CHAPTER - IV

The fixed point theorems of reasonable wanderer maps in Hilbert space

1 Introduction

The concept of 'REASONABLE WANDERER MAP' (I-Definition 1.1.8) in Hilbert space was first considered by Browder and Petryshyn [7] and obtained the results (I-Theorems 1.3.4 and 1.3.6). These results are extended in generalised Hilbert space by Hicks and Huffman [15]. In this chapter theorem 1.3.4 is extended to show that the demicontractive (I-1.1.11) and hemicontractive (I - 1.1.12) self-maps defined on a closed convex subset C of a Hilbert space H are reasonable wanderer in C. Further theorem (I-1.1.6) is extended to obtain a fixed point of Lipschitzian (I-1.1.10) demicontractive, demicompact (I-1.1.13) self-mapping of a bounded closed convex subset of a Hilbert space H. Finally some of our results and theorem (I-1.3.4) along with its corollary (I-1.3.5) are generalised by considering generalised contraction mapping (III-1.1) introduced by us in Chapter-III.

Theorem 1.1 : Let C be a closed convex subset of a Hilbert space H. If T is a demicontractive mapping of C into C with contraction coefficient K and having non-empty set F(T) of fixed points of T in C, then the mapping defined by $T_\lambda = \lambda T + (1-\lambda) I$, where I is identity mapping on C and for a given λ with $0 < \lambda < 1$ and $K < 1 - \lambda$ is reasonable wanderer from C in to C with the same fixed points as T.

Proof : For any $x \in C$, set $x_n = T_\lambda^n x$, $0 < \lambda < 1$.
Let $P \in F(T)$, hence $P \in F(T_\lambda)$ (see [7]).

It follows from (I-1.3.12), where t stands for λ that

$$\| x_{n+1} - P \|^2 = \| \lambda T x_n + (1-\lambda) x_n - P \|^2.$$

$$= \lambda \| T x_n - P \|^2 + (1-\lambda) \| x_n - P \|^2 -$$

$$- \lambda (1-\lambda) \| T x_n - x_n \|^2. \qquad \ldots (1.2)$$

Using the fact that T is demicontractive, we have

$$\| T x_n - P \|^2 \leq \| x_n - P \|^2 + K \| x_n - T x_n \|^2 \qquad \ldots (1.3)$$

From equations (1.2) and (1.3), we obtain

$$\| x_{n+1} - P \|^2 \leq \| x_n - P \|^2 - \lambda (1-\lambda-K) \| T x_n - x_n \|^2,$$

$$K < 1 - \lambda.$$

Summing these inequalities from $n = 0$ to $n = j$, j being positive integer, we have

$$\lambda(1-\lambda-K) \sum_{n=0}^{j} \| T x_n - x_n \|^2 \leq \sum_{n=0}^{j} \left\{ \| x_n - P \|^2 - \right.$$

$$\left. - \| x_{n+1} - P \| \right\}^2 .$$

$$\leq \| x_o - P \|^2 - \| x_{j+1} - P \|^2$$

$$\leq \| x_o - P \|^2. \qquad \ldots (1.4)$$

which implies that $\sum_{n=0}^{\infty} \| Tx_n - x_n \|^2 < \infty.$

Now since $x_{n+1} - x_n = \lambda [Tx_n - x_n]$

$$\sum_{n=0}^{\infty} \| x_{n+1} - x_n \|^2 = \lambda^2 \sum_{n=0}^{\infty} \| Tx_n - x_n \|^2$$

$$\leq \frac{\lambda}{1-\lambda-K} \| x_o - P \|^2 \text{ by (1.4),} \quad \ldots (1.5)$$

$$K < 1 - \lambda.$$

This means (by definition I-1.1.8) that T is reasonable wanderer in C and hence T_λ is reasonable wanderer in C.

<u>Theorem 1.6</u> : Let C be closed convex subset of a Hilbert space H. Let $T : C \longrightarrow C$ such that

(i) $F(T) \neq \emptyset$,

(ii) T is hemicontractive.

If $T_\lambda = \lambda T + (1-\lambda) I$ where I is identity mapping and for a given λ with $0 < \lambda < 1$, then T_λ is reasonable wanderer in C with same fixed points as T.

<u>Proof</u> : Since T is hemicontractive, we have

$$\| Tx_n - P \|^2 \leq \| x_n - P \|^2 + \| x_n - Tx_n \|^2 .$$

In equation (1.2) using the fact that T is hemicontractive we obtain

$$\| x_{n+1} - P \|^2 + \lambda(1-\lambda) \| Tx_n - x_n \|^2 \leq \| x_n - P \|^2 + \\ + \lambda \| Tx_n - x_n \|^2 \qquad \ldots (1.7)$$

Replacing $\| Tx_n - x_n \|^2$ by $\frac{1}{\lambda^2} \| x_{n+1} - x_n \|^2$ on right hand side of (1.7) and then summing over n = 0 to n = j, we get

$$\lambda(1-\lambda) \sum_{n=0}^{j} \| Tx_n - x_n \|^2 \leq \sum_{n=0}^{j} \left\{ \| x_n - P \|^2 - \| x_{n+1} - P \|^2 \right\} + \\ + \frac{1}{\lambda} \sum_{n=0}^{j} \| x_{n+1} - x_n \|^2 .$$

$$\leq \| x_o - P \|^2 - \| x_{j+1} - P \|^2 + \\ + \frac{1}{\lambda} \| x_{j+1} - x_o \|^2 .$$

$$\leq \| x_o - P \|^2 + \frac{1}{\lambda} \| P - x_o \|^2 .$$

$$\leq \frac{1+\lambda}{\lambda} \| x_o - P \|^2 .$$

Thus $\sum_{n=0}^{\infty} \| Tx_n - x_n \|^2 < \infty$,

and $$\sum_{n=0}^{\infty} \| x_{n+1} - x_n \|^2 \leq \frac{1+\lambda}{1-\lambda} \| x_o - P \|^2 \qquad \ldots (1.8)$$

which implies that T is reasonable wanderer in C.

Now we extend corollary (I-1.3.5) for the mappings considered in theorems (1.1. and 1.6) in the following manner.

<u>Corollary 1.9</u> : If T is demicontractive (or hemicontractive) self map of C, where C is closed convex subset of a Hilbert space H and the set F(T) of fixed points of T in C is non-empty, then the mapping $T_\lambda = \lambda T + (1-\lambda)\, I$, I is identity map on C and for given λ with $0 < \lambda < 1$, $K < 1 - \lambda$, maps C into C, has same fixed points as T and it is asymptotically regular (I-definition 1.1.9) at x.

<u>Proof</u> : Suppose T is demicontractive mapping with contraction coefficient K. Then the mapping $T_\lambda = \lambda T + (1-\lambda)\, I$ is reasonable wanderer in C and has same fixed points as T. This implies inequality (1.5) i.e.

$$\sum_{n=0}^{\infty} \| x_{n+1} - x_n \|^2 \leqslant \frac{\lambda}{1-\lambda-K} \| x_o - P \|^2, \quad K < 1-\lambda.$$

Setting $x_n = T_\lambda^n x$, $\lambda = \frac{1}{1+n}$, we obtain

$$\sum_{n=0}^{\infty} \| T_\lambda^{n+1} x - T_\lambda^n x \| \leqslant \frac{1}{n-k(n+1)} \| x_o - P \|^2$$

which implies that

$$\lim_{n\to\infty} \| T_\lambda^{n+1} x - T_\lambda^n x \| = 0.$$

i.e. T is asymptotically regular at x.

2. Construction of Fixed Points of a Reasonable Wanderer map.

Theorem 2.1 : Suppose T is (i) Lipschitzian demicontractive selfmap of a bounded closed convex subset C of a Hilbert space H,

(ii) reasonable wanderer in C,

(iii) demicompact,

(iv) $F(T) \neq \emptyset$, where F(T) denote set of fixed points of T in C.

Then F(T) is convex set and for any given x_o in C and any fixed λ with $0 < \lambda < 1$, the sequence $\{x_n\} = \{T_\lambda^n x_o\}$ determined by the process

$$x_n = \lambda T x_{n-1} + (1-\lambda)\, x_{n-1}, \; n = 1, 2, \ldots \qquad \ldots(2.2)$$

converges strongly to a fixed point of T in C.

Proof : Let P be a fixed point of T and hence it is of T_λ (by hypothesis).

Since T is liptschitzian, then there exists a constant $L > 0$ such that

$$\| Tx - Ty \| \leq L \| x-y \| , \; x, \; y \in C.$$

For P_o, $P_1 \in F(T)$, we have

$$\|TP_\lambda - P_o\| = \|TP_\lambda - T P_o\| \leqslant L \| P_\lambda - P_o\|$$

and $\|TP_\lambda - P_1\| = \|TP_\lambda - TP_1\| \leqslant L \|P_\lambda - P_1\|$.

From which it follows that

$$\| P_1 - P_o\| \leqslant \|P_1 - TP_\lambda\| + \| TP_\lambda - P_o\|$$

$$< L \left\{ \|P_1 - P_\lambda\| + \|P_\lambda - P_o\| \right\}.$$

$$< L \| P_1 - P_o\| \qquad \ldots (2.3)$$

Thus for some a, b with $0 \leqslant a, b \leqslant 1$, it follows that

$$TP_\lambda - P_o = aL (P_o - P_\lambda). \qquad \ldots (2.4)$$

$$TP_\lambda - P_1 = bL (P_1 - P_\lambda). \qquad \ldots (2.5)$$

Adding (2.4) and (2.5), we obtain

$$2TP_\lambda = (1-aL) P_o + (1-bL) P_1 + L(a+b) P_\lambda.$$

Now setting $L(a+b) = 1$

$$1 - aL = t \quad \text{with } 0 \leqslant t \leqslant 1$$

we obtain

$$TP_\lambda = P_\lambda$$

where $P_\lambda = tP_o + (1-t) P_1$.

This implies that $P_\lambda \in F(T)$ and $F(T)$ is a convex set.

Now, the sequence $\{x_n\}$ lying in C is bounded and since T is reasonable wanderer in C by corollary 1.9 it is asymptotically regular at x_o in C. It follows that

$$\lim_{n \to \infty} \| T_\lambda^{n+1} x_o - T_\lambda^n x_o \| = 0$$

i.e. $\lim_{n \to \infty} \| T_\lambda x_n - x_n \| = 0$, where $T_\lambda = \lambda T + (1-\lambda)I$.

which implies that the sequence $\{Tx_n - x_n\} - \{1/\lambda . (x_n - x_{n+1})\}$ strongly converges to zero. Now by demicompactness of T, there exists a strongly convergent subsequence $\{x_{n_i}\}$ such that

$$x_{n_i} \longrightarrow q \in F(T).$$

Since T is demicontractive, there exists a constant K, $0 \leq K < 1$ such that

$$\| Tx_{n_i} - q \| \leq \|x_{n_i} - q\| + K \|Tx_{n_i} - x_{n_i}\| \longrightarrow 0, \text{ as } i \longrightarrow \infty.$$

It follows that $Tx_{n_i} \longrightarrow q$, as $i \longrightarrow \infty$.

Using the fact that T is of Lip. class, we have

$$\| Tq - q \| \leq \| Tq - Tx_{n_i} \| + \| Tx_{n_i} - q \|$$

$$< L \| q - x_{n_i} \| + \| Tx_{n_i} - q \| \longrightarrow 0, \text{ as } i \to \infty.$$

i.e. $\|Tq - q\| = 0$ which implies that $Tq = q$.

The inequality $\|x_n - q\| \leq \|x_{n-1} - q\|$ valid for each n implies that the sequence $\{\|x_n-q\|\}$ is monotonically decreasing sequence. This coment along with the fact $x_{n_i} \rightarrow q$ implies the convergence of entire sequence $\{x_n\}$ to a fixed point q of T.

This completes the proof of the theorem.

3. <u>Generalisation Of Reasonable Wanderer Maps</u> :

The following Theorem 3.1 generalize Theorems 1.1, 1.6 proved in the first section of this chapter and the Theorem (I-1.3.4) of [7] . The proof is given in the light of proof of Theorem 1.1.

<u>Theorem 3.1</u> : Let C be a closed convex subset of a Hilbert space H. Let T be a generalised contraction mapping of C into C (i.e. $\|Tx-Ty\|^2 \leq a_1 \|x-y\|^2 + a_2 \|x-Ty\|^2 +$

$$+ a_3 \|y-Tx\|^2 + a_4 \|(I-T)x-(I-T)y\|^2.$$

where $a_i \geq 0$, $\sum_{i=1}^{4} a_i < 1$, for all x and y in C.)

with further assumptions $a_1 + a_2 + a_3 = 1$, $a_3 + a_4 < 1$. Suppose set F(T) of fixed points of T in C is non-empty. Then the mapping defined by $T_\lambda = \lambda T + (1-\lambda) I$, where I is identity mapping on C and for a given λ with $0 < \lambda < 1$, is reasonable from C into C with the same fixed points as T.

Proof : For any $x \in C$, set $x_n = T_\lambda^n x$, $0 < \lambda < 1$.
Let $P \in F(T)$, hence $P \in F(T_\lambda)$. [7].

Using Ishikawa technique (I-1.3.12), where t stands for λ, we have

$$\| x_{n+1} - P \|^2 = \| \lambda T x_n + (1-\lambda) x_n - P \|^2 .$$

$$= \lambda \| Tx_n - P \|^2 + (1-\lambda) \| x_n - P \|^2 -$$

$$- \lambda(1-\lambda) \| Tx_n - x_n \|^2 \qquad \ldots (3.2)$$

since T is generalised contraction mapping

$$\| Tx_n - P \|^2 = \| Tx_n - TP \|^2 \leq a_1 \| x_n - P \|^2 +$$

$$+ a_2 \| x_n - P \|^2 + a_3 \| P - Tx_n \|^2 +$$

$$+ a_4 \| x_n - Tx_n \|^2 .$$

$$\| Tx_n - P \|^2 \leq \frac{a_1 + a_2}{1 - a_3} \| x_n - P \|^2 + \frac{a_4}{1-a_3} \| x_n - Tx_n \|^2 .$$

$$\leq \| x_n - P \|^2 + K \| x_n - Tx_n \|^2 \qquad \ldots (3.3)$$

since $a_1 + a_2 + a_3 = 1$ and setting

$$K = \frac{a_4}{1-a_3} < 1$$

From (3.2) and (3.3) we obtain

$$\| x_{n+1} - P \|^2 \leqslant \lambda \left\{ \| x_n - P \|^2 + K \| x_n - Tx_n \|^2 \right\} + \\ + (1-\lambda) \| x_n - P \|^2 - \lambda(1-\lambda) \| Tx_n - x_n \|^2.$$

$$\leqslant \| x_n - P \|^2 - \lambda(1-\lambda-K) \| Tx_n - x_n \|^2,$$

$$K = \frac{a_4}{1-a_3} < 1.$$

Now letting $K < 1-\lambda$ and developing further this theorem on the similar lines of Theorem 1.1, we obtain the desired result.

Remark 3.4 (i) If we put $a_2 = a_3 = a_4 = 0$, $\sqrt{a_1} = 1$, we obtain theorem (I-1.3.4) of Browder and Petryshyn [7] as a corollary to our theorem.

(ii) If we put $a_1 + a_2 + a_3 = 1$, $a_3 + a_4 < 1$ and $y = P = TP$, we obtain Theorem 1.1.

(iii) If we put $a_1 + a_2 + a_3 = 1$, $a_3 + a_4 = 1$ and $y = P = TP$, we obtain Theorem 1.6.

(iv) It is obvious that the mappings defined by I-1.3.4, o.3.6 and 1.3.7 have been taken care of.

Finally, we formulate the following corollary 3.5 which generalize corollary 1.9 and corollary (I-1.3.5) of Browder and Petryshyn [7].

<u>Corollary 3.5</u>: If T is a self map of a closed convex subset C of a Hilbert space H and satisfies conditions of Theorem 3.1, Suppose further that T has atleast one fixed point in C. Then the mapping $T_\lambda = \lambda T + (1-\lambda)I$, I is identity map on C and for a given λ with $0 < \lambda < 1$, maps C into C, has same fixed points as T and it is asymptotically regular at x.

<u>Proof</u> : Proof may be given in the light of proof of corollary 1.1.