the topology of $H_{c,d,\lambda,\mu}$ is stronger than the topology induced on it by $H_{a,b,\lambda,\mu}$. This follows from the inequality

$$T^{\lambda,\mu,a,b}_{k_1,k_2}(\phi) \leqslant T^{\lambda,\mu,c,d}_{k_1,k_2}(\phi)$$

for $\phi \in H_{a,b,\lambda,\mu}$.

Let $o < e^{-ax-by} < e^{-cx-dy}$ on I .

Then $\left| e^{-ax-by} \Delta^{k_1,k_2}_{\lambda,\mu,x,y} (\phi(x,y)) \right| \leqslant$

$$\leqslant \left| e^{-cx-dy} \Delta^{k_1,k_2}_{\lambda,\mu,x,y} (\phi(x,y)) \right|$$

so that, $T^{\lambda,\mu,a,b}_{k_1,k_2}(\phi(x,y)) \leqslant T^{\lambda,\mu,c,d}_{k_1,k_2}(\phi(x,y))$.

Thus, our assertion is implied by the last inequality. Hence, the restriction of $f \in H'_{a,b,\lambda,\mu}$ to $H_{c,d,\lambda,\mu}$ is in $H'_{c,d,\lambda,\mu}$.

(iv) $D(I) \subset H_{a,b,\lambda,\mu}$, and the topology of $D(I)$ is stronger than that induced on it by $H_{a,b,\lambda,\mu}$. Hence, the restriction of

$f \in H'_{a,b,\lambda,\mu}$ to $D(I)$ is in $D'(I)$. Thus, members of $H'_{a,b,\lambda,\mu}$ are distributions in Zemanian's sense [5, p.39].

(v) Let $f(x,y)$ be locally integrable function on I and such that $\int_0^\infty \int_0^\infty e^{ax+by} |f(x,y)| \, dxdy < \infty$.

Then f generates a regular generalized function in $H'_{a,b,\lambda,\mu}$ defined by

$$\langle f, \phi \rangle = \int_0^\infty \int_0^\infty f(x,y) \, \phi(x,y) \, dxdy \qquad (2.2\text{-}6)$$

Indeed,

$$|\langle f, \phi \rangle| = \left| \int_0^\infty \int_0^\infty \frac{f(x,y)}{e^{-ax-by}} \, e^{-ax-by} \, \phi(x,y) dxdy \right|$$

$$\leqslant T^{\lambda,\mu,a,b}_{o,o} (\phi(x,y)) \int_0^\infty \int_0^\infty \left| \frac{f(x,y)}{e^{-ax-by}} \right| dxdy$$

which shows that (2.2-6) truely defines a functional f on $H_{a,b,\lambda,\mu}$. This functional is clearly a linear one. Moreover, if $\{\phi_m\}_{m=1}^{\infty}$ converges in $H_{a,b,\lambda,\mu}$ to zero, then $T^{\lambda,\mu,a,b}_{o,o} (\phi_m) \to 0$. So that $|\langle f, \phi_m \rangle| \to 0$. Thus, f is also continuous on $H_{a,b,\lambda,\mu}$. Hence, f generates a regular generalized function in $H'_{a,b,\lambda,\mu}$.

(vi) For each $f \in H'_{a,b,\lambda,\mu}$, there exist a nonnegative integer r and a positive constant C such that, for all

$\phi \in H_{a,b,\lambda,\mu}$,

$$|\langle f, \phi \rangle| \leq C \max_{\substack{0 \leq k_1 \leq r \\ 0 \leq k_2 \leq r}} T^{\lambda,\mu,a,b}_{k_1,k_2}(\phi)$$

The proof of this statement is similar to that of [6, Theorem 3.3-1].

2.3 The Distributional Hankel Type Transformation

Let $-\frac{1}{2} \leq \lambda < \infty$, $-\frac{1}{2} \leq \mu < \infty$, $a, b > 0$. In view of note (iii) Sec. 2.2, to every $f \in H'_{a,b,\lambda,\mu}$ there exist the unique reals σ_f , $\rho_f > 0$ (possibly $\sigma_f = \infty$, $\rho_f = \infty$) such that $f \in H'_{c,d,\lambda,\mu}$ if $c < a < \sigma_f$, $d < b < \rho_f$ and $f \notin H'_{c,d,\lambda,\mu}$ if $c > \sigma_f$, $d > \rho_f$. We define the $(\lambda,\mu)^{th}$ order Hankel type transform $h'_{\lambda,\mu}f$ of f as the application of f to the kernel

$(u/x)^{\lambda/2} (v/y)^{\mu/2} J_\lambda(2\sqrt{ux})\, J_\mu(2\sqrt{vy})$; i.e.

$$F(u,v) = \langle f(x,y), (u/x)^{\lambda/2} (v/y)^{\mu/2} J_\lambda(2\sqrt{ux})\, J_\mu(2\sqrt{vy}) \rangle , \tag{2.3-1}$$

where $(u,v) \in \Omega_f = \{(u,v) \in C^2 / u,v \neq 0$ or a negative number $\}$.

The right hand side of (2.3-1) has a sense because, by note (i),

$(u/x)^{\lambda/2}(v/y)^{\mu/2}J_\lambda(2\sqrt{ux})J_\mu(2\sqrt{vy}) \in H_{c,d,\lambda,\mu}$

for every $c < a < \sigma_f$, $d < b < \rho_f$ and $(u,v) \in \Omega_f$. If

$f(x,y)$ satisfies the conditions of note (v), Sec.2.2 for every $c < a < \sigma_f$, $d < b < \rho_f$, then we may write

$$F(u,v) = (h_{\lambda,\mu}f)(y) = \int_0^\infty \int_0^\infty (u/x)^{\lambda/2} (v/y)^{\mu/2} \cdot$$

$$\cdot J_\lambda(2\sqrt{ux})\, J_\mu(2\sqrt{vy}) f(x,y)\,dxdy, \quad (u,v) \in \Omega_f \; . \qquad (2.3\text{-}2)$$

Lemma 2.3-1 : Let a, b, σ_f and ρ_f be fixed real numbers such that $0 < a < \sigma_f$, $0 < b < \rho_f$. For all (u,v) in the strip $\Omega = \{(u,v) \in C^2 \,/\, u, v \neq 0$ or a negative number$\}$, for $0 < x < \infty$, $0 < y < \infty$ and for $\lambda \geqslant -\frac{1}{2}$, $\mu \geqslant -\frac{1}{2}$,

$$\left| e^{-ax-by} (2\sqrt{ux})^{-\lambda} (2\sqrt{vy})^{-\mu} J_\lambda(2\sqrt{ux})\, J_\mu(2\sqrt{vy}) \right| < A_{\lambda,\mu} ,$$

where $A_{\lambda,\mu}$ is the constant with respect to u,v,x and y .

<u>Proof</u> : $w^{-\lambda}z^{-\mu}J_\lambda(w)J_\mu(z)$ is entire and hence bounded on any bounded domain. Moreover, from the asymptotic expansion of $J_\lambda(w)J_\mu(z)$ as $|w| \to \infty$, $|z| \to \infty$, we see that there exists a constant $C_{\lambda,\mu}$ such that, for $|w|>1$, $|z|>1$,

$$\left| w^{-\lambda}z^{-\mu}J_\lambda(w)J_\mu(z) \right| < C_{\lambda,\mu} |w|^{-\lambda-\frac{1}{2}} |z|^{-\mu-\frac{1}{2}} (e^{-\mathrm{Im}w} + e^{\mathrm{Im}w}) \cdot$$

$$\cdot (e^{-\mathrm{Im}z} + e^{\mathrm{Im}z}) < C_{\lambda,\mu} (e^{-\mathrm{Im}w} + e^{\mathrm{Im}w})(e^{-\mathrm{Im}z} + e^{\mathrm{Im}z})$$

Consequently, there exists another constant $A_{\lambda,\mu}$ such that, for all w, z,

$$\left| w^{-\lambda} z^{-\mu} J_\lambda(w) J_\mu(z) \right| < A_{\lambda,\mu} \, (e^{-\mathrm{Im} w} + e^{\mathrm{Im} w}) \, (e^{-\mathrm{Im} z} + e^{\mathrm{Im} z}) \, .$$

It follows that, for x, y, u, v, and λ, μ restricted as stated,

$$\left| e^{-ax-by} (2\sqrt{ux})^{-\lambda} (2\sqrt{vy})^{-\mu} J_\lambda(2\sqrt{ux}) \, J_\mu(2\sqrt{vy}) \right| <$$

$$< A \, (e^{-(a\sqrt{x} + \mathrm{Im}\sqrt{u})\sqrt{x}} + e^{(\mathrm{Im}\sqrt{u} - a\sqrt{x})\sqrt{x}}) \, .$$

$$\cdot \, (e^{-(b\sqrt{y} + \mathrm{Im}\sqrt{v})\sqrt{y}} + e^{(\mathrm{Im}\sqrt{v} - b\sqrt{y})\sqrt{y}}) < A_{\lambda,\mu} \, .$$

Hence the proof.

We shall now prove the analyticity theorem for the generalized Hankel type transform.

<u>Theorem 2.3-1</u> . If $(h'_{\lambda,\mu} f)(u,v) = F(u,v)$ for $(u,v) \in \Omega_f$, then $F(u,v)$ is analytic in u and for fixed v, $(u,v) \in \Omega_f$,

$$\frac{\partial}{\partial u} F(u,v) = \left\langle f(x,y), \; \frac{\partial}{\partial u} [(u/x)^{\lambda/2} (v/y)^{\mu/2} J_\lambda(2\sqrt{ux}) J_\mu(2\sqrt{vy})] \right\rangle . \tag{2.3-3}$$

<u>Proof</u> : Let (u,v) be an arbitrary but fixed point of Ω_f.

Since $(u/x)^{(\lambda-1)/2} (v/y)^{\mu/2} \, J_{\lambda-1}(2\sqrt{ux}) \, J_\mu(2\sqrt{vy})$

$$= \frac{\partial}{\partial u} [(u/x)^{\lambda/2} (v/y)^{\mu/2} J_\lambda(2\sqrt{ux}) \, J_\mu(2\sqrt{vy})] \in H_{a,b,\lambda,\mu}$$

for fixed $(u,v) \in \Omega_f$, the right-hand side of the equation (2.3-3) has meaning. Fix v. With u as center, construct two concentric circles of radii r and r_1 such that both circles are in Ω_f . Let $r < r_1$. Let $|\Delta u|$ be a nonzero complex increment in u-plane such that $|\Delta u| < r$. Consider

$$\frac{F(u+\Delta u,v) - F(u,v)}{\Delta u} - \Big\langle f(x,y),\ \frac{\partial}{\partial u}\Big[(u/x)^{\lambda/2} (v/y)^{\mu/2} \cdot$$

$$\cdot J_{\lambda}(2\sqrt{ux})\, J_{\mu}(2\sqrt{vy}) \Big]\Big\rangle = \Big\langle f(x,y),\ \Psi_{\Delta u}(x,y)\Big\rangle \qquad (2.3\text{-}4)$$

where

$$\Psi_{\Delta u}(x,y) = \frac{1}{\Delta u}\Big\{ \Big[\Big(\frac{u+\Delta u}{x}\Big)^{\lambda/2} \Big(\frac{v}{y}\Big)^{\mu/2} J_{\lambda}\big(2\sqrt{(u+\Delta u)x}\big) \cdot$$

$$\cdot J_{\mu}(2\sqrt{vy}) \Big] - \Big[\Big(\frac{u}{x}\Big)^{\lambda/2} \Big(\frac{v}{y}\Big)^{\mu/2} J_{\lambda}(2\sqrt{ux})\, J_{\mu}(2\sqrt{vy}) \Big] \Big\} -$$

$$- \frac{\partial}{\partial u}\Big[\Big(\frac{u}{x}\Big)^{\lambda/2} \Big(\frac{v}{y}\Big)^{\mu/2} J_{\lambda}(2\sqrt{ux})\, J_{\mu}(2\sqrt{vy}) \Big] .$$

Our theorem will be proven when we show that (2.3.4) converges to zero as $|\Delta u| \to 0$. This can be done by showing that $\Psi_{\Delta u}(x)$ converges in $H_{a,b,\lambda,\mu}$ to zero as $|\Delta u| \to 0$. Using the fact that

$$\Delta^{k_1}_{\lambda,x}(\phi(x,y)) = (-1)^{k_1} u^{k_1} \Big(\frac{u}{x}\Big)^{\lambda/2} \Big(\frac{v}{y}\Big)^{\mu/2} J_{\lambda}(2\sqrt{ux})\, J_{\mu}(2\sqrt{vy})$$

$$(2.3\text{-}5)$$

and by interchanging differentiation on u with differentiation on x, we may write $\Delta^{k_1}_{\lambda,x}\Psi_{\Delta u}(x,y)$ using Cauchy's integral formulas [3] as follows :

$$\Delta^{k_1}_{\lambda,x}\Psi_{\Delta u}(x,y) = \eta^{k_1}(-1)^{k_1}\Big[\frac{1}{2\pi i}\int_{C} \frac{(\eta/x)^{\lambda/2}(v/y)^{\mu/2} J_{\lambda}(2\sqrt{\eta x})J_{\mu}(2\sqrt{vy})}{(\eta - u - \Delta u)\Delta u}$$

$$\cdot\, d\eta - \frac{1}{2\pi i}\int_{C} \frac{(\eta/x)^{\lambda/2}(v/y)^{\mu/2} J_{\lambda}(2\sqrt{\eta x})\, J_{\mu}(2\sqrt{vy})}{(\eta - u)\,\Delta u}\, d\eta\ -$$

$$- \frac{1}{2\pi i} \int_C \frac{(\eta/x)^{\lambda/2} (v/y)^{\mu/2} J_\lambda(2\sqrt{\eta x})\, J_\mu(2\sqrt{vy})}{(\eta - u)^2} \, d\eta \;]$$

$$= \frac{(-1)^{k_1}}{2\pi i} \int_C \frac{[(\eta-u) - (\eta-u-\Delta u) - (\eta-u-\Delta u)\,]}{(\eta - u - \Delta u)(\eta - u)^2} \, .$$

$$\cdot\, \eta^{k_1} (\eta/x)^{\lambda/2} (v/y)^{\mu/2} J_\lambda(2\sqrt{\eta x})\, J_\mu(2\sqrt{vy})\, d\eta$$

$$= \frac{(-1)^{k_1}}{2\pi i} \int_C \frac{\Delta u}{(\eta-u)^2(\eta-u-\Delta u)} \eta^{k_1} \left(\frac{\eta}{x}\right)^{\lambda/2} \left(\frac{v}{y}\right)^{\mu/2} J_\lambda(2\sqrt{\eta x})\, J_\mu(2\sqrt{vy})\, d\eta \, .$$

Now, for all $\eta \in C$ and $0 < x < \infty$, $0 < y < \infty$,

$$\left| e^{-ax-by} \Delta^{k_1}_{\lambda,x} \psi_{\Delta u}(x,y) \right| \leq \frac{|\Delta u|}{2\pi} \int_C \frac{\eta^{|k_1+\lambda|} v^{\mu}\, 2^{\lambda+\mu}}{(\eta-u)^2(\eta-u-\Delta u)} \, .$$

$$\cdot \left| e^{-ax-by} (2\sqrt{\eta x})^{-\lambda} (2\sqrt{vy})^{-\mu} J_\lambda(2\sqrt{\eta x})\, J_\mu(2\sqrt{vy}) \right| d\eta$$

$$\leq \frac{K\,|\Delta u|\, A_{\lambda,\mu}}{r_1^2\,(r_1 - r)} \, .$$

Here, $A_{\lambda,\mu}$ be a bound on $\left| e^{-ax-by} (2\sqrt{\eta x})^{-\lambda} (2\sqrt{vy})^{-\mu} \right.$. $\left. J_\lambda(2\sqrt{\eta x})\, J_\mu(2\sqrt{vy}) \right|$ and K is a constant independent of u and x. Moreover, $|\eta - u - \Delta u| > r_1 - r > 0$ and $|\eta - u| = r_1$. Thus, as $|\Delta u| \to 0$, $\psi_{\Delta u}(x,y)$ converges to zero in $H_{a,b,\lambda,\mu}$. Consequently,

$$\langle f(x,y), \psi_{\Delta u}(x,y) \rangle \to 0 \text{ as } |\Delta u| \to 0 \, .$$

<u>Theorem 2.3-2</u> If $(h'_{\lambda,\mu}f)(u,v) = F(u,v)$ for $(u,v) \in \Omega_f$, then $F(u,v)$ is analytic in V and for fixed u, $(u,v) \in \Omega_f$,

$$\frac{\partial}{\partial v} F(u,v) = \left\langle f(x,y), \frac{\partial}{\partial v} \left[\left(\frac{u}{x}\right)^{\lambda/2} \left(\frac{v}{y}\right)^{\mu/2} J_\lambda(2\sqrt{ux})\, J_\mu(2\sqrt{vy}) \right] \right\rangle . \tag{2.3-6}$$

<u>Proof</u> : The proof of this theorem is very similar to that above and Theorem 1 [5].

<u>Theorem 2.3-3</u>. If $(h'_{\lambda,\mu} f)(u,v) = F(u,v)$ for $(u,v) \in \Omega_f$, then $F(u,v)$ is analytic on Ω_f and

$$DF(u,v) = \left\langle f(x,y), D\left[\left(\frac{u}{x}\right)^{\lambda/2} \left(\frac{v}{y}\right)^{\mu/2} J_\lambda(2\sqrt{ux})\, J_\mu(2\sqrt{vy}) \right] \right\rangle \tag{2.3-7}$$

where $D = \frac{\partial}{\partial u}$ or $\frac{\partial}{\partial v}$.

<u>Proof</u> : By the Theorems 2.3-1 and 2.3-2, at every point $(u',v') \in \Omega_f$, each of the functions $F(u,v')$ and $F(u',v)$ is analytic in the single variable u and v respectively. Therefore, invoking the Hartog's theorem [2, p.140], we see that $F(u,v)$ is analytic on Ω_f .

<u>Theorem 2.3-4</u>. (Boundedness of $F(u,v)$). If $F(u,v) = (h'_{\lambda,\mu}f)(u,v)$ for $(u,v) \in \Omega_f$, then $F(u,v)$ is bounded on any subset $\Omega'_f = \{(u,v) \in C^2 / u,v \neq 0$ or a negative number$\}$ of Ω_f according to

$$|F(u,v)| \leq |u|^\lambda |v|^\mu P_{a,b}(|uv|)$$

where $P_{a,b}(|uv|)$ is a polynomial in uv depending on a and b .

<u>Proof</u> : In view of a general result [5. Theorem 1.8.11], there exists a constant $C > 0$ and a nonnegative integer r such that

$$|F(u,v)| = |\langle f(x,y), \phi(x,y)\rangle|$$

$$\leq C \left\{ \max_{\substack{0\leq k_1 \leq r \\ 0 \leq k_2 \leq r}} \sup_{\substack{0<x<\infty \\ 0<y<\infty}} \left| e^{-ax-by} \right. \right. .$$

$$\left. \left. \cdot \Delta^{k_1,k_2}_{\lambda,\mu,x,y} \left[\left(\frac{u}{x}\right)^{\lambda/2} \left(\frac{v}{y}\right)^{\mu/2} J_\lambda(2\sqrt{ux})\, J_\mu(2\sqrt{vy}) \right] \right| \right\} .$$

By (2.2-5), the right hand side is equal to

$$C \left\{ \max_{\substack{0\leq k_1 \leq r \\ 0 \leq k_2 \leq r}} \sup_{\substack{0<x<\infty \\ 0<y<\infty}} \left| e^{-ax-by}\, 2^{\lambda+\mu}\, u^{k_1+\lambda}\, v^{k_2+\mu} \right. \right. .$$

$$\left. \left. \cdot (2\sqrt{ux})^{-\lambda} (2\sqrt{vy})^{-\mu} J_\lambda(2\sqrt{ux})\, J_\mu(2\sqrt{vy}) \right| \right\} .$$

since for all u, v in Ω'_f ,

$$\left| e^{-ax-by} (2\sqrt{ux})^{-\lambda} (2\sqrt{vy})^{-\mu} J_\lambda(2\sqrt{ux})\, J_\mu(2\sqrt{vy}) \right| <$$

$$< A_{\lambda,\mu} \quad [4] ,$$

where $A_{\lambda,\mu}$ is constant with respect to u, v, x and y, and the theorem follows.

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