CHAPTER I

FUZZY SETS, RELATIONS AND GRAPHS

In this chapter, we state the basic definitions and results which will be needed in the succeeding chapters. Throughout the discussion, X stands for the universal set and I for the closed unit interval $[0, 1]$ of the real line.

1.1 FUZZY SETS

Definition 1.1.1 : A fuzzy set A on a universal set X is a function $A : X \longrightarrow I$.

Definition 1.1.2 : If $A : X \longrightarrow I$ and $B : X \longrightarrow I$ are two fuzzy sets on X, then A is fuzzy subset of B if and only if $A(x) \leqslant B(x), \ \forall x \in X$.

Definition 1.1.3 : Two fuzzy sets, A and B on X are equal if and only if they are equal as functions, i.e. $A(x)=B(x), \forall x \in X$.

Definition 1.1.4 : If A and B are fuzzy sets on X, then their union $A \cup B$ and intersection $A \cap B$ are fuzzy sets on X, defined by

$$(A \cup B)(x) = \max \{A(x), B(x)\} \quad \text{and}$$
$$(A \cap B)(x) = \min \{A(x), B(x)\} \ \forall x \in X.$$

If $\{A_i\}_{i \in I}$ is a family of fuzzy sets on X, then $(\bigcup_{i \in I} A_i)$ and $(\bigcap_{i \in I} A_i)$ are defined by,

$(\bigcup_{i \in I} A_i)(x) = \sup_{i \in I} A_i(x)$ and

$(\bigcap_{i \in I} A_i)(x) = \inf_{i \in I} A_i(x) \;\forall\; x \in X.$

<u>Definition 1.1.5</u> : The complement of a fuzzy set A on X is a fuzzy set on X, defined by

$$A'(x) = 1 - A(x) \;\forall\; x \in X.$$

Similar to crisp sets, fuzzy sets satisfy De-Morgan's laws and distributive laws.

<u>Theorem 1.1.6</u> : If A, B and C are fuzzy sets on the same universe X, then

1) $(A \cup B)' = A' \cap B'$

2) $(A \cap B)' = A' \cup B'$

3) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and

4) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

<u>Proof</u> : 1) $(A \cup B)'(x) = 1-(A \cup B)(x) = 1 - \max\{A(x), B(x)\}$

$= \min\{1-A(x), 1-B(x)\} = \min\{A'(x), B'(x)\}$

$= (A' \cap B')(x) \;\forall\; x \in X.$

Therefore, $(A \cup B)' = A' \cap B'$.

2) $(A \cap B)'(x) = 1-\min\{A(x), B(x)\} = \max\{1-A(x), 1-B(x)\}$

$= \max\{A'(x), B'(x)\} = (A' \cup B')(x) \;\forall\; x \in X.$

Thus $(A \cap B)' = A' \cup B'$.

3) $(A \cup (B \cap C))(x) = \max\{A(x), \min[B(x), C(x)]\}$

$= \min\{\max[A(x), B(x)], \max[A(x), C(x)]\}$

$= ((A \cup B) \cap (A \cup C))(x) \;\forall\; x \in X.$

Hence, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

4) $(A \cap (B \cup C))\ (x) = \min\{A(x), \max[B(x), C(x)]\}$

$= \max\{\min[A(x), B(x)], \min[A(x), C(x)]\}$

$= [(A \cap B) \cup (A \cap C)](x) \quad \forall x \in X.$

Therefore, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

<u>Remarks 1.1. 7</u>: 1) The operations '$\cup$' and '$\cap$' are commutative and associative.

2) We have idempotency i.e. $A \cup A = A$ and $A \cap A = A$.

3) If $\emptyset$ and X are the fuzzy sets, mapping every element of X to zero and one respectively, then $A \cup \emptyset = A$ and $A \cap X = A$ so that $\emptyset$ and X are the identities under '$\cup$' and '$\cap$'.

4) Obviously $A \cap \emptyset = \emptyset$ and $A \cup X = X$.

5) $(A')' = A$.

6) Absorption laws are satisfied.
i.e. $A \cup (A \cap B) = A$ and $A \cap (A \cup B) = A$.

7) For a fuzzy set A, $A \cup A' \neq X$ and $A \cap A' \neq \emptyset$ always, for, if $A(x) = a$, $a \neq 0$ and $a \neq 1$, then $(A \cup A')\ (x) \neq 1$ and $(A \cap A')\ (x) \neq 0$.

For this reason, the collection of all fuzzy sets on X does not form a complemented distributive lattice, but forms a pseudocomplemented distributive lattice.

Lastly, we define the cartesian product AxB of two fuzzy sets as follows :

<u>Definition 1.1.8</u> : If $A : X \longrightarrow I$ and $B : Y \longrightarrow I$ are fuzzy

sets on X and Y respectively, then the cartesian product $A \times B$, $A \times B : X \times Y \longrightarrow I$ is defined as

$(A \times B)(x,y) = \min \{A(x), B(y)\} \quad x \in X$ and $y \in Y$.

1.2. Relations

Definition 1.2.1 : A binary relation R from a set X to a set Y is a subset of $X \times Y$, i.e. $R \subseteq X \times Y$.

When $X = Y$, R is said to be a relation on X.

More generally, we have the following definition.

Definition 1.2.2 : An n-ary relation is a set of ordered n-tuples which is a subset of the cartesian product $X_1 \times X_2 \times \ldots \times X_n$, n being a positive integer.

Definition 1.2.3 : If $R_1 \subseteq X \times Y$ and $R_2 \subseteq Y \times Z$, then the composition $R_1 \circ R_2$ is a relation from X to Z, defined by

$$R_1 \circ R_2 = \{(x,z) : (x,y) \in R_1 \text{ and } (y,z) \in R_2 \text{ for some } y \in Y\}.$$

We now define some important types of relations.

Definition 1.2.4 : If R is a relation on X, then

1) R is reflexive if $(x, x) \in R \;\; \forall x \in X$,
2) R is antireflexive if $(x, x) \notin R \;\forall x \in X$.
3) R is non-reflexive if $(x, x) \notin R$ for some $x \in X$.
4) R is symmetric, if $(x,y) \in R$ implies $(y,x) \in R$, $\forall x,y \in X$.
5) R is antisymmetric, if $(x,y) \in R$ and $(y,x) \in R$ imply $x = y \;\; \forall x, y \in X$.

6) R is asymmetric, if $(x,y) \in R$ implies $(y,x) \notin R$ for every $x, y \in X$.

7) R is transitive, if $(x,y) \in R$ and $(y,z) \in R$, imply $(x,z) \in R$ $\forall x, y, z \in X$.

Definition 1.2.5 : A relation R on X which is reflexive and transitive is called a pre-ordering relation.

Definition 1.2.6 : A pre-ordering relation which is symmetric is called an equivalence relation.

Definition 1.2.7 : A pre-ordering relation which is anti-symmetric is called an ordering relation.

Definition 1.2.8 : The relation $R^+ = RUR^2UR^3U \ldots$ is called the transitive closure of R.

If the set X contains n elements, then $R^+ = RUR^2U\ldots UR^n$.

The following result is well-known in literature.

Theorem 1.2.9 : R^+ is the smallest transitive relation containing R.

Definition 1.2.10 : The symmetric closure of a relation R is the relation RUR^{-1}, where R^{-1} is the inverse relation defined as

$$R^{-1} = \{(y,x) : (x,y) \in R\}.$$

Theorem 1.2. 11: The symmetric closure of a relation R is the smallest symmetric relation containing R.

1.3. Graphs and digraphs

Definition 1.3.1 : A graph G is an ordered pair (V,E) where V is a nonempty set of elements called vertices and E is a finite set of unordered pairs of elements of V called edges.

Definition 1.3.2 : In a graph (V,E), if $\{u,v\} \in E$ then $\{u,v\}$ is called an edge joining the vertices u and v, and u and v are said to be adjacent vertices.

Remark 1.3.3 : As $\{u,v\}$ is unordered pair, the edges $\{u,v\}$ and $\{v, u\}$ are the same. Graphically, an edge in a graph is represented by an undirected arc joining the two vertices.

Definition 1.3.4 : A path from u to v of length n in a graph G = (V,E) is a sequence of vertices $u, a_1, a_2, \ldots, a_{n-1}, v$ such that each pair $\{u, a_1\}$, $\{a_1, a_2\}$, ..., $\{a_{n-1}, v\}$ is in E.

u and v are called the endpoints.

Definition 1.3.5 : If $\{u,u\} \in E$, then there is an arc from u to u, known as a loop.

Definition 1.3.6 : If the end points u and v of a path $u, a_1, a_2, \ldots, a_{n-1}, v$ in a graph (V,E) are equal, then the path forms a circuit.

We now define digraphs and their adjacency matrices.

Definition 1.3.7 : A digraph is an ordered pair (V,E) where V is a nonempty set and E is a relation on V.

Definition 1.3.8 : The members of V are called vertices and the members of E which are ordered pairs are called directed edges.

Definition 1.3.9 : A directed path from u to v of length n in a digraph G = (V,E) is a sequence of vertices $u, a_1, a_2, \ldots, a_{n-1}, v$ such that $(u,a_1) \in E$, $(a_1,a_2) \in E, \ldots, (a_{n-1},v) \in E$.

u and v are called the end points of the directed path.

Remarks 1.3.10 : 1) A directed edge (u,v) in a digraph is represented by an directed arc which starts from u and ends in v.

2) Relations and digraphs are equivalent concepts. For, if G = (V,E) is a digraph, then E is a relation on V and if, $R \subseteq A \times B$ is a relation, then G = (AUB, R) is a digraph.

3) If the relation E in a digraph G = (V,E) is anti-reflexive, then G is a loopfree graph.

4) A relation E in a digraph G = (V,E) is transitive if and only if every directed path has shortcut, i.e. if and only if for every non-trivial directed path from a vertex x to a vertex y, there exists an edge (x,y).

The following result is well known in graph theory.

Theorem 1.3.11 : If G = (V,E) is a digraph, then for $n \geqslant 1$,

$(u,v) \in E^n$ if and only if there exists a directed path of length n from u to v in G.

Corollary 1.3.12 : For any two vertices u and v in a digraph (V,E), $(u,v) \in E^+$ if and only if there exists a nontrivial directed path from u to v where E^+ is the transitive closure of E.

Remark 1.3.13 : The fuzzy analogues of the above two results are proved in the Theorem 3.1.8 and Corollary 3.1.9 in the Chapter III.

We now define the adjacency matrix of a digraph and state a few well-known basic results involving it.

Definition 1.3.14 : Let G = (V,R) be a digraph where $V = \{v_1, v_2, \ldots, v_n\}$ and R be a binary relation on V. The adjacency matrix A of R is defined to be the n x n Boolean matrix where

$$A(i,j) = \begin{cases} 1 \text{ if } (v_i, v_j) \in R \\ 0, \text{ otherwise.} \end{cases}$$

Remark 1.3.15 : For a given ordering on V, the adjacency matrix is unique.

As the adjacency matrix is a Boolean matrix, we define the operators $\oplus$ and $\otimes$ on $\{0, 1\}$ as,

$$x \oplus y = \max (x,y) \text{ and}$$
$$x \otimes y = \min (x,y), \text{ where } x, y \in \{0,1\} .$$

Definition 1.3.16 : If A and B are two n x n Boolean matrices over $\{0, 1\}$, then their inner product $A \oplus \otimes B$ is an n x n matrix given by

$$(A \oplus \otimes B)(i,j) = [A(i,1) \otimes B(1,j)] \oplus [A(i,2) \otimes B(2,j)] \oplus \dots \oplus [A(i,n) \otimes B(n,j)] .$$

Definition 1.3.17 : For any $k \geq 1$, $(\oplus \otimes)^k A$ is defined inductively as

$$(\oplus \otimes)^1 A = A,$$
$$(\oplus \otimes)^2 A = A \oplus \otimes A, \dots,$$
$$(\oplus \otimes)^k A = ((\oplus \otimes)^{k-1} A) \oplus \otimes A,$$

where A is a Boolean matrix.

Theorem 1.3.18 : If A and B are adjacency matrices of the relations R and Q on $V = \{v_1, v_2, \dots, v_n\}$, then $A \oplus \otimes B$ is the adjacency matrix of the relation R o Q.

Theorem 1.3.19 : If A is the adjacency matrix of a relation R then $(\oplus \otimes)^n A$ is the adjacency matrix of R^n.

Theorem 1.3.20 : If A and B are adjacency matrices of the relations R and Q on V then the adjacency matrix of R U Q is $A \oplus B$, where $(A \oplus B)(i,j) = A(i,j) \oplus B(i,j)$.

Theorem 1.3.21 : If A is the adjacency matrix of a relation R on $V = \{v_1, v_2, \dots, v_n\}$, then the adjacency matrix of the transitive closure of R is $A \oplus (\oplus \otimes)^2 A \oplus (\oplus \otimes)^3 A \oplus \dots \oplus (\oplus \otimes)^n A$.

Remark 1.3.22 : The adjacency matrix of a relfexive relation R is the identity matrix and that of a symmetric relation is symmetric.

The proof of the following theorem is well known in literature. The fuzzy analogue of this theorem is proved in the Chapter III.

Theorem 1.3.23 : Let G = (V,E) be a digraph and A is the adjacency matrix. Let $\oplus$ and $\otimes$ denote the operations,

$$x \oplus y = \begin{cases} x & \text{if } x > y \\ y & \text{otherwise} \end{cases},$$

$$x \otimes y = \begin{cases} x + y & \text{if } x > 0,\ y > 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$\text{and } \quad L^k = \begin{cases} A \text{ if } k = 1 \\ L^{k-1} \oplus (L^{k-1} \oplus \otimes A), \text{if } k > 1, \end{cases}$$

Then $L^k(i,j)$ is the length of the longest nontrivial directed path from v_i to v_j that has length less than or equal to k. $L^k(i,j) = 0$ if there is no such path.

oOo