C H A P T E R - I I I

INVERSION OF THE DISTRIBUTIONAL HANKEL TYPE TRANSFORMATION

CHAPTER - III

3.1 Introduction :

In this chapter the Inversion Theorem has been derived for generalized two-dimensional Hankel type transformation. This work is based on [1], [2], [3].

3.2 Inversion Theorem :

The inversion formula for distributional Hankel type transformation determines the restriction to D(I) of any $h_{\lambda,\mu}$ - transformable generalized function from its Hankel type transform. From this we will obtain an incomplete version of a uniqueness theorem, which states that two $h_{\lambda,\mu}$ - transformable generalized functions having the same transform must have the same restriction to D(I).

Theorem 3.2 Let $F(u,v) = h_{\lambda,\mu}(f)(u,v)$, $(u,v) \in \Omega_f$ as in (2.3-1) where (u,v) is restricted to the real positive axis. Let $\lambda \geqslant -\frac{1}{2}$, $\mu \geqslant -\frac{1}{2}$. Then for each $\phi \in D(I)$,

$$\left\langle \int_0^R \int_0^{R'} F(u,v)(x/u)^{\lambda/2}(y/v)^{\mu/2} J_\lambda(2\sqrt{ux})\, J_\mu(2\sqrt{vy})\,du\,dv,\ \phi(x,y)\right\rangle \to \langle f, \phi\rangle \text{ as } R, R' \to \infty. \qquad (3.2\text{-}1)$$

Proof : Let $\phi \in D(I)$. Choose real numbers a and b such that $0 < a < \sigma_f$, $0 < b < \rho_f$. Since the integral in (3.2-1) is a continuous function of (x,y), it generates a regular distribution in D(I). Hence, we have

$$\left\langle \int_0^R \int_0^{R'} F(u,v)(x/u)^{\lambda/2}(y/v)^{\mu/2} J_\lambda (2\sqrt{ux})\, J_\mu(2\sqrt{vy})\,du\,dv,\ \phi(x,y)\right\rangle$$

$$= \int_0^\infty \int_0^\infty \phi(x,y)\,dx\,dy \int_0^R \int_0^{R'} F(u,v)\ \left(\frac{x}{u}\right)^{\lambda/2}\left(\frac{y}{v}\right)^{\mu/2} J_\lambda(2\sqrt{ux}) J_\mu(2\sqrt{vy})\,du\,dv .$$

Since ϕ is of bounded support and the integrand on the right-hand side is a continuous function of (x,y,u,v), we can change the order of integration and obtain

$$\left\langle \int_0^R \int_0^{R'} F(u,v)\left(\frac{x}{u}\right)^{\lambda/2}\left(\frac{y}{v}\right)^{\mu/2} J_\lambda(2\sqrt{ux})\ J_\mu(2\sqrt{vy})\,du\,dv,\ \phi(x,y)\right\rangle$$

$$= \int_0^R \int_0^{R'} \left\langle f(t_1,t_2),\ \left(\frac{u}{t_1}\right)^{\lambda/2}\left(\frac{v}{t_2}\right)^{\mu/2} J_\lambda(2\sqrt{ut_1}) J_\mu(2\sqrt{vt_2})\right\rangle du\,dv \ .$$

$$\cdot \int_0^\infty \int_0^\infty \phi(x,y)\left(\frac{x}{u}\right)^{\lambda/2}\left(\frac{y}{v}\right)^{\mu/2} J_\lambda\ (2\sqrt{ux}),\ J_\mu\ (2\sqrt{vy})\,dx\,dy \ . \quad (3.2\text{-}2)$$

Now, let

$$\Phi(u,v) = \int_0^\infty \int_0^\infty \phi(x,y)\left(\frac{x}{u}\right)^{\lambda/2}\left(\frac{y}{v}\right)^{\mu/2} J_\lambda\ (2\sqrt{ux}) J_\mu(2\sqrt{vy})\,dx\,dy$$

and

$$M_{R,R'}^{(t_1,t_2)} = \int_0^R \int_0^{R'} \Phi(u,v)\left(\frac{u}{t_1}\right)^{\lambda/2}\left(\frac{v}{t_2}\right)^{\mu/2} J_\lambda(2\sqrt{ut_1}) J_\mu(2\sqrt{vt_2})\,du\,dv \ .$$

Since

$$\left| e^{-at_1-bt_2}\ \Delta^{k_1,k_2}_{\lambda,\mu,t_1,t_2}\ M_{R,R'}^{(t_1,t_2)} \right| = \left| \int_0^R \int_0^{R'} u^{k_1} v^{k_2}\ e^{-at_1-bt_2} \right. .$$

$$\cdot \left(\frac{u}{t_1}\right)^{\lambda/2}\left(\frac{v}{t_2}\right)^{\mu/2} J_\lambda\ (2\sqrt{ut_1}) J_\mu(2\sqrt{vt_2})\ \Phi\ (u,v)\ du\,dv \left.\right|$$

$$\leq B \int_0^R \int_0^{R'} |\Phi(u,v)| \, u^{k_1} v^{k_2} \, du dv$$

for some suitable constant B, $M_{R,R'}^{(t_1,t_2)} \in H_{a,b,\lambda,\mu}$ for each $R, R' > 0$. Using the Riemann sum technique, we can write equation (3.2-2) as

$$\left\langle \int_0^R \int_0^{R'} F(u,v) \left(\frac{x}{u}\right)^{\lambda/2} \left(\frac{y}{v}\right)^{\mu/2} J_\lambda(2\sqrt{ux}) \, J_\mu(2\sqrt{vy}) du dv, \right.$$

$$\left. \phi(x,y) \right\rangle = \left\langle f(t_1,t_2), M_{R,R'}^{(t_1,t_2)} \right\rangle , \quad R, R' > 0 .$$

This has sense because $M_{R,R'}^{(t_1,t_2)} \in H_{a,b,\lambda,\mu}$. Hence, now, proof of the theorem will be complete, if we show that $M_{R,R'}^{(t_1,t_2)} \longrightarrow \phi(t_1,t_2)$ in $H_{a,b,\lambda,\mu}$ as $R, R' \to \infty$.

Since $\Phi(u,v) \left(\frac{u}{t_1}\right)^{\lambda/2} \left(\frac{v}{t_2}\right)^{\mu/2} (2\sqrt{ut_1})$. $J_\mu(2\sqrt{vt_2})$ is smooth and $\phi \in D(I)$, we may repeatedly differentiate under the integral sign and use the equation (2.2-5) to write

$$\Delta_{\lambda,\mu,t_1,t_2}^{k_1,k_2} M_{R,R'}^{(t_1,t_2)} = \int_0^R \int_0^{R'} \Phi(u,v) \, .$$

$$\cdot \, \Delta_{\lambda,\mu,t_1,t_2}^{k_1,k_2} \left[\left(\frac{u}{t_1}\right)^{\lambda/2} \left(\frac{v}{t_2}\right)^{\mu/2} J_\lambda(2\sqrt{ut_1}) \, J_\mu(2\sqrt{vt_2}) \right] du dv$$

$$= \int_0^R \int_0^{R'} (-1)^{k_1+k_2} u^{k_1} v^{k_2} \left(\frac{u}{t_1}\right)^{\lambda/2} \left(\frac{v}{t_2}\right)^{\mu/2} J_\lambda(2\sqrt{ut_1}) J_\mu(2\sqrt{vt_2}) du dv \, .$$

$$\cdot\left\{\int_0^\infty\int_0^\infty \phi(x,y)\left(\frac{x}{u}\right)^{\lambda/2}\left(\frac{y}{v}\right)^{\mu/2} J_\lambda(2\sqrt{ux})\, J_\mu(2\sqrt{vy})\, dxdy\right\}$$

$$= \int_0^R\int_0^{R'} u^{-\lambda/2}\, t_1^{-\lambda/2}\, v^{-\mu/2}\, t_2^{-\mu/2}\, J_\lambda(2\sqrt{ut_1})\, J_\mu(2\sqrt{vt_2})\, dudv\,\cdot$$

$$\cdot\int_0^\infty\int_0^\infty x^\lambda\, y^\mu\, \phi(x,y)\,(-1)^{k_1+k_2}\, u^{k_1}\, v^{k_2}\left(\frac{u}{x}\right)^{\lambda/2}\left(\frac{v}{y}\right)^{\mu/2}\cdot$$

$$\cdot\; J_\lambda(2\sqrt{ux})\, J_\mu(2\sqrt{vy})\, dxdy$$

$$= \int_0^R\int_0^{R'} u^{-\lambda/2}\, v^{-\mu/2}\, t_1^{-\lambda/2}\, t_2^{-\mu/2}\, J_\lambda(2\sqrt{ut_1}) J_\mu(2\sqrt{vt_2})\, dudv\,\cdot$$

$$\cdot\int_0^\infty\int_0^\infty x^\lambda\, y^\mu\, \phi(x,y)\, \Delta^{k_1,k_2}_{\lambda,\mu,x,y}\left[\left(\frac{u}{x}\right)^{\lambda/2}\left(\frac{v}{y}\right)^{\mu/2} J_\lambda(2\sqrt{ux}) J_\mu(2\sqrt{vy})\right] dxdy$$

$$= \int_0^R\int_0^{R'} t_1^{-\lambda/2}\, t_2^{-\mu/2}\, J_\lambda(2\sqrt{ut_1})\, J_\mu(2\sqrt{vt_2})\, dudv\;.$$

$$\cdot\int_0^\infty\int_0^\infty x^{\lambda/2}\, y^{\mu/2} J_\lambda(2\sqrt{ux})\, J_\mu(2\sqrt{vy})\, \Delta^{k_1,k_2}_{\lambda,\mu,x,y}(\phi(x,y))\, dxdy\quad\cdot$$

The last equality is obtained by integrating by parts the inner integral $2k_1$ times with respect to x first and then $2k_2$ times with respect to y and noting that ϕ is of compact support. Let us reverse the order of integration and use the formula

$$L_{R,R'}^{(x,y,t_1,t_2)} = x^{\lambda/2} y^{\mu/2} t_1^{-\lambda/2} t_2^{-\mu/2} \cdot$$

$$\cdot \int_0^R \int_0^{R'} J_\lambda(2\sqrt{ux})\, J_\lambda(2\sqrt{ut_1})\, J_\mu(2\sqrt{vy})\, J_\mu(2\sqrt{vt_2})\, dudv$$

$$= \frac{\sqrt{R}\sqrt{R'}\, x^{\lambda/2} y^{\mu/2} t_1^{-\lambda/2} t_2^{-\mu/2}}{(x - t_1)(y - t_2)} \Big[\Big\{ \sqrt{x}\, J_{\lambda+1}(2\sqrt{xR})\, J_\lambda(2\sqrt{t_1 R}) -$$

$$- \sqrt{t_1}\, J_\lambda(2\sqrt{xR})\, J_{\lambda+1}(2\sqrt{t_1 R}) \Big\} \Big\{ \sqrt{y}\, J_{\mu+1}(2\sqrt{yR'}) \cdot$$

$$\cdot J_\mu(2\sqrt{t_2 R'}) - \sqrt{t_2}\, J_\mu(2\sqrt{yR'})\, J_{\mu+1}(2\sqrt{t_2 R'}) \Big\} \Big], \qquad (3.2\text{-}3)$$

then we obtain,

$$\Delta_{\lambda,\mu,t_1,t_2}^{k_1,k_2} M_{R,R'}^{(t_1,t_2)} = \int_0^\infty \int_0^\infty L_{R,R'}^{(x,y,t_1,t_2)} \cdot$$

$$\cdot \Delta_{\lambda,\mu,x,y}^{k_1,k_2} (\emptyset(x,y))\, dxdy \quad .$$

Denote $\Delta_{\lambda,\mu,x,y}^{k_1,k_2} (\emptyset(x,y))$ by $\emptyset_K(x,y)$. (3.2-4)

Now, suppose that the support of $\emptyset(x,y)$ is contained in $[A,B] \times [C,D]$, where $o < A < B < \infty$, $o < C < D < \infty$. Let us break the integral in (3.2-4) into

$$\Delta_{\lambda,\mu,t_1,t_2}^{k_1,k_2} M_{R,R'}^{(t_1,t_2)} = \int_0^{t_2-\delta} \int_0^\infty + \int_{t_2-\delta}^\infty \int_0^{t_1-\delta} + \int_{t_2-\delta}^{t_2+\delta} \int_{t_1-\delta}^{t_1+\delta} +$$

$$+ \int_{t_2-\delta}^{t_2+\delta} \int_{t_1+\delta}^\infty + \int_{t_2+\delta}^\infty \int_{t_1-\delta}^\infty$$

$$= V_1(t_1,t_2) + V_2(t_1,t_2) + V_3(t_1,t_2) + V_4(t_1,t_2) + V_5(t_1,t_2).$$

We shall first show that

$$N_{R,R'}^{(t_1,t_2)} = e^{-at_1-bt_2} [V_3(t_1,t_2) - \Delta^{k_1,k_2}_{\lambda,\mu,t_1,t_2}(\phi(t_1,t_2))]$$

converges uniformly to zero on $o < t_1 < \infty$, $o < t_2 < \infty$, as $R, R' \to \infty$. If either $o < t_1 + \delta \leqslant A$, $o < t_2 < \infty$ or $t_1 - \delta \geqslant B$, $o < t_2 < \infty$ and either $o < t_1 < \infty$, $o < t_2 + \delta \leqslant C$ or $o < t_1 < \infty$, $t_2 - \delta \geqslant D$, then $V_3(t_1,t_2) \equiv 0$ and $\Delta^{k_1,k_2}_{\lambda,\mu,t_1,t_2}\phi(t_1,t_2) \equiv 0$.

Therefore, we have to consider the rectangle $A - \delta < t_1 < B + \delta$, $C - \delta < t_2 < D + \delta$. Moreover, since the support of ϕ i.e. supp $\phi \subset [A, B] \times [C, D]$, we take the integral in (3.2-4) on $(A,B) \times (C,D)$. For $R,R' > 0$, using the asymptotic expansions of the Bessel functions, we have for large R, R',

$$N_{R,R'}^{(t_1,t_2)} = \frac{1}{4\pi^2} e^{-at_1-bt_2} \int_{t_2-\delta}^{t_2+\delta} \int_{t_1-\delta}^{t_1+\delta} x^{\lambda/2-1/4} y^{\mu/2-1/4} \cdot t_1^{-\lambda/2-1/4} t_2^{-\mu/2-1/4} \phi_K(x,y) \,.$$

$$\cdot \frac{\sin(2\sqrt{xR} - 2\sqrt{t_1R}) \sin(2\sqrt{yR'} - 2\sqrt{t_2R'})}{(\sqrt{x} - \sqrt{t_1})(\sqrt{y} - \sqrt{t_2})} dydx -$$

$$- \frac{1}{4\pi^2} e^{-at_1-bt_2} \int_{t_2-\delta}^{t_2+\delta} \int_{t_1-\delta}^{t_1+\delta} x^{\lambda/2-1/4} y^{\mu/2-1/4} t_1^{-\lambda/2-1/4} t_2^{-\mu/2-1/4} \,.$$

$$\cdot \phi_K(x,y) \frac{\sin(2\sqrt{xR} - 2\sqrt{t_1R}) \cos(2\sqrt{yR'} + 2\sqrt{t_2R'} - \mu\pi)}{(\sqrt{x} - \sqrt{t_1})(\sqrt{y} + \sqrt{t_2})} .$$

$$. \, dydx - \frac{1}{4\pi^2} e^{-at_1-bt_2} \int_{t_2-\delta}^{t_2+\delta} \int_{t_1-\delta}^{t_1+\delta} x^{\lambda/2-1/4} y^{\lambda/2-1/4} \, .$$

$$. \, t_1^{-\lambda/2-1/4} t_2^{-\mu/2-1/4} \phi_K (x,y) \, .$$

$$\frac{\sin (2\sqrt{yR'} - 2\sqrt{t_2R'}) \cos (2\sqrt{xR} + 2\sqrt{t_1R} - \lambda\pi)}{(\sqrt{y} - \sqrt{t_2})(\sqrt{x} + \sqrt{t_1})} dydx \; +$$

$$+ \frac{1}{4\pi^2} e^{-at_1-bt_2} \int_{t_2-\delta}^{t_2+\delta} \int_{t_1-\delta}^{t_1+\delta} x^{\lambda/2-1/4} y^{\mu/2-1/4} t_1^{-\lambda/2-1/4} t_2^{-\mu/2-1/4} \, .$$

$$. \, \phi_K(x,y) \frac{\cos(2\sqrt{xR} + 2\sqrt{t_1R} - \lambda\pi) \cos (2\sqrt{yR'} + 2\sqrt{t_2R'} - \mu\pi)}{(\sqrt{x} + \sqrt{t_1})(\sqrt{y} + \sqrt{t_2})} dydx -$$

$$- e^{-at_1-bt_2} \Delta^{k_1,k_2}_{\lambda,\mu,t_1,t_2} (\phi (t_1,t_2)) \, . \qquad (3.2\text{-}5)$$

First, consider the 4th term of (3.2-5). Since $\operatorname{supp} \phi (x,y) \subset [A,B] \times [C,D]$, the function is bounded on $\{ (x,y,t_1,t_2) \, / \, A < x < B, \; C < y < D, \; A-\delta < t_1 < B + \delta, \; C - \delta < t_2 < D + \delta \}$.

Hence for any given $\epsilon > 0$, the magnitude of this term can be made less than $\epsilon/2$ for all $R, R' > 1$ by choosing δ small enough, say $\delta = \delta_1$.

Now, consider the sum of the first and last term in (3.2-5). We can write this sum as

$$\frac{1}{\pi^2} \int_{\sqrt{t_2-\delta}-\sqrt{t_2}}^{\sqrt{t_2+\delta}-\sqrt{t_2}} \int_{\sqrt{t_1-\delta}-\sqrt{t_1}}^{\sqrt{t_1+\delta}-\sqrt{t_1}} H(T_1,T_2,t_1,t_2) \sin (2\sqrt{R}\, T_1) \sin \cdot$$

$$\cdot (2\sqrt{R'}\, T_2)\, dT_2 dT_1 + e^{-at_1-bt_2} \Delta^{k_1,k_2}_{\lambda,\mu,t_1,t_2}(\phi(t_1,t_2)) \cdot$$

$$\cdot \left[\frac{1}{\pi^2} \int_{-2\sqrt{R'}(\sqrt{t_2}-\sqrt{t_2-\delta})}^{2\sqrt{R'}(\sqrt{t_2+\delta}-\sqrt{t_2})} \frac{\sin v}{v}\, dv \int_{-2\sqrt{R}(\sqrt{t_1}-\sqrt{t_1-\delta})}^{2\sqrt{R}(\sqrt{t_1+\delta}-\sqrt{t_1})} \frac{\sin u}{u}\, du - 1 \right] \tag{3.2-6}$$

where

$$H(T_1,T_2,t_1,t_2) = e^{-at_1-bt_2} \left[\phi_K\left((T_1+\sqrt{t_1})^2, (T_2+\sqrt{t_2})^2 \right) \cdot \right.$$

$$\cdot (T_1+\sqrt{t_1})^{\lambda+1/2}\, t_1^{-\lambda/2-1/4} (T_2+\sqrt{t_2})^{\mu+1/2}\, t_2^{-\mu/2-1/4} -$$

$$\left. - \Delta^{k_1,k_2}_{\lambda,\mu,t_1,t_2} (\phi(t_1,t_2)) \right] / T_1 T_2 .$$

Since $H(T_1,T_2,t_1,t_2)$ is a continuous function of (T_1,T_2,t_1,t_2) and supp. $\phi(x,y) \subset [A,B] \times [C,D]$, $H(T_1,T_2,t_1,t_2)$ is bounded function of (T_1,T_2) on $\sqrt{t_1-\delta} - \sqrt{t_1} < T_1 < \sqrt{t_1+\delta} - \sqrt{t_1}$, $\sqrt{t_2-\delta} - \sqrt{t_2} < T_2 < \sqrt{t_2+\delta} - \sqrt{t_2}$ for all $o < t_1 < \infty$, $o < t_2 < \infty$. Hence choosing δ very small, say $\delta = \delta_2$, the first term in (3.2-6) can be made less than $\epsilon/2$ for all $R,R' > 1$. Now, fix $\delta = \min(\delta_1,\delta_2)$. The second term in (3.2-6) converges uniformly to zero on $o < t_1 < \infty$, $o < t_2 < \infty$.

Let us consider the second term in (3.2-5). Put $\sqrt{x} = \sqrt{x} + \sqrt{t_1}$, $\sqrt{y} = \sqrt{y} + \sqrt{t_2}$. Then the second term in (3.2-5) can be written as

$$\frac{1}{4\pi^2} e^{-at_1-bt_2} \int_{\sqrt{t_2-\delta}+\sqrt{t_2}}^{\sqrt{t_2+\delta}+\sqrt{t_2}} \int_{\sqrt{t_1-\delta}+\sqrt{t_1}}^{\sqrt{t_1+\delta}+\sqrt{t_1}} (\sqrt{x}+\sqrt{t_1})^{\lambda-1/2}\, t_1^{-\lambda/2-1/4} \cdot$$

$$\cdot (\sqrt{y}+\sqrt{t_2})^{\mu-1/2}\, t_2^{-\mu/2-1/4}\, \phi_K((\sqrt{x}+\sqrt{t_1})^2, (\sqrt{y}+\sqrt{t_2})^2) \cdot$$

$$\cdot \sin(2\sqrt{Rx}) \cos(2\sqrt{yR'} - \mu\pi) \Big/ \sqrt{x}(\sqrt{y} + 2\sqrt{t_2}) \ .$$

The function

$$e^{-at_1-bt_2} (\sqrt{x}+\sqrt{t_1})^{\lambda-1/2}\, t_1^{-\lambda/2-1/4}\, (\sqrt{y}+\sqrt{t_2})^{\mu-1/2}\, t_2^{-\mu/2-1/4} \cdot$$

$$\cdot \phi_K((\sqrt{x}+\sqrt{t_1})^2, (\sqrt{y}+\sqrt{t_2})^2) \Big/ \sqrt{x}(\sqrt{y} + 2\sqrt{t_2})$$

is continuous for $\sqrt{x}+\sqrt{t_1} > 0$, $\sqrt{y}+\sqrt{t_2} > 0$, $t_1 > 0$, $t_2 > 0$, Since supp. $\phi(x,y) \subset [A,B] \times [C,D]$, the function is bounded on

$$\{(\sqrt{x}, \sqrt{y}, t_1, t_2) \,/\, \sqrt{t_1-\delta} + \sqrt{t_1} < \sqrt{x} < \sqrt{t_1+\delta} + \sqrt{t_1}, \sqrt{t_2-\delta} +$$

$$+ \sqrt{t_2} < \sqrt{y} < \sqrt{t_2+\delta} + \sqrt{t_2}\ ,\ A - \delta < t_1 < B+\delta,\ C-\delta < t_2 < D + \delta\} \ .$$

Hence for any given $\epsilon > 0$, the magnitude of this second term can be made less than $\epsilon/2$ for all R, $R' > 1$ by choosing δ small enough, say $\delta = \delta_3$.

Similarly, we can show the convergence of the third term in (3.2-5). Thus

$$\left| N_{R,R'}^{(t_1,t_2)} \right| < \epsilon$$

on $o < t_1 < \infty$, $o < t_2 < \infty$. Since $\epsilon > o$ is arbitrary, we conclude that $N_{R,R'}^{(t_1,t_2)}$ converges uniformly to zero on $o < t_1 < \infty$, $o < t_2 < \infty$ as $R, R' \to \infty$.

Now, consider

$$P_{R,R'}(t_1,t_2) = e^{-at_1 - bt_2} \; V_1(t_1,t_2)$$

$$= e^{-at_1, bt_2} \int_o^{t_2-\delta} \int_o^{\infty} L_{R,R'}^{(x,y,t_1,t_2)} \, \phi_K(x,y) \, dydx$$

$$P_{R,R'}^{(t_1,t_2)} \equiv 0 \quad \text{for } o < t_1 < \infty, \; t_2 - \delta \leqslant C .$$

Therefore, we have to consider the range $o < t_1 < \infty$, $C < t_2 - \delta < \infty$. Let $D' = \min(D, t_2 - \delta)$. For $R > o$ and large R', the asymptotic expansions of the Bessel's functions implies that

$$P_{R,R'}^{(t_1,t_2)} = \frac{1}{4\pi^2} e^{-at_1 - bt_2} \int_o^{t_2-\delta} \int_o^{\infty} x^{\lambda/2-1/4} \; y^{\mu/2-1/4} \, .$$

$$\cdot \; t_1^{-\lambda/2-1/4} \; t_2^{-\mu/2-1/4} \; \phi_K^{(x,y)} \; .$$

$$\cdot \frac{\sin(2\sqrt{xR} - 2\sqrt{t_1 R}) \sin(2\sqrt{yR'} - 2\sqrt{t_2 R'})}{(\sqrt{x} - \sqrt{t_1})(\sqrt{y} - \sqrt{t_2})} \, dydx \; -$$

$$- \frac{1}{4\pi^2} e^{-at_1 - bt_2} \int_o^{t_2-\delta} \int_o^{\infty} x^{\lambda/2-1/4} \; y^{\mu/2-1/4} \; t_1^{-\lambda/2-1/4} \; t_2^{-\mu/2-1/4} \, .$$

$$\cdot \; \phi_K^{(x,y)} \; \frac{\sin(2\sqrt{xR} - 2\sqrt{t_1 R}) \cos(2\sqrt{yR'} + 2\sqrt{t_2 R'} - \mu\pi)}{(\sqrt{x} - \sqrt{t_1})(\sqrt{y} + \sqrt{t_2})} \, dydx \; -$$

$$- \frac{1}{4\pi^2} e^{-at_1-bt_2} \int_0^{t_2-\delta} \int_0^{\infty} x^{\lambda/2-1/4} y^{\mu/2-1/4} t_1^{-\lambda/2-1/4} t_2^{-\mu/2-1/4} \cdot$$

$$\cdot \varnothing_K(x,y) \frac{\sin(2\sqrt{yR'} - 2\sqrt{t_2R'}) \cos(2\sqrt{xR} + 2\sqrt{t_1R} - \lambda\pi)}{(\sqrt{y} - \sqrt{t_2})(\sqrt{x} + \sqrt{t_1})} dydx +$$

$$+ \frac{1}{4\pi^2} e^{-at_1-bt_2} \int_0^{t_2-\delta} \int_0^{\infty} x^{\lambda/2-1/4} y^{\mu/2-1/4} t_1^{-\lambda/2-1/4} t_2^{-\mu/2-1/4} \cdot$$

$$\cdot \varnothing_K(x,y) \frac{\cos(2\sqrt{xR} + 2\sqrt{t_1R} - \lambda\pi) \cos(2\sqrt{yR'} + 2\sqrt{t_2R'} - \mu\pi)}{(\sqrt{x} + \sqrt{t_1})(\sqrt{y} + \sqrt{t_2})} dydx \cdot \tag{3.2-7}$$

Upon integrating by parts the inner integral in the first term in (3.2-7) and noting that for any $y > o$, the limits at A and at B are zero, the integral in the first term equals

$$\frac{1}{R} \int_C^{D'} \frac{y^{\mu/2-1/4}}{(\sqrt{y} - \sqrt{t_2})} \left\{ \int_A^B \left[D_x \left(\frac{\varnothing_K(x,y)\, x^{\lambda/2-1/4}}{(\sqrt{x} - \sqrt{t_1})} \right) \right] \cdot \right.$$

$$\left. \cdot \left[\frac{1}{2} \sin(2\sqrt{xR} - 2\sqrt{t_1R}) - \sqrt{xR} \cos(2\sqrt{xR} - 2\sqrt{t_1R}) \right] dx \right\} \cdot$$

$$\cdot \sin(2\sqrt{yR'} - 2\sqrt{t_2R'})dy \cdot$$

Again, integrating by parts with respect to y, we have

$$\frac{1}{RR'} \left\{ \frac{y^{\mu/2-1/4}}{(\sqrt{y} - \sqrt{t_2})} \int_A^B \left[D_x \left(\frac{x^{\lambda/2-1/4} \varnothing_K(x,y)}{(\sqrt{x} - \sqrt{t_1})} \right) \right] \cdot \right.$$

$$\left. \cdot \left[\frac{1}{2} \sin(2\sqrt{xR} - 2\sqrt{t_1R}) - \sqrt{xR} \cos(2\sqrt{xR} - 2\sqrt{t_1R}) \right] dx \right\} \cdot$$

$$\left\{ - \sqrt{yR'} \cos (2\sqrt{yR'} - 2\sqrt{t_2 R'}) + \frac{1}{2} \sin (2\sqrt{yR'} - 2\sqrt{t_2 R'}) \right\}_C^{D'} -$$

$$- \frac{1}{RR'} \int_C^{D'} dy \int_A^B \left[\frac{\partial^2}{\partial x \partial y} \left(\frac{x^{\lambda/2-1/4} y^{\mu/2-1/4} \phi_K (x,y)}{(\sqrt{x} - \sqrt{t_1})(\sqrt{y} - \sqrt{t_2})} \right) \right] .$$

$$. \left[\frac{1}{2} \sin (2\sqrt{xR} - 2\sqrt{t_1 R}) - \sqrt{xR} \cos (2\sqrt{xR} - 2\sqrt{t_1 R}) \right] .$$

$$. \left[- \sqrt{yR'} \cos (2\sqrt{yR'} - 2\sqrt{t_2 R'}) + \frac{1}{2} \sin (2\sqrt{yR'} - 2\sqrt{t_2 R'}) \right] dx \quad .$$

(3.2-8)

The lower limit term is zero. If $D \leqslant t_2 - \delta$, the upper limit term is zero. Since

$$D_x \left(\frac{x^{\lambda/2-1/4} \phi_K (x,y)}{(\sqrt{x} - \sqrt{t_1})} \right)$$ is bounded say by some constant

M on $\left\{ (x,y,t_1) \,/\, A < x < B,\ C < y < D,\ o < t_1 < \infty \right\}$,

for $t_2 - \delta < D$, the upper term is bounded by $\frac{1}{RR'\delta} M (B - A)$

consequently, the upper limit term converges to zero for $o < t_1 < \infty$, $C < t_2 - \delta < \infty$ as $R, R' \to \infty$. Moreover,

$$\frac{\partial^2}{\partial x \partial y} \left(\frac{x^{\lambda/2-1/4} y^{\mu/2-1/4} \phi_K (x,y)}{(\sqrt{x} - \sqrt{t_1})(\sqrt{y} - \sqrt{t_2})} \right)$$

is also bounded on the domain

$$Ⓗ = \left\{ (x,y,t_1,t_2) \,/\, A < x < B,\ C < y < D,\ o < t_1 < \infty,\ C < t_2 - \delta < \infty \right\}$$

which implies that the second term in (3.2-8) also converges uniformly to zero for $o < t_1 < \infty$, $C < t_2 - \delta < \infty$ as $R, R' \to \infty$.

Now, consider the second term in (3.2-7). Upon integrating by parts the inner integral we have

$$\left\{ \frac{\phi_K(x,y)\; x^{\lambda/2-1/4}}{(\sqrt{x}-\sqrt{t_1}\,)} \left[\frac{1}{2R} \sin\,(2\sqrt{xR} - 2\sqrt{t_1R}) - \frac{\sqrt{xR}}{R} \cos\,\cdot \right.\right.$$

$$\left.\left. \cdot\,(2\sqrt{xR} - 2\sqrt{t_1R})\ \right] \right\}_A^B + \int_A^B \left[D_x\left(\frac{\phi_K(x,y)\; x^{\lambda/2-1/4}}{(\sqrt{x}-\sqrt{t_1})} \right)\right] .$$

$$\cdot \left[\frac{1}{2R} \sin\,(2\sqrt{xR} - 2\sqrt{t_1R}) - \frac{\sqrt{xR}}{R} \cos(2\sqrt{xR} - 2\sqrt{t_1R})\right] dx\ .$$

For any $y > o$, both the upper and lower limits are zero. Therefore, the second term in (3.2-7) is equal to

$$t_1^{-\lambda/2-1/4}\; t_2^{-\mu/2-1/4}\; y^{\mu/2-1/4}\; \frac{e^{-at_1-bt_2}}{4\pi^2} \int_C^{D'} dy\,\cdot$$

$$\cdot \int_A^B \left[D_x \left(\frac{\phi_k(x,y)\; x^{\lambda/2-1/4}}{(\sqrt{x}-\sqrt{t_1})} \right)\right] \left[\frac{1}{2R} \sin\,(2\sqrt{xR} - 2\sqrt{t_1R}) - \right.$$

$$\left. - \frac{\sqrt{xR}}{R}\ \cos(2\sqrt{xR} - 2\sqrt{t_1R})\ \right] \cos\,\frac{(2\sqrt{yR'} + 2\sqrt{t_2R'} - \mu\pi)}{(\sqrt{y} + \sqrt{t_2})}\,dx\ ,$$

since

$$\frac{e^{-at_1-bt_2}\; t_1^{-\lambda/2-1/4}\; t_2^{-\mu/2-1/4}\; y^{\mu/2-1/4}}{(\sqrt{y} + \sqrt{t_2}\,)}\ .$$

$$.\ D_x\left(\frac{\phi_K(x,y)\; x^{\lambda/2-1/4}}{(\sqrt{x}-\sqrt{t_1}\,)} \right)$$

is bounded on the domain (H), the second term in (3.2.7) converges to zero uniformly on $o < t_1 < \infty$, $C < t_2 - \delta < \infty$ as $R \to \infty$ for all $R' > 1$. Similarly, we can show the convergence of the third term.

Let us consider the convergence of the fourth term in (3.2-7).

For all $R, R' > 1$, the integrand is bounded on $\{(x,y,t_1, t_2) \,/\, A < x < B,\ C < y < D,\ o < t_1 < \infty,\ C < t_2-\delta < \infty\}$ by a constant independent of R, R'. Therefore, given $\epsilon > 0$, we can choose δ so small, say $\delta = \delta_4$, that the magnitude of the term can be made less than $\epsilon / 2$ for all $R, R' > 1$.

Thus, $P_{R,R'}^{(t_1,t_2)}$ converges uniformly to zero on $o < t_1 < \infty$, $o < t_2 < \infty$. Let,

$$P'_{R,R'}(t_1,t_2) = e^{-at_1-bt_2} V_4^{(t_1,t_2)}$$

$$= e^{-at_1-bt_2} \int_{t_2-\delta}^{t_2+\delta} \int_{t_1+\delta}^{\infty} L_{R,R'}^{(x,y,t_1,t_2)} \emptyset_K^{(x,y)} \, dydx \quad .$$

For the ranges $B \leqslant t_1 + \delta < \infty$, $o < t_2 < \infty$; $o < t_1 < \infty$, $o < t_2 + \delta \leqslant C$ and $o < t_1 < \infty$, $D \leqslant t_2 - \delta < \infty$, $P'_{R,R'}(t_1,t_2) \equiv 0$.

So, we consider the range $o < t_1 + \delta < B$, $C - \delta < t_2 < D + \delta$.

Let $A' = \max(A, t_1 + \delta)$. Using the asymptotic expansions of the Bessel's functions, for $R > o$ and large R', we have

$$P'_{R,R'}(t_1,t_2) = \frac{1}{4\pi^2} e^{-at_1-bt_2} \int_{t_2-\delta}^{t_2+\delta} \int_{A'}^{B} x^{\lambda/2-1/4} y^{\mu/2-1/4} \; .$$

$$\cdot \; t_1^{-\lambda/2-1/4} t_2^{-\lambda/2-1/4} \emptyset_K(x,y) \; \frac{\sin(2\sqrt{xR}-2\sqrt{t_1R}) \sin(2\sqrt{yR'}-2\sqrt{t_2R})}{(\sqrt{x}-\sqrt{t_1})(\sqrt{y}-\sqrt{t_2})} .$$

$$\cdot dydx \frac{1}{4\pi^2} e^{-at_1-bt_2} \int_{t_2-\delta}^{t_2+\delta} \int_{A'}^{B} x^{\lambda/2-1/4} y^{\mu/2-1/4} t_1^{-\mu/2-1/4} \cdot$$

$$\cdot t_2^{-\mu/2-1/4} \not{O}_K(x,y) \frac{\sin(2\sqrt{xR} - 2\sqrt{t_1R})\cos(2\sqrt{yR'}+2\sqrt{t_2R'}-\mu\pi)}{(\sqrt{x}-\sqrt{t_1})(\sqrt{y}+\sqrt{t_2})} dydx -$$

$$-\frac{1}{4\pi^2} e^{-at_1-bt_2} \int_{t_2-\delta}^{t_2+\delta} \int_{A'}^{B} x^{\lambda/2-1/4} y^{\mu/2-1/4} t_1^{-\lambda/2-1/4} t_2^{-\mu/2-1/4} \cdot$$

$$\cdot \not{O}_K(x,y) \frac{\sin(2\sqrt{yR'} - 2\sqrt{t_2R'}) \cos(2\sqrt{xR} + 2\sqrt{t_1R} - \lambda\pi)}{(\sqrt{y}-\sqrt{t_2})(\sqrt{x}+\sqrt{t_1})} dydx +$$

$$+\frac{1}{4\pi^2} e^{-at_1-bt_2} \int_{t_2-\delta}^{t_2+\delta} \int_{A'}^{B} x^{\lambda/2-1/4} y^{\mu/2-1/4} t_1^{-\lambda/2-1/4} t_2^{-\mu/2-1/4}$$

$$\cdot \frac{\cos(2\sqrt{xR} + 2\sqrt{t_1R} - \lambda\pi) \cos(2\sqrt{yR'} + 2\sqrt{t_2R'} - \mu\pi)}{(\sqrt{x}+\sqrt{t_1})(\sqrt{y}+\sqrt{t_2})} dydx \quad . \qquad (3.2\text{-}9)$$

Consider the inner integral in the first term in (3.2-9) i.e.

$$\int_{A'}^{B} x^{\lambda/2-1/4} \not{O}_K(x,y) \frac{\sin(2\sqrt{xR} - 2\sqrt{t_1R})}{(\sqrt{x}-\sqrt{t_1})} dx \quad .$$

Upon integrating by parts the above integral we have

$$\left\{ \frac{x^{\lambda/2-1/4} \not{O}_K(x,y)}{(\sqrt{x}-\sqrt{t_1})} \left[\frac{1}{2R} \sin(2\sqrt{xR} - 2\sqrt{t_1R}) - \right.\right.$$

$$\left.\left. - \frac{\sqrt{xR}}{R} \cos(2\sqrt{xR} - 2\sqrt{t_1R}) \right] \right\}_{A'}^{B} - \frac{1}{R} \int_{A'}^{B} \left[D_x \left(\frac{x^{\lambda/2-1/4} \not{O}_K(x,y)}{(\sqrt{x}-\sqrt{t_1})} \right) \right] \cdot$$

$$\cdot\left[\tfrac{1}{2}\sin(2\sqrt{xR}-2\sqrt{t_1R})-\sqrt{xR}\cos(2\sqrt{xR}-2\sqrt{t_1R}\right]dx \quad (3.2\text{-}10)$$

The upper limit term is zero so is the lower limit term if $A > t_1 + \delta$ and on the other hand if $t_1 + \delta > A$, the lower limit term is bounded by

$$(R\delta)^{-1} \sup_{\substack{A<x<B \\ C<y<D}} \left| \frac{x^{\lambda/2-1/4}\,\phi_K(x,y)}{(\sqrt{x}-\sqrt{t_1})} \right| .$$

Consequently, the lower limit term converges to zero for $o < t_1 + \delta < A$, $C < y < D$ as $R \to \infty$. Moreover,

$$D_x\left(\frac{x^{\lambda/2-1/4}\,\phi_K(x,y)}{(\sqrt{x}-\sqrt{t_1})}\right)$$

is bounded on the domain

$$\left\{ (x,y,t_1) \;/\; A < x < B,\;\; C < y < D,\;\; o < t_1 + \delta < B \right\}$$

which implies that the second term in (3.2-10) also converges uniformly to zero for $o < t_1 + \delta < B$, $C < y < D$ as $R \to \infty$. Hence for large R', first, second and third term in (3.2-9) converges to zero uniformly for $o < t_1 + \delta < B$, $C - \delta < t_2 < D + \delta$ as $R \to \infty$.

Now, for all R, R' > 1, the integrand in the fourth term is bounded on

$$\left\{ (x,y,t_1,t_2) \;/\; A < x < B,\;\; C < y < D,\;\; o < t_1 + \delta < B,\; C - \delta < t_2 < D + \delta \right\}$$

by a constant independent of R, R'.

Thus, $P'_{R,R'}(t_1,t_2)$ converges uniformly to zero on $o < t_1 < \infty$, $o < t_2 < \infty$.

Again, using similar arguments as in the previous cases, we can show that $N'_{R,R'}(t_1,t_2)$ and $Q_{R,R'}(t_1,t_2)$ converges to zero uniformly for $o < t_1 < \infty$, $o < t_2 < \infty$ as $R, R' \to \infty$, where $N'_{R,R'}(t_1,t_2)$ and $Q_{R,R'}(t_1,t_2)$ are given by

$$N'_{R,R'}(t_1,t_2) = e^{-at_1-bt_2} \int_{t_2-\delta}^{\infty} \int_{o}^{t_1-\delta} L_{R,R'}(x,y,t_1,t_2) \phi_K(x,y)\,dydx$$

and

$$Q_{R,R'}(t_1,t_2) = e^{-at_1-bt_2} \int_{t_2+\delta}^{\infty} \int_{t_1-\delta}^{\infty} L_{R,R'}(x,y,t_1,t_2) \phi_K(x,y)\,dydx \, .$$

Thus,

$$e^{-at_1-bt_2} \Delta^{k_1,k_2}_{\lambda,\mu,t_1,t_2} \left[M^{(t_1,t_2)}_{R,R'} - \phi(t_1,t_2) \right] \to 0$$

uniformly on $o < t_1 < \infty$, $o < t_2 < \infty$ as $R, R' \to \infty$ which implies that $M^{(t_1,t_2)}_{R,R'} \longrightarrow \phi(t_1,t_2)$ in $H_{a,b,\lambda,\mu}$ as $R, R' \to \infty$, and the theorem is proved.

As a result of the inversion theorem, we have the following uniqueness theorem.

<u>Theorem 3.2.2</u> Let $F(u,v) = (h_{\lambda,\mu}f)(u,v)$ for $(u,v) \in \Omega_f$ and $G(u,v) = (h_{\lambda,\mu}g)(u,v)$ for $(u,v) \in \Omega_g$. If $\Omega_f \cap \Omega_g \neq \phi$ and $F(u,v) = G(u,v)$ for $(u,v) \in \Omega_f \cap \Omega_g$ then $f = g$ in the sense of equality in $D'(I)$.

<u>Proof</u> : By the inversion theorem, in the sense of convergence in $D'(I)$, we have

$$f(x,y) - g(x,y) = \lim_{R,R' \to \infty} \int_0^R \int_0^{R'} [F(u,v) - G(u,v)] \, .$$

$$\cdot \left(\frac{x}{u}\right)^{\lambda/2} \left(\frac{y}{v}\right)^{\mu/2} J_\lambda (2\sqrt{ux}) \, J_\mu (2\sqrt{vy}) \, du dv = 0 \, .$$

Thus, $f = g$ in the sense of equality in $D'(I)$.

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