CHAPTER - I

RAY ANALYSIS OF GRAVITATIONAL FIELDS

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§1. INTRODUCTION :

For investigating radiation fields in empty space, an incisive technique has been inaugurated in 1962 by Newman and Penrose. The technique is often referred as the ray analysis of gravitational fields. At each point of the space time permeated by the gravitational field, four rays are introduced. Two of them are real and two of them are complex. The most distinguishing feature is that they can be chosen as mutually orthogonal, thus providing a basis for the four dimensional complex vector space.

THE COMPLEXITY OF EINSTEIN'S FIELD EQUATIONS FOR GRAVITATION :

There are more than three score theories of gravitation but the most successful theory of Gravitation is the one proposed by Einstein in 1916.

Einstein's field equations

$$R_{ab} - \frac{1}{2} R g_{ab} = \frac{-8\pi G}{c^4} T_{ab} \qquad (1.1)$$

are ten nonlinear simultaneous partial differential equations in the variables $\overset{1}{x}, \overset{2}{x}, \overset{3}{x}, \overset{4}{x}$.

Accordingly they provide a stumbling block for analytic treatment of (1.1) for obtaining exact solutions. Besides this mathematical difficulty the following hurdles in experimental validations and understanding exist :

1) the minuteness of the magnitude of physical quantities amenable for experimental detection in the laboratory.

2) the invariance of field equations under arbitrary continuous coordinate transformations.

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3) the nonintegrability of tensors over finite regions of spacetime.

In Sec. 2 we briefly refer to the several formalism invougue among research workers in Relativity. An exposition of the Newman Penrose formalism is given in four sections. Sec. 3 contains Algebraic relations, Completeness relation, Special Ray Scalars, Differential Relation, Transparency of Newman formalism, Five Weyl Scalars, Ricci Scalars. The ray analysis of Einstein field equations and Ricci Identities is described in Sec. 4. (The appendix contains a comparision of the formalism used in this dissertation with classical method.) Last section contains the enumeration of the eleven Bianchi Identities in the NP formalism. This chapter does not contain original results and it is primarily an exposition of the mathematical preliminaries for the dissertation.

2. FORMALISMS IN RELATIVITY :

To solve the Einstein's field equations of Gravitation in the presence of matter or in the absence of matter many formalisms have been developed. The most prominent one acclaimed as "amazingly useful" (FLAHARTY 1964) formalism is the one proposed by Newman & Penrose

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(1962). The discovery of the Kerr Newman black hole metric is the proof of the efficiency of this ray formalism. Newman Unti (1962) Ludwig (1980 a, b) Newman and Tod (1980) have studied the asymptotic flatness of solutions using this formalism. With the help of this formalism Collinson and Morris (1972); Griffiths (1976 a, b) have studied Newtrino radiation fields.

Other formalisms after (1962) are mentioned below :-

Author of Formalism	Year	Nature of tetrad.
Synge	1972	One null and three space like vector fields
Geroch, Held and Penrose	1973	Two null and two unit spacelike vector fields
Hall	1977	Two null and two spacelike vector fields

In 1980, two more formalisms have been introduced (1) Penrose Conformal formalism, (2) Edgar Rotational Invariant Formalism.

NP formalism has the following advantages :-

a) It is suitable for computational work.

b) It is adaptable to other formalisms (Vide 3e).

c) It makes the Einstein's field equations - transparent.

d) It thoroughly utilizes the Bianchi identities.

3. EXPOSITION :

3 a) Algebraic Relation :

Newman and Penrose (1962) invented a set of four rays

$$X_i^a = \{ l^a, m^a, \bar{m}^a, n^a \}, \quad i = 1,2,3,4, \qquad (1.2)$$

where l^a, n^a are two real rays and m^a, $\bar{m}^a$ (an overhead bar denotes complex conjugate) are complex rays which satisfy the following conditions.

The Four Null Rays :

$$l^a l_a = m^a m_a = n^a n_a = \bar{m}^a \bar{m}_a = 0.$$

The Orthonormal Relations :

$$\left.\begin{array}{l} (i)\ l^a m_a = l^a \bar{m}_a = n^a m_a = n^a \bar{m}_a = 0, \\ (ii)\ l^a n_a = -m^a \bar{m}^a = 1. \end{array}\right\} \qquad (1.3)$$

Thus out of the ten inner products of the four rays, as many as eight vanish . The appearance of zeros in (1.3) is responsible for the computational advantage of this formalism over the other.

3.b) Completeness Relation :

The spacetime metric gives the relationship between this tetrad and the geometry of space-time, viz.,

$$g_{ab} = l_a n_b + n_a l_b - m_a \bar{m}_b - \bar{m}_a m_b. \qquad (1.4)$$

c) Special Ray Scalars :

Newman and Penrose have utilised the 12 Greek symbols $\alpha, \beta, \gamma, \epsilon, \kappa, \lambda, \mu, \nu, \pi, \rho, \sigma, \tau$ with the following specific identification :

$$\begin{aligned}
\alpha &= \tfrac{1}{2}\left(l_{a;b} n^a \bar{m}^b - m_{a;b} \bar{m}^a \bar{m}^b\right),\\
\beta &= \tfrac{1}{2}\left(l_{a;b} n^a m^b - m_{a;b} \bar{m}^a m^b\right),\\
\gamma &= \tfrac{1}{2}\left(l_{a;b} n^a n^b - m_{a;b} \bar{m}^a n^b\right),\\
\epsilon &= \tfrac{1}{2}\left(l_{a;b} n^a l^b - m_{a;b} \bar{m}^a l^b\right),\\
\kappa &= l_{a;b} m^a l^b,\\
\rho &= l_{a;b} m^a \bar{m}^b,\\
\sigma &= l_{a;b} m^a m^b,\\
\tau &= l_{a;b} m^a n^b,\\
\nu &= -n_{a;b} \bar{m}^a n^b,\\
\mu &= -n_{a;b} \bar{m}^a m^b,\\
\lambda &= -n_{a;b} \bar{m}^a \bar{m}^b,\\
\pi &= -n_{a;b} \bar{m}^a l^b.
\end{aligned} \qquad (1.5)$$

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3 d <u>On the simplification of differential relations in the ray formalism</u> :

The fact that the covariant derivative of a ray (null vector field) is expressible as an algebraic (linear) combination of the basis rays, means that every differential relation becomes an algebraic relation
For example,

$$\begin{aligned} n_{a;b} = {} & \nu m_a l_b - \lambda m_a m_b - \mu m_a \bar{m}_b + \pi m_a n_b + \\ & + \bar{\nu} \bar{m}_a l_b - \bar{\mu} \bar{m}_a m_b - \bar{\lambda} \bar{m}_a \bar{m}_b + \bar{\pi} \bar{m}_a n_b - \\ & - (\gamma + \bar{\gamma}) n_a l_b + (\alpha + \bar{\beta}) n_a m_b + (\bar{\alpha} + \beta) n_a \bar{m}_b - \\ & - (\epsilon + \bar{\epsilon}) n_a n_b \end{aligned} \tag{1.6}$$

implies that the differential equations

$$n_{a;b} = 0 \tag{1.7}$$

are equivalent to the seven algebraic relations

$$\nu = 0, \ \lambda = 0, \ \mu = 0, \ \pi = 0, \ \gamma + \bar{\gamma} = 0, \ \alpha + \bar{\beta} = 0, \ \epsilon + \bar{\epsilon} = 0. \tag{1.8}$$

3 e <u>Transition from rays to timelike vectors and spacelike vectors</u> :

The accessibility of this ray formalism to non-null vector fields is easy, since any vector can be expressed as the unique linear combination of the basis ray fields ($l^a, m^a, \bar{m}^a, n^a$). We write the unit time like unit vector u^a as :

$$u^a = A l^a + B m^a + \bar{B} \bar{m}^a + C n^a \tag{1.9}$$

together with

$$u^a u_a = 1 \tag{1.10}$$

implies by virtue of orthonormal relations (1.3)

$$2(AC - B\bar{B}) = u^a u_a = 1 \tag{1.11}$$

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This is only one condition on A, C, B. Therefore, there exist infinitely many choices for A, C, B. The most famous choice is

$$A = C = (2)^{-1/2}, \quad B = 0. \tag{1.12}$$

Accordingly,

$$u^a = (2)^{-1/2} (l^a + n^a)$$

$$\left(\text{or } u^a = (2)^{-1/2} (m^a - \bar{m}^a), \text{ for } A = C = 0, B \neq 0\right) \tag{1.13}$$

expresses the popular connection between timelike and null vectors.

The unit spacelike vector field h^a can be expressed as

$$\text{either } h^a = (2)^{-1/2} (m^a + \bar{m}^a),$$

$$\text{or } h^a = (2)^{-1/2} (l^a - n^a),$$

$$\text{or } h^a = -i (2)^{-1/2} (m^a - \bar{m}^a), \tag{1.14}$$

$$\text{with } h^a h_a = -1$$

3 f <u>The five ray components of Weyl Tensor</u> :

C_{abcd} is the Weyl tensor which is the free gravitational part of the curvature tensor R_{abcd}.

$$R_{abcd} = C_{abcd} - \tfrac{1}{2} (g_{ac} R_{bd} - g_{ad} R_{bc} + g_{bd} R_{ac} - g_{bc} R_{ad}) - \tfrac{R}{6} (g_{ad} g_{bc} - g_{ac} g_{bd}) \tag{1.15}$$

where

$R_{ab} = R^c{}_{abc}$ is the Ricci tensor, and

$R^a{}_a = R$ is the scalar curvature.

In general, Rab has ten independent components and Rabcd has twenty independent components. Hence C_{abcd}

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has ten independent components. The five complex ray components of C_{abcd} are always denoted by Ψ_a, $(a=0,1,2,3,4)$ in this Newman-Penrose formalism.

$$\begin{aligned}
&\underline{\text{W 1}} \quad \Psi_0 = -C_{abcd}\, l^a m^b l^c m^d \\
&\underline{\text{W 2}} \quad \Psi_1 = -C_{abcd}\, l^a n^b l^c m^d \\
&\underline{\text{W 3}} \quad \Psi_2 = -C_{abcd}\, \bar{m}^a n^b l^c m^d \qquad\qquad (1.16)\\
&\underline{\text{W 4}} \quad \Psi_3 = -C_{abcd}\, \bar{m}^a n^b l^c n^d \\
&\underline{\text{W 5}} \quad \Psi_4 = -C_{abcd}\, \bar{m}^a n^b \bar{m}^c n^d .
\end{aligned}$$

3 g <u>The ray components of Ricci Tensor</u> :

Ricci tensor R_{ab} can be expressed in terms of physical components or optical projections which are denoted by

$$\begin{aligned}
&\underline{\text{R 1}} \quad \Phi_{00} = -\tfrac{1}{2} R_{ab}\, l^a l^b \\
&\underline{\text{R 2}} \quad \Phi_{01} = -\tfrac{1}{2} R_{ab}\, l^a m^b \\
&\underline{\text{R 3}} \quad \Phi_{02} = -\tfrac{1}{2} R_{ab}\, m^a m^b \\
&\underline{\text{R 4}} \quad \Phi_{12} = -\tfrac{1}{2} R_{ab}\, n^a m^b \qquad\qquad (1.17)\\
&\underline{\text{R 5}} \quad \Phi_{11} = -\tfrac{1}{4} R_{ab}\, (l^a n^b + m^a \bar{m}^b) \\
&\underline{\text{R 6}} \quad \Phi_{22} = -\tfrac{1}{2} R_{ab}\, n^a n^b \\
&\underline{\text{R 7}} \quad \Phi_{20} = -\tfrac{1}{2} R_{ab}\, m^a \bar{m}^b \\
&\underline{\text{R 8}} \quad \Phi_{21} = -\tfrac{1}{2} R_{ab}\, n^a \bar{m}^b
\end{aligned}$$

R 9 $\quad \phi_{10} = -\frac{1}{2} R_{ab} l^a \bar{m}^b$

and a curvature scalar

R 10 $\quad \Lambda = \frac{1}{24} R.$

§ 4. RAY VERSION OF EINSTEIN FIELD EQUATION AND RICCI IDENTITIES (NP EQUATIONS) :

The Newman Penrose equations are obtained from the Ricci identities.

$$Z_{a;bc} - Z_{a;cb} = Z^d R_{dabc}, \qquad (1.18)$$

together with Einstein's field equations.

4 a. Equations for the free gravitational field (Ψ's)

G 1 $\quad \Psi_0 = D\sigma - \delta\kappa - \sigma(3\epsilon - \bar\epsilon) - \sigma(\rho + \bar\rho) + \kappa(\tau - \bar\pi + \bar\alpha + 3\beta),$

G 2 $\quad \Psi_1 = D\beta - \delta\epsilon - \sigma(\alpha + \pi) + \kappa(\gamma + \mu) - \beta(\bar\rho - \bar\epsilon) - \epsilon(\bar\pi - \bar\alpha), \qquad (1.19)$

G 3 $\quad \Psi_3 = \bar\delta\gamma - \Delta\alpha - \lambda(\beta + \tau) + \gamma(\rho + \epsilon) + \alpha(\bar\gamma - \bar\mu) + \gamma(\bar\beta - \bar\tau),$

G 4 $\quad \Psi_4 = \bar\delta\nu - \Delta\lambda - \lambda(\mu + \bar\mu) - \lambda(3\nu - \bar\nu) - \nu(\bar\tau - \pi - 3\alpha - \bar\beta).$

4 b. Equations characterizing the matter field (ϕ_{ab})

M 1 $\quad \phi_{00} = D\rho - \bar\delta\kappa - \rho^2 - \sigma\bar\sigma - \rho(\epsilon + \bar\epsilon) + \bar\kappa\tau + \kappa(3\alpha + \bar\beta - \pi)$

M 2 $\quad \phi_{10} = D\alpha - \bar\delta\epsilon - \alpha(\rho + \bar\epsilon - 2\epsilon) - \beta\bar\sigma + \bar\beta\epsilon - \pi(\rho + \epsilon) + \kappa\lambda + \bar\kappa\gamma,$

M 3 $\quad \phi_{20} = D\lambda - \bar\delta\pi - \rho\lambda - \bar\sigma\mu - \pi^2 - \pi(\alpha - \bar\beta) - \lambda(\bar\epsilon - 3\epsilon) + \bar\kappa\nu,$

$$\begin{aligned}
&\underline{\text{M 4}}\ \Phi_{02} = \delta\tau - \Delta\sigma - \mu\sigma - \rho\bar{\lambda} - \tau(\tau - \bar{\alpha} + \beta) - \sigma(\bar{\gamma} - 3\gamma) + \\
&\qquad + \kappa\bar{\nu}, \\
&\underline{\text{M 5}}\ \Phi_{12} = \delta\gamma - \Delta\beta - \mu\tau + \sigma\bar{\nu} + \epsilon\nu - \gamma(\tau - \bar{\alpha} - \beta) + \\
&\qquad + \beta(\gamma - \bar{\gamma} - \mu) - \bar{\lambda}\alpha, \\
&\underline{\text{M 6}}\ \Phi_{22} = \delta\nu - \Delta\mu - \mu^2 - \lambda\bar{\lambda} - \mu(\gamma + \bar{\gamma}) - \nu(\tau - \bar{\alpha} - 3\beta) + \\
&\qquad + \bar{\nu}\pi.
\end{aligned} \tag{1.20}$$

4 c. Equations for the coupling of matter and free gravitation (ϕ_{ab} and Ψ_a)

$$\begin{aligned}
&\underline{\text{MG 1}}\ \Psi_2 + 2\Lambda = D\mu - \delta\mu - \bar{\rho}\mu - \sigma\lambda - \pi\bar{\pi} + \mu(\epsilon + \bar{\epsilon}) + \\
&\qquad + \pi(\bar{\alpha} - \beta) + \kappa\nu, \\
&\underline{\text{MG 2}}\ \Psi_1 + \Phi_{01} = D\tau - \Delta\kappa - \rho(\tau + \bar{\pi}) - \sigma(\bar{\tau} + \pi) - \\
&\qquad - \tau(\epsilon - \bar{\epsilon}) + \kappa(\bar{\gamma} + 3\gamma), \\
&\underline{\text{MG 3}}\ \Psi_2 + \Phi_{11} - \Lambda = D\gamma - \Delta\epsilon - \alpha(\tau + \bar{\pi}) - \beta(\bar{\tau} + \pi) - \\
&\qquad - \gamma(\epsilon + \bar{\epsilon}) + \epsilon(\gamma + \bar{\gamma}) - \tau\pi + \kappa\nu, \\
&\underline{\text{MG 4}}\ \Psi_3 + \Phi_{21} = D\nu - \Delta\pi - \lambda(\tau + \bar{\pi}) - \mu(\bar{\tau} + \pi) + \\
&\qquad + \nu(\bar{\epsilon} + 3\epsilon) - \pi(\nu - \bar{\nu}), \\
&\underline{\text{MG 5}}\ \Psi_1 - \Phi_{01} = \bar{\delta}\sigma - \delta\rho + \rho(\bar{\alpha} + \beta) - \sigma(3\alpha - \bar{\beta}) + \\
&\qquad + \tau(\rho - \bar{\rho}) + \kappa(\mu - \bar{\mu}), \\
&\underline{\text{MG 6}}\ \Psi_2 - \Phi_{11} - \Lambda = \bar{\delta}\beta - \delta\alpha + \rho\mu - \sigma\lambda + \alpha\bar{\alpha} + \beta\bar{\beta} - \\
&\qquad - 2\alpha\beta - \gamma(\rho - \bar{\rho}) + \epsilon(\mu - \bar{\mu}), \\
&\underline{\text{MG 7}}\ \Psi_3 - \Phi_{21} = \bar{\delta}\mu - \delta\lambda - \nu(\rho - \bar{\rho}) + \pi(\mu - \bar{\mu}) + \\
&\qquad + \mu(\alpha + \bar{\beta}) + \lambda(\bar{\alpha} - 3\beta) \\
&\underline{\text{MG 8}}\ \Psi_2 + 2\Lambda = \bar{\delta}\tau - \Delta\rho - \rho\bar{\mu} - \sigma\lambda + \tau(\bar{\beta} - \alpha - \bar{\tau}) + \\
&\qquad + \rho(\gamma + \bar{\gamma}) + \kappa\nu.
\end{aligned} \tag{1.21}$$

Where
$$D = l^a \frac{\partial}{\partial x^a}, \quad \Delta = n^a \frac{\partial}{\partial x^a},$$
$$\delta = m^a \frac{\partial}{\partial x^a}, \quad \bar{\delta} = \bar{m}^a \frac{\partial}{\partial x^a}.$$

4 d. <u>The Commutative Relations</u> :

The explicit equations obtained by applying the commutators of successive intrinsic derivatives (on any scalar ϕ) can be given as :

$$\text{CR 1} \quad (\Delta D - D\Delta)\phi = [(\gamma+\bar{\gamma})D + (\epsilon+\bar{\epsilon})\Delta - (\bar{\tau}+\pi)\delta - (\tau+\bar{\pi})\bar{\delta}]\phi,$$

$$\text{CR 2} \quad (\delta D - D\delta)\phi = [(\bar{\alpha}+\beta-\bar{\pi})D + \kappa\Delta - (\bar{\rho}+\epsilon-\bar{\epsilon})\delta - \sigma\bar{\delta}]\phi,$$

$$\text{CR 3} \quad (\delta\Delta - \Delta\delta)\phi = [-\bar{\nu} D + (\tau-\bar{\alpha}-\beta)\Delta + (\mu-\gamma+\bar{\gamma})\delta + \bar{\lambda}\bar{\delta}]\phi, \qquad (1.22)$$

$$\text{CR 4} \quad (\bar{\delta}\delta - \delta\bar{\delta})\phi = [(\bar{\mu}-\mu)D + (\bar{\rho}-\rho)\Delta + (\alpha-\bar{\beta})\delta - (\bar{\alpha}-\beta)\bar{\delta}]\phi$$

If in these equations, we substitute x^a for the arbitrary function ϕ and use
$$Dx^a = l^a, \quad \delta x^a = m^a, \quad \bar{\delta} x^a = \bar{m}^a, \quad \Delta x^a = n^a$$
we get

$$\text{I} \quad [Dn^a - \Delta l^a] = [-(\gamma+\bar{\gamma})l^a + (\pi+\bar{\tau})m^a + (\bar{\pi}+\tau)\bar{m}^a - (\epsilon+\bar{\epsilon})n^a],$$

$$\text{II} \quad [Dm^a - \delta \bar{m}^a] = [(\bar{\pi}-\bar{\alpha}-\beta)l^a + (\bar{\rho}+\epsilon-\bar{\epsilon})m^a + \sigma\bar{m}^a - \kappa n^a], \qquad (1.23)$$

$$\text{III} \quad [\delta n^a - \Delta m^a] = [-\bar{\nu} l^a + (\mu-\gamma+\bar{\gamma})m^a + \bar{\lambda}\bar{m}^a + (\tau-\bar{\alpha}-\beta)n^a],$$

$$\text{IV} \quad [\delta\bar{m}^a - \bar{\delta} m^a] = [(\mu-\bar{\mu})l^a + (\bar{\beta}-\alpha)m^a + (\bar{\alpha}-\beta)\bar{m}^a + (\rho-\bar{\rho})n^a].$$

These equations, with their complex conjugates are referred to as the 'metric equations' and constitute the first set of the spin coefficient equations (Newman and Tod, 1980)

§5. RAY FORMALISM FOR THE INTERACTION OF THE FREE GRAVITATIONAL FIELD WITH SOURCES :

From the four energy balance equations.

$$T^{ab}{}_{;b} = 0 \tag{1.24}$$

we cannot get the idea of the interaction of the free gravitational field characterized by C_{abcd} with the sources. We use twenty four Bianchi identities to get that information.

Interaction Equations (Bianchi identities) :

The enumeration (STEWART 1984) follows the computer generated version developed in Cambridge University.

BI 1 $$D\Psi_1 - \bar{\delta}\Psi_0 - D\Phi_{01} + \delta\Phi_{00} = (\pi - 4\alpha)\Psi_0 + 2(2\rho + \epsilon)\Psi_1 - 3\kappa\Psi_2 - (\bar{\pi} - 2\alpha - 2\beta)\Phi_{00} - 2(\bar{\rho} + \epsilon)\Phi_{01} - 2\sigma\Phi_{10} + 2\kappa\Phi_{11} + \bar{\kappa}\Phi_{02},$$

BI 2 $$\Delta\Psi_0 - \delta\Psi_1 + D\Phi_{02} - \delta\Phi_{01} = (4\gamma - \mu)\Psi_0 - 2(2\tau + \beta)\Psi_1 + 3\sigma\Psi_2 - \bar{\lambda}\Phi_{00} + 2(\bar{\pi} - \beta)\Phi_{01} + 2\sigma\Phi_{11} + (\bar{\rho} + 2\epsilon - 2\bar{\epsilon})\Phi_{02} - 2\kappa\Phi_{12},$$

BI 3
$$\begin{aligned} & D\Psi_2 - \bar\delta\Psi_1 + \Delta\Phi_{00} - \bar\delta\Phi_{01} + 2D\Lambda = -\lambda\Psi_0 + \\ & + 2(\pi-\alpha)\Psi_1 + 3\rho\Psi_2 - 2\kappa\Psi_3 + (2\gamma+2\bar\gamma-\bar\mu)\Phi_{00} - \\ & - 2(\alpha+\bar\tau)\Phi_{01} - 2\tau\Phi_{10} + 2\rho\Phi_{11} + \bar\sigma\Phi_{02}, \end{aligned}$$

BI 4
$$\begin{aligned} & \Delta\Psi_1 - \delta\Psi_2 - \Delta\Phi_{01} + \bar\delta\Phi_{02} - 2\delta\Lambda = \nu\Psi_0 + 2(\gamma-\mu)\Psi_1 - \\ & - 3\tau\Psi_2 + 2\sigma\Psi_3 - \bar\nu\Phi_{00} + 2(\bar\mu-\gamma)\Phi_{01} + \\ & + (2\alpha+\bar\tau-2\bar\beta)\Phi_{02} + 2\tau\Phi_{11} - 2\rho\Phi_{12}, \end{aligned}$$

BI 5
$$\begin{aligned} & D\Psi_3 - \bar\delta\Psi_2 - D\Phi_{21} + \delta\Phi_{20} - 2\bar\delta\Lambda = -2\lambda\Psi_1 + \\ & 3\pi\Psi_2 + 2(\rho-\epsilon)\Psi_3 - \kappa\Psi_4 + 2\mu\Phi_{10} - 2\pi\Phi_{11} - \\ & - (2\beta+\bar\pi-2\bar\alpha)\Phi_{20} - 2(\bar\rho-\epsilon)\Phi_{21} + \bar\kappa\Phi_{22}, \end{aligned}$$

BI 6
$$\begin{aligned} & \Delta\Psi_2 - \delta\Psi_3 + D\Phi_{22} - \delta\Phi_{21} + 2\Delta\Lambda = 2\nu\Psi_1 - \\ & - 2\mu\Psi_2 + 2(\beta-\tau)\Psi_3 + \sigma\Psi_4 - 2\mu\Phi_{11} - \bar\lambda\Phi_{20} + \\ & + 2\pi\Phi_{12} + 2(\beta+\bar\pi)\Phi_{21} + (\bar\rho-2\epsilon-2\bar\epsilon)\Phi_{22}, \end{aligned}$$

BI 7
$$\begin{aligned} & D\Psi_4 - \bar\delta\Psi_3 + \Delta\Phi_{20} - \bar\delta\Phi_{21} = -3\lambda\Psi_2 + 2(\alpha+2\pi)\Psi_3 + \\ & + (\rho-4\epsilon)\Psi_4 + 2\nu\Phi_{10} - 2\lambda\Phi_{11} - (2\gamma-2\bar\gamma+\bar\mu)\Phi_{20} - \\ & - 2(\bar\tau-\alpha)\Phi_{21} + \bar\sigma\Phi_{22}, \end{aligned}$$

BI 8
$$\begin{aligned} & \Delta\Psi_3 - \delta\Psi_4 - \Delta\Phi_{21} + \bar\delta\Phi_{22} = 3\nu\Psi_2 - 2(\gamma+2\mu)\Psi_3 + \\ & + (4\beta-\tau)\Psi_4 - 2\nu\Phi_{11} - \bar\nu\Phi_{20} + 2\lambda\Phi_{12} + 2(\gamma+\bar\mu)\Phi_{21} + \\ & + (\bar\tau-2\bar\beta-2\alpha)\Phi_{22}, \end{aligned}$$

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BI 9 $D\Phi_{11} - \delta\Phi_{10} + \Delta\Phi_{00} - \bar{\delta}\Phi_{01} + 3D\Lambda =$

$(2\gamma + 2\bar{\gamma} - \mu - \bar{\mu})\Phi_{00} + (\pi - 2\alpha - \bar{\tau})\Phi_{01} +$

$+ (\bar{\pi} - 2\alpha - 2\tau)\Phi_{10} + 2(\rho + \bar{\rho})\Phi_{11} + \bar{\sigma}\Phi_{02} +$

$+ \sigma\Phi_{20} - \bar{\kappa}\Phi_{12} - \kappa\Phi_{21}$,

BI 10 $D\Phi_{12} - \delta\Phi_{11} + \Delta\Phi_{01} - 2\bar{\delta}\Phi_{02} + 3\delta\Lambda =$

$(2\gamma - \mu - 2\bar{\mu})\Phi_{01} + \bar{\nu}\Phi_{00} - \bar{\lambda}\Phi_{10} + 2(\bar{\pi} - \tau)\Phi_{11} +$

$(\pi + 2\bar{\beta} - 2\alpha - \bar{\tau})\Phi_{02} + (2\rho + \bar{\rho} - 2\epsilon)\Phi_{12} +$

$\sigma\Phi_{21} - \kappa\Phi_{22}$,

BI 11 $D\Phi_{22} - \delta\Phi_{21} + \Delta\Phi_{11} - \bar{\delta}\Phi_{12} + 3\Delta\Lambda = \nu\Phi_{01} +$

$\bar{\nu}\Phi_{10} - 2(\mu + \bar{\mu})\Phi_{11} - \lambda\Phi_{02} - \bar{\lambda}\Phi_{20} +$

$+ (2\pi - \bar{\tau} + 2\beta)\Phi_{12} + (2\beta - \tau + 2\bar{\pi})\Phi_{21} +$

$+ (\rho + \bar{\rho} - 2\epsilon - 2\bar{\epsilon})\Phi_{22}$.

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APPENDIX

. THE NP METHOD COMPARED WITH THE CLASSICAL METHOD :

Campbell and Wain Wright (1977) find that NP method saves 60% of the computer time needed by the classical method in evaluating tensor quantities.

The relative simplicity of the NP method can be given as :

1) Classical method has 10 components of metric tensor while NP method has 6 independent components of the tetrad (16 components can be reduced using 10 orthonormal conditions).

2) Classical method gives 40 christoffel symbols while NP method gives 12 complex spin coefficients.

3) Classical method has 20 components of curvature tensor while NP method has 12 tetrad components of the various curvature tensors (5 components of Weyl tensor, 6 components of Ricci tensor, 1 component of Ricci scalar).

4) Classical method gives 24 differential Bianchi identities while NP method gives 11 complex Bianchi identities.