C H A P T E R - I V

SPIN COEFFICIENT FORMALISM FOR THE THREE SPACE-LIKE CONGRUENCES ON THE WORLD LINE OF A PARTICLE

1. Introduction :

In this chapter we exploit the 'amazingly useful' (Flaherty 1976) formalism invented by Newman and Penrose for studying special space-like congruences associated with the stream line. Under the conditions that the null vector field m^a is shear-free and expansion-free, we obtain an elegant criterion for the existence of a time-like helix as the world line of a particle. When the null vector field m^a is shear-free, expansion free and harmonic yields that the path of a particle is time-like circle.

2. Spin coefficient formalism :

Newman and Penrose in 1962 have introduced the complex null tetrad

$$Z^a_{(i)} = (l^a,\ m^a,\ \bar{m}^a,\ n^a) \quad i = 0,\ 1,\ 2,\ 3 \tag{4.1}$$

where l^a, n^a are real and m^a is complex with $\bar{m}^a$ as it's conjugate, such that the following ten relations are satisfied :

The two normal relations :

$$l^a n_a = 1,\ m^a \bar{m}_a = -1. \tag{4.2a}$$

The four null relations :

$$l^a\, l_a = n^a n_a = m^a m_a = \bar{m}^a\, \bar{m}_a = 0. \tag{4.2b}$$

The four orthogonal relations :

$$l^a m_a = l^a \bar{m}_a = n^a m_a = n^a \bar{m}_a = 0. \qquad (4.2c)$$

The relation between this tetrad $Z^a_{(i)}$ and space-time metric tensor g^{ab} is known as completeness relation, viz.

$$g^{ab} = l^a n^b + n^a l^b - m^a \bar{m}^b - \bar{m}^a m^b \qquad (4.3)$$

Spin coefficients :

One of the simplifications existing in the NP - formalism is that the usual 24 Ricci rotation coefficients are reduced to 12 (complex) spin coefficients and the following specific Greek letters enumerate them :

$$\kappa = l_{a;b}\, m^a l^b$$

$$\rho = l_{a;b}\, m^a \bar{m}^b$$

$$\sigma = l_{a;b}\, m^a m^b$$

$$\tau = l_{a;b}\, m^a n^b$$

$$\nu = -n_{a;b}\, \bar{m}^a n^b$$

$$\mu = -n_{a;b}\, \bar{m}^a m^b$$

$$\lambda = -n_{a;b}\, \bar{m}^a m^b$$

$$\pi = -n_{a;b}\, \bar{m}^a l^b$$

$$\alpha = \frac{1}{2}\,(l_{a;b}\, n^a \bar{m}^b - m_{a;b}\, \bar{m}^a \bar{m}^b)$$

$$\beta = \frac{1}{2} (l_{a;b}\, n^a m^b - m_{a;b}\, \bar{m}^a m^b)$$

$$\gamma = \frac{1}{2} (l_{a;b}\, n^a n^b - m_{a;b}\, \bar{m}^a\, n^b)$$

$$\epsilon = \frac{1}{2} (l_{a;b}\, n^a l^b - m_{a;b}\, \bar{m}^a\, l^b)\,. \qquad (4.4)$$

It is interesting to note that the covariant derivatives of the null congruences can be expressed as linear combination of (the outer product of) the null congruences themselves with coefficients chosen from (4.4),

$$\begin{aligned} l_{a;b} \;=\; & (\gamma+\bar{\gamma}) l_a l_b + (\epsilon + \bar{\epsilon})\, l_a n_b - (\alpha + \bar{\beta})\, l_a m_b - \bar{\tau}\, m_a l_b \\ & + \bar{\sigma}\, m_a m_b + \bar{\rho}\, m_a \bar{m}_b - \bar{k}\, m_a n_b - (\bar{\alpha} + \beta)\, l_a \bar{m}_b \\ & - \tau\, \bar{m}_a l_b + \sigma\, \bar{m}_a \bar{m}_b + \rho\, \bar{m}_a m_b - k\, \bar{m}_a n_b \end{aligned} \qquad (4.5)$$

$$\begin{aligned} n_{a;b} \;=\; & \nu\, m_a l_b - \lambda\, m_a m_b - \mu\, m_a \bar{m}_b + \bar{\pi}\, m_a n_b + \bar{\nu}\, \bar{m}_a l_b \\ & - \bar{\lambda}\, \bar{m}_a \bar{m}_b - \bar{\mu}\, \bar{m}_a m_b + \bar{\pi}\, \bar{m}_a n_b - (\gamma + \bar{\gamma})\, n_a l_b \\ & - (\epsilon + \bar{\epsilon})\, n_a n_b + (\alpha + \bar{\beta})\, n_a m_b + (\bar{\alpha} + \beta)\, n_a \bar{m}_b \end{aligned} \qquad (4.6)$$

$$\begin{aligned} m_{a;b} \;=\; & \bar{\nu}\, l_a l_b - \bar{\mu}\, l_a m_b - \bar{\lambda}\, l_a \bar{m}_b + \bar{\pi}\, l_a n_b + (\gamma - \bar{\gamma})\, m_a l_b \\ & + (\beta - \bar{\alpha})\, m_a m_b + (\bar{\alpha} - \beta)\, m_a \bar{m}_b + (\epsilon - \bar{\epsilon})\, m_a n_b \\ & - \tau\, n_a l_b + \rho\, n_a m_b + \sigma\, n_a \bar{m}_b - k\, n_a n_b \end{aligned} \qquad (4.7)$$

$$
\begin{aligned}
\bar{m}_{a;b} = {} & \nu\, l_a l_b - \mu\, l_a \bar{m}_b - \lambda\, l_a m_b + \pi\, l_a n_b + (\bar{\gamma} - \gamma)\, \bar{m}_a l_b \\
& + (\bar{\beta} - \alpha)\, \bar{m}_a \bar{m}_b + (\alpha - \bar{\beta})\, \bar{m}_a m_b + (\bar{\epsilon} - \epsilon)\, \bar{m}_a n_b \\
& - \bar{\tau}\, n_a l_b + \bar{\rho}\, n_a \bar{m}_b + \bar{\sigma}\, n_a m_b - \bar{k}\, n_a n_b \,. \qquad (4.8)
\end{aligned}
$$

The divergence of the null congruences :

$$
\begin{aligned}
l^a{}_{;a} &= \epsilon + \bar{\epsilon} - \rho - \bar{\rho} \\
n^a{}_{;a} &= \mu + \bar{\mu} - \gamma - \bar{\gamma} \\
m^a{}_{;a} &= \bar{\pi} - \bar{\alpha} + \beta - \tau \\
\bar{m}^a{}_{;a} &= \pi - \alpha + \bar{\beta} - \bar{\tau} \,. \qquad (4.9)
\end{aligned}
$$

3. Application of the NP - formalism for the triad P^a, Q^a, R^a and the curvature scalars K_1, K_2, K_3 :

The great advantage of this null formalism is that we can change to other formalisms easily. For instance if u^a is the unit time-like congruence, without loss of generality, we can choose

$$u^a = 2^{-\frac{1}{2}} (l^a + n^a) \qquad (4.10)$$

since

$$
\begin{aligned}
u^a u_a &= 2^{-1} (l^a + n^a)(l_a + n_a) \\
&= +1 \quad \text{by } (4.2\text{ a}), (4.2\text{ b})
\end{aligned}
$$

We compute the expression for $\dot{u}^a$ as

$$\dot{u}^a = 2^{-1} (l^a{}_{;b} + n^a{}_{;b}) (l^b + n^b)$$

$$= 2^{-1} \Big[(\epsilon + \bar{\epsilon} + \gamma + \bar{\gamma}) l^a + (\pi + \nu - \bar{k} - \bar{\tau}) m^a$$

$$+ (\bar{\pi} + \bar{\nu} - k - \tau) \bar{m}^a - (\epsilon + \bar{\epsilon} + \gamma + \bar{\gamma}) n^a \Big]$$

$$\text{by } (4.5)\ (4.6) \qquad (4.11)$$

The expressions for P^a, Q^a, R^a are too cumbersome for interpretation in the NP - formalism and so we have to consider some special cases as a pilot study for more general situations. A non-trivial investigation follows -

<u>The case of shear-free and expansion-free complex congruence m^a</u> :

The expansion and the shear for the complex null congruence m^a are given by

$$\underset{(m)}{\theta} = \bar{\pi} - \tau$$

$$\underset{(m)}{\sigma}_{ab} = \bar{\nu}\, l_a l_b - k\, n_a n_b + (\gamma - \bar{\gamma})\, l_{(a} m_{b)} + (\epsilon - \bar{\epsilon})\, m_{(a} n_{b)} .$$

When the complex null congruence m^a is shear-free and expansion free, i.e. $\underset{(m)}{\theta} = 0$ and $\underset{(m)}{\sigma} = 0$, gives that

$$\bar{\pi} - \tau = \bar{\nu} = k = \epsilon - \bar{\epsilon} = \gamma - \bar{\gamma} = 0 \qquad (4.12)$$

and hence (4.12) becomes

$$\dot{u}^a = 2^{-1}\,(\epsilon + \bar{\epsilon} + \gamma + \bar{\gamma})\,(l^a - n^a) \tag{4.13}$$

From (2.6) and (4.13), we have the expression of the first curvature scalar k_1 in terms of spin coefficients

$$K_1 = 2^{-\frac{1}{2}}\,(\epsilon + \bar{\epsilon} + \gamma + \bar{\gamma})$$

or

$$\boxed{K_1 = 2^{\frac{1}{2}}\,(\epsilon + \gamma)}\ . \tag{4.14}$$

Therefore the space-like congruence P^a is given by

$$P^a = 2^{-\frac{1}{2}}\,(l^a - n^a). \tag{4.15}$$

For the second space-like congruence Q^a and the second curvature scalar K_2, we require $\ddot{u}^a$, accordingly we compute

$$\ddot{u}^a = 2^{-3/2}\,(\epsilon + \bar{\epsilon} + \gamma + \bar{\gamma})\,(l^a{}_{;b} - n^a{}_{;b})\,(l^b + n^b) \quad \text{by (4.13)}$$

$$= 2^{-3/2}\,(\epsilon + \bar{\epsilon} + \gamma + \bar{\gamma})\,\Big[(\epsilon + \bar{\epsilon} + \gamma + \bar{\gamma})\,l^a - (\bar{k} + \bar{\tau} + \pi + \nu)\,m^a - (K + \tau + \bar{\pi} + \bar{\nu})\,\bar{m}^a + (\epsilon + \bar{\epsilon} + \gamma + \bar{\gamma})\,n^a\Big] \quad \text{by (4.5)(4.6)}$$

$$= 2^{1/2}\,\Big[(\epsilon + \gamma)l^a - \pi\,m^a - \bar{\pi}\,\bar{m}^a + (\epsilon + \gamma)n^a\Big] \quad \text{by(4.12)}$$

..(4.16)

Therefore

$$\ddot{u}^a \ddot{u}_a = 2 \{ 2(\epsilon + \gamma)^4 + (\epsilon + \gamma)^2 \pi \bar{\pi} \} \tag{4.17}$$

and

$$\ddot{u}^a \dot{u}_a = 0 \quad \text{by (4.13) (4.16)} . \tag{4.18}$$

From (2.20), (4.14), (4.17), (4.18) we write

$$K_2 = 2^{1/2} (\epsilon + \gamma) (- \pi \bar{\pi})^{1/2}$$

or

$$\boxed{K_2 = K_1 (- \pi \bar{\pi})^{1/2}} \quad \text{by (4.14)} . \tag{4.19}$$

Adopting (2.19), (4.14), (4.16), (4.19) we readily get the expression of space-like congruence Q^a in terms of the spin coefficients.

$$Q^a = -2^{-1/2} (\epsilon + \gamma)^{-1} (- \pi \bar{\pi})^{-1/2} (\pi m^a + \bar{\pi} \bar{m}^a) . \tag{4.20}$$

Equations (2.26), (4.10), (4.13), (4.16) gives expression of space-like congruence R^a.

$$R^a = -2^{-3/2} (\epsilon + \gamma)^{-1} (- \pi \bar{\pi})^{-1/2} \eta^{abcd} l_b n_c \Omega_d$$

with $\Omega = (\pi m_d + \bar{\pi} \bar{m}_d)$. (4.21)

For the expression K_3, we require $\dddot{u}^a$, so we compute

$$\dddot{u}^a = -2 \left| \pi (\epsilon - \bar{\epsilon} + \gamma - \bar{\gamma}) m^a + \bar{\pi} (\epsilon - \bar{\epsilon} + \gamma - \bar{\gamma}) \bar{m}^a \right|$$

$$= 0 \quad \text{by (4.12)} .$$

Therefore we have

$$K_3 = -2^{-5/2} (\epsilon + \gamma)^{-2} (\bar{\epsilon} - \epsilon + \bar{\gamma} - \gamma) \eta^{abcd} l_b n_c m_d \bar{m}_a$$

$$= 0 \quad \text{by (4.12)} . \qquad (4.22)$$

4. <u>Special cases</u> :

(i) <u>Time-like helix</u> : It is characterized by $K_3 = 0$ and K_1 = constant, K_2 = constant.

We have the theorem : If m^a is shear-free, expansion free, $\epsilon + \gamma$ = constant and π = constant then the path of the particle is a time-like helix.

Proof : Obvious from (4.14), (4.19), (4.22).

(ii) <u>Time-like circle</u> : The defining relations are $K_2 = K_3 = 0$, K_1 = constant. If the null congruence m^a is shear-free, expansion free and harmonic besides $(\epsilon + \gamma)$ = constant and $\pi = 0$, then the world line of a particle is time-like circle.

The proof follows from (4.14), (4.19), (4.22).