C H A P T E R - I

HISTORICAL INTRODUCTION

Section 1 :

Introduction :

A family of uncountably many non-intersecting space-filling curves is called a congruence of curves. The famous example of a congruence is the lines of force defined by a magnetic field. In hydrodynamics the stream lines characterized by the velocity field of a moving fluid form a congruence.

In the general theory of relativity there exist three types of congruences. The first type is called 'time-like' congruence. If the curves of a congruence have always time-like tangent vectors then the congruence is called time-like. For instance, the world lines of the particles in a continuum provide a time-like congruence. This congruence is the most popular one among research workers, since they were studied by Ehlers and Kundt (1962), Mc Vittie (1965), Ozsvath (1966), Ellis (1967, 1971), Vaidya (1968, 1973), Greenberg (1970), Date (1973), Krasinski (1975), Rao (1978), Raychaudhari (1979), Duggal and Sharma (1986) and Duggal (1987). Radhakrishna et.al. (1975, 1976, 1980 a, 1980 b, 1981) have obtained conservation laws as concomitants of Lie invariance, Jaumann invariance of certain important tensor fields with respect to a time-like congruence in electrodynamics.

Next in popularity are the null congruences which have no counterpart in Newtonian mechanics. When the curves of a congruence have always null tangent vectors then the congruence is called a null contruence. For instance, the path of a photon constitutes a member of a null congruence. Extensive applications of this congruence to the exploration of gravitational radiation have been initiated by Newman and Penrose (1962). Such types of congruences are studied by Takeno (1957), Peres (1960), Kundt (1961), Debney and Zund (1971), Geroch, Held and Penrose (1973), Hull (1977), Edgar (1980), Lukacs et. al. (1981), Radhakrishna and Singh (1984) and Radhakrishna (1988). Null congruences are especially suited for studying electromagnetic null fields interacting with gravitational null fields (Radhakrishna and Gumaste, 1984).

The last type of congruence which has received least attention from relativists, is the space-like ones. The recent text by Stephani (1982) refers only to the work on time-like and null congruences. When the curves of congruence have always space-like tangent vectors then the congruence is called space-like. According to Narlikar (1978), the paths of tachyons form a space-like geodesic congruence. (Tachyons are hypothetical particles supposed to travel with velocity greater than velocity of light).

Section-2 :

Space-like congruences :

For a given time-like vector field, there exist many space-like vector fields which are orthogonal to it. It follows that the number of different space-like congruences is much more than the time-like congruences in relativistic continuum mechanics. Due to this proliferation of space-like congruences in relativistic mechanics, their study is imperative. This forms the motivation for the investigations in this dissertation. It is observed that there have not been many investigations of space-like congruences till 1970, although the concept of space-like vectors was initiated in 1905 by Einstein through his special theory of relativity.

We give a brief survey of space-like congruences in relativistic hydrodynamics, thermodynamics and magnetohydrodynamics.

2.1 : Space-like congruences in relativistic hydrodynamics :

Relativistic hydrodynamics deals with super massive objects

$$M \sim \frac{R}{G} C^2 \qquad \ldots \qquad (1.1)$$

at high pressure

$$p \sim \rho C^2 \qquad \ldots \qquad (1.2)$$

and moving with velocity comparable to the velocity of light C,

$$\text{i.e.} \quad V \sim C . \qquad \ldots \qquad (1.3)$$

Here M is the mass of a body of radius R and G is the universal constant of gravitation. Such situations exist only on neutron stars for which

$$M = 0.91 \frac{R}{G} C^2 . \qquad \ldots \qquad (1.4)$$

For earth we have

$$M_E \sim 6 \times 10^{-10} C^2 \frac{R_E}{G} \qquad \ldots \qquad (1.5)$$

and for Sun we have (Narlikar 1978)

$$M_{\odot} \sim 2 \times 10^{-6} C^2 \frac{R_{\odot}}{G} \qquad \ldots \qquad (1.6)$$

and so relativistic mechanics is not pertinent to the two planets.

The acceleration vector field defined by

$$\dot{u}^a = u^a{}_{;b} u^b \qquad \ldots \qquad (1.7)$$

where u^a is the unit velocity vector field and the semicolon denotes covariant derivative, is a space-like vector field, since

$$\dot{u}^a u_a = 0 \qquad \ldots \qquad (1.8)$$

due to the relation

$$u^a u_a = 1. \qquad \ldots \qquad (1.9)$$

Also the magnitude of $\dot{u}^a$ is negative due to the signature of the metric ($-$, $-$, $-$, +). The vorticity vector field

$$\omega^a = \frac{1}{2} \eta^{abcd} u_b u_{c;d} \quad \ldots \qquad (1.10)$$

where η^{abcd} is the Levi-Civita tensor, is a space-like congruence, because

$$\omega^a u_a = \frac{1}{2} \eta^{abcd} u_a u_b u_{c;d}$$

leads to $\omega^a u_a = 0$

due to the skewsymmetry of η^{abcd} in (a,b) and the symmetry of $u_a u_b$ in (a,b). This congruence has been extensively studied recently by Tsamparlis and Mason (1983).

2.2 : Space-like congruences in relativistic thermodynamics :

i) Heat flux congruence :

The expression for the space-like vector field Q^a - the heat flux, has raised lot of controversy in the history of thermodynamics. The earliest investigations in the relativistic domain are due to Eckart (1940), who gave the following relation between temperature gradient and heat flux,

$$Q^a = -\lambda \gamma^{ab} \{ \frac{1}{c^2} \dot{u}_b \Theta^{-1} + \Theta^{-1}{}_{,b} \} \quad \ldots \qquad (1.12)$$

where Θ is temperature, $\gamma^{ab} = g^{ab} - u^a u^b$, .. (1.13)

λ is the coefficient of conductivity. When Θ = constant, we note that

$$q^a = -\frac{\lambda}{c^2} \gamma^{ab} \dot{u}_b \Theta^{-1}. \quad .. \quad (1.14)$$

This means that even though temperature does not change, there is heat flux which is due to the acceleration field $\dot{u}_b$. This feature is the peculiarity of relativity. It does not exist in classical thermodynamics, since when $C \longrightarrow \infty$ (in the Newtonian limit) we get from (1.12)

$$q^a = -\lambda \gamma^{ab} \Theta^{-1}{}_{,b}. \quad .. \quad (1.15)$$

The objection by relativists to this definition of Eckart is that it leads to the conclusion that 'heat propagates with infinite velocity'. However, according to the theory of relativity no interaction can propagate faster than light, whose speed is 1,86,000 miles per second. Thus the expression for q^a was not acceptable to relativists. Landau-Lifschitz in 1958 proposed another expression for the heat flux, viz.,

$$q^a = \lambda \gamma^{ab} \Theta^{-1}{}_{,b} + u^a u^b \Theta^{-1}{}_{,b} \quad .. \quad (1.16)$$

This also did not predict sub-luminal speed for the heat propagation. This defect has been resolved only in 1988.

Cambridge University Press is publishing (December, 1988) the proceedings of the latest International Conference on Gravitation and Cosmology, wherein Professor Carter has exposed a theory of causal thermodynamics under the caption 'Conductivity with Causality in Relativistic Hydrodynamics'. This is referred as'Regular' thermodynamics since it has successfully overcome the pathological behaviour of acausality, contained in the earlier theories of high speed contimuum mechanics developed by Eckart and Landau-Lifschitz, as relativistic generalizations of classical FOURIER-EULER conducting fluid models. This regular theory can be safely adopted for a wide range of astrophysical applications. An exposition of this theory follows.

Differential relations :

Let $\rho(n,s)$, be the density of the fluid, $k(n,s)$ conductivity scalar where n is the particle density (independent of the flow vector u^a), s is the entropy. Thus ρ, k are the primary equations of state functions. The secondary equation of state functions are

μ : chemical potential

Θ : the temperature

p : the pressure .

Standard equilibrium theory gives

$$\left.\begin{aligned} \mu &= \frac{P}{n}, \qquad \Theta = \frac{P}{s} \\ P &= n\mu + s\Theta - \rho . \end{aligned}\right\} \qquad \ldots \quad (1.17)$$

Basic conductivity equation is

$$Q^{a} = -k\,(\gamma^{ab}\,\Theta_{,b} + \dot{u}^{b}\,\Theta) \qquad \ldots \quad (1.18)$$

Here

$$\left.\begin{aligned} \gamma^{ab} &= g^{ab} - u^{a}u^{b}, \\ \dot{u}^{a} &= u^{a}{}_{;b}\,u^{b}. \end{aligned}\right\} \qquad \ldots \quad (1.19)$$

The heat flux vector Q^{a} is defined in terms of an appropriate heat transport velocity vector v^{a} by the expression.

$$Q^{a} = \Theta s v^{a} \qquad \ldots \quad (1.20)$$

$$v^{a}u_{a} = 0 \qquad \ldots \quad (1.21)$$

Carter finally obtains in 1988

$$T_{a}{}^{b} = n^{b}\chi_{a} + s^{b}\Theta_{a} + P\,g_{a}{}^{b} \qquad \ldots \quad (1.22)$$

where χ_{a} is the chemical 4-momentum and Θ_{a} is the thermal 4-momentum given by the relation

$$\Theta_{a} = \Theta u_{a}. \qquad \ldots \quad (1.23)$$

The entropy current vector σ^{a} is defined by

$$\sigma^{a} = s\,(u^{a} + u^{b}\,\frac{\chi_{b}}{\mu}\cdot\frac{\chi^{a}}{\mu}) \qquad \ldots \quad (1.24)$$

$$\sigma^{a}\chi_{a} = 0. \qquad \ldots \quad (1.25)$$

The resistivity scalar Z is given by

$$Z = \frac{\mu^{2}}{K(\chi^{a}u_{a})^{2}}. \qquad \ldots \quad (1.26)$$

The complete set of equations of the 'Regular' model are

$$2\,n^{a}\,\chi_{[b;a]} + Z\,s^{a}\,\Theta_{a}\,\sigma_{b} = 0 \qquad \ldots \qquad (1.27)$$

and

$$2\,s^{a}\,(\Theta_{[b;a]} + Z\,\sigma_{[a}\,\Theta_{b]}) = 0 . \qquad \ldots \qquad (1.28)$$

Entropy creation formula is

$$\left.\begin{aligned} s^{a}{}_{;a} &= Z\,\sigma^{a}\sigma_{a} \\ s^{a} &= s\,u^{a}. \end{aligned}\right\} \qquad \ldots \qquad (1.29)$$

The particle conservation formula is

$$\left.\begin{aligned} n^{a}{}_{;a} &= 0 \\ n^{a} &= n\,u^{a} \end{aligned}\right\} \qquad \ldots \qquad (1.30)$$

and the stress-energy momentum conservation is

$$T_{a}{}^{b}{}_{;b} = 0 . \qquad \ldots \qquad (1.31)$$

This theory by Carter yields a hyperbolic type of equation for the propagation of heat and this is the correct form for finite speed of heat propagation.

ii) The other space-like congruences for relativistic thermodynamics, viz., specific current vector field, gradient of Clebesch potential for rotational motion, gradient of potential for irrotational motion, gradient of specific entropy, gradient of charge-mass density ratio, specific vorticity pseudo-vector field, etc., have been identified by Ghunakikar (1974). Some of these congruences have been studied by Schutz (1972).

2.3 : Space-like congruences in relativistic magnetohydrodynamics :

The occurrence of magnetic field in solar winds (Parkar, 1964), spiral arms (Hewish, 1969) and sun-spots (Wilson, 1968) emphasize the necessity of relativistic magnetohydrodynamics for the development of astrophysics. The magnetic vector field, the electric vector field, Poynting flux are the space-like vector fields in relativistic magnetohydrodynamics, whose field equations together with the existance and uniqueness of solutions are studied by Lichncrowicz (1967). This formed the basis of several investigations on space-like congruences by Date (1974), Ghunakikar (1974), Jangam (1982) and Gumaste (1984).

Section 3 :

GREENBERG'S parameters for a space-like congruence :

In Chapter-III we propose to study special space-like congruences and their geometrical, ~~kinematical~~ as well as physical ~~as well as physical~~ significances. We take full advantage of the formalism developed by Greenberg for the parameters of space-like congruences, an exposition of which follows.

The parametric expression of the space-like congruence is given by the relations

$$x^a = x^a (\xi^\alpha , \tau) \qquad \ldots \qquad (1.32)$$

where the parameters ξ^α ($\alpha = 1,2,3$) specify the particular space-like curve of the congruence and where τ is some arc-length parameter along the curve of the space-like congruence. At any point (i.e. ξ^α is fixed) on any one curve of the space-like congruence, the unit tangent vector n^a is defined by

$$n^a = \left(\frac{dx^a}{d\tau} \right)_{\xi^\alpha} \qquad \ldots \qquad (1.33)$$

where $n^a n_a = + 1$. Obviously we have

$$n_{a;b}\, n^b = 0 \qquad \ldots \qquad (1.34)$$

Greenberg (1970) defined the three parameters of space-like congruence n^a relative to time-like congruence u^a for the metric signature (+,+,+,-).

$$\text{(i)}\quad \theta = \frac{1}{2}\left(n^a{}_{;a} + n_{a;b}\, u^a u^b\right) \quad \ldots \quad (1.35)$$

$$\text{(ii)}\quad \sigma_{ab} = \frac{1}{2} \perp_a^c \perp_b^d \left(n_{c;d} - n_{d;c}\right) - \perp_{ab}\theta \quad (1.36)$$

$$\text{(iii)}\quad \omega_{ab} = \frac{1}{2} \perp_a^c \perp_b^d \left(n_{c;d} - n_{d;c}\right) \quad \ldots \quad (1.37)$$

where θ is called the expansion, the σ_{ab} is called shear and the ω_{ab} is known as rotation of the space-like congruence and $\perp_{ab}$ is projection operator :

$$\perp_{ab} = g_{ab} + u_a u_b - n_a n_b \quad \ldots \quad (1.38)$$

with properties

$$\perp_{ab} = \perp_{ba}, \perp_b^a \perp_c^b = \perp_c^a, \perp_a^a = 2 \quad (1.40)$$

$$\perp_{ab} n^a = 0, \perp_{ab} u^a = 0 .$$

<u>Transport laws governing the definition of parameters</u> :

Two more space-like vector fields q^a, r^a are introduced satisfying the orthonormal relations.

$$u^a u_a = -1 \quad q^a q_a = r^a r_a = n^a n_a = 1 \quad (1.41)$$

$$u^a q_a = u^a r_a = u^a n_a = n^a q_a = q^a r_a = n^a r_a = 0$$

The definitions of the three parameters are subject to Greenbergs transport laws which are essential for preserving the orthonormal relations. They are three in number as enumerated below

$$\text{(i)} \quad \frac{Du^a}{d\tau} = n^a{}_{;c}\, \dot{u}^c + u^a\, n_{b;c}\, u^b u^c - n^a\, \frac{Dn_b}{D\tau}\, u^b \qquad ..(1.42)$$

$$\text{(ii)} \quad \frac{Dq^a}{d\tau} = u^a\, n_{b;c}\, q^b u^c - n^a\, \frac{Dn_b}{d\tau}\, q^b \qquad ..(1.43)$$

$$\text{(iii)} \quad \frac{Dr^a}{d\tau} = u^a\, n_{b;c}\, r^b u^c - n^a\, \frac{Dn_b}{d\tau}\, r^b\, . \qquad ..(1.44)$$

Several exact solutions of Einstein's field equations for gravitating fields admitting special types of space-like congruences have been delineated (in their comprehensive book) by Kramer et. al. (1980).

Section 4 :

Space-like congruences in generalized Serret-Frenet (GSF) Formulae :

The basis for new work on the space-like congruences reported in Chapter-II is described in this section.

In the 3-dimensional differential geometry the relations between the three vector fields tangent, normal, binormal and the two scalar fields curvature and torsion are well known as Serret-Frenet formulae (O'Neill 1970). The extension of these concepts to the 4-dimensional space-time of general relativity have been accomplished by Synge (1960). These are referred as generalized Serret-Frenet (GSF) formulae. For the world line (space-time curve).

$$x^a = x^a\,(\underline{s})$$

where $\underline{s}$ is the arc-length parameter along the curve, the GSF formulae are (Davis 1970).

$$\frac{D\ \rho^{a}{}_{(n)}}{D\underline{s}} = K^{(m)}_{(n)}\ \rho^{a}{}_{(m)} \qquad \ldots \qquad (1.45)$$

where a,m,n range over (0,1,2,3) and $\frac{D}{D\underline{s}}$ is the covariant derivative along the world line and $\rho^{a}{}_{(n)}$ is the orthonormal tetrad on the curve, with $\underline{a}$ as the tensor index and (n) as the tetrad index, $e^{a}_{(1)}$, $e^{a}_{(2)}$, $e^{a}_{(3)}$ are the three space-like congruences. The matrix of coefficients $K^{(m)}_{(n)}$ is described by

$$K^{(m)}_{(n)} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix}$$

where k_1, k_2, k_3 are called the first, the second and the third curvatures. It should be noted that the matrix $K^{(m)}_{(n)}$ is neither symmetric nor skew-symmetric (as distinguished from it's counterpart in 3-dimensions which is skew-symmetric).