# CHAPTER - II

## SOME SUBSTITUTION THEOREMS FOR LAPLACE - HANKEL TRANSFORMATIONS

### 2.1 Introduction :

The Laplace-Hankel transform [2] will be taken as

$$(2.1.1) \qquad F(u,v) = \int_0^\infty \int_0^\infty e^{-ux} \sqrt{vy}\; J_\lambda(vy)\, f(x,y)\, dx dy ,$$

in which f(x,y) will be referred to as the original; F(u,v) as the image. This transformation will be denoted by

$$\mathcal{L}\mathcal{H}_\lambda \left[ f(x,y) \right] = F(u,v) ,$$

or by

$$f(x,y) \overset{\mathcal{L}\mathcal{H}_\lambda}{\circ\!\!-\!\!\circ} F(u,v) .$$

In this chapter formulae for the Laplace-Hankel image of $K(x) f(g(x), h(y))$ in terms of f(x,y) are derived with certain restrictions on f(x,y). g(x) and h(y), similar to the substitution theorems of Buschman [1].

The first substitution theorem involves the representation of $\mathcal{L}\mathcal{H}_\lambda \{ K(x)\, f(g(x), h(y)) \}$ in terms of $\mathcal{L}\mathcal{H}_\lambda \{ f(g(x), h(y)) \}$. The convergence of integral involved in the above transform formula will depend upon k(x), g(x), h(x) and f(x,y). For validity of the transform formula general conditions are imposed on these functions.

$$\leq \int_0^\infty \int_0^\infty \left| \int_0^\infty \int_0^\infty \left| e^{-pw} \sqrt{qz}\, J_\lambda(qz) \phi(u,v;w,z) f(p,q) \right| dwdz \right| dpdq ,$$

then by the substitution from (iii) the first iterated integral becomes

$$\left| \int_0^\infty \int_0^\infty \right| e^{-uG(p)} \sqrt{vH(q)}\; J_\lambda\,(vH(q))\, K\,(G(p))\; |G'(p)|\,|H'(q)| \Big|$$

$$|f(p,q)|\, dpdq$$

Thus

$$\int_0^\infty \int_0^\infty e^{-uG(p)} \sqrt{vH(q)}\; J_\lambda\, (vH(q))\; K(G(p))\; |G'(p)|\; |H'(q)| f(p,q) dpdq$$

is absolutely convergent for $Re\, u > 0$ and $0 < v < \infty$ .

There are two cases to be considered.

<u>Case 1</u> :

If $g(0) = 0$, $h(0) = 0$ and $g(\infty) = \infty$, $h(\infty) = \infty$, then $G'(p) \geq 0$ , $H'(q) \geq 0$ so that if the substitutions $x = G(p)$ and $y = H(q)$ are made, then

$$K(x) f\;(g(x),h(y)) \overset{\mathcal{L}\mathcal{H}_\lambda}{O\!-\!O} \int_0^\infty \int_0^\infty e^{-ux} \sqrt{vy}\; J_\lambda(vy) k(x) f(g(x),h(y)) dxdy$$

becomes

$$K(x)\; f(g(x),h(y)\;) \overset{\mathcal{L}\mathcal{H}_\lambda}{O\!-\!O} \int_0^\infty \int_0^\infty e^{-uG(p)} \sqrt{vH(q)}\; J_\lambda(vH(q)\; K(G(p)\,)$$

$$|G'(p)|\;\; |H'(q)|\;\; f(p,q)\; dpdq .$$

<u>Case 2</u> :

If $g(0) = \infty$ , $h(0) = \infty$ and $g(\infty) = 0$, $h(\infty) = 0$ then $G'(p) \leq 0$, $H'(q) \leq 0$ so that if the substitutions $x = G(p)$, $y = H(q)$ are made, then

$$K(x)\ f(g(x),h(y)) \overset{\mathcal{H}_\lambda}{\circ\!-\!\circ} \int_0^\infty\int_0^\infty e^{-ux}\sqrt{vy}\ J_\lambda(vy)K(x)f(g(x),h(y))\,dx\,dy$$

becomes

$$K(x)f(g(x),h(y)) \overset{\mathcal{H}_\lambda}{\circ\!-\!\circ} \int_{\infty}^{0}\int_{\infty}^{0} e^{-uG(p)}\sqrt{vH(q)}\ J_\lambda(vH(q))K(G(p))$$

$$G'(p)\ H'(q)\ f(p,q)\ dpdq\ ,$$

which is equivalent to

$$K(x)f(g(x),h(y)) \overset{\mathcal{H}_\lambda}{\circ\!-\!\circ} \int_0^\infty\int_0^\infty e^{-uG(p)}\sqrt{vH(q)}\ J_\lambda\ (vH(q))K(G(p))$$

$$|\ G'(p)|\ |H'(q)|\ f(p,q)\ dpdq\ .$$

Thus in either case $\mathcal{H}_\lambda\{K(x)\ f\ (g(x),h(y))\}$ is absolutely convergent for $\operatorname{Re} u > 0$ and $0 < v < \infty$ .

Now the substitution from (iii) is used in either of these cases; the result is

$$K(x)f((g(x),h(y)) \overset{\mathcal{H}_\lambda}{\circ\!-\!\circ} \int_0^\infty\int_0^\infty \left[\int_0^\infty\int_0^\infty e^{-pw}\sqrt{qz}\ J_\lambda(qz)\phi(u,v;w,z)\right.$$

$$\left. dwdz\right]\ f(p,q)\ dpdq\ .$$

which is an absolutely convergent iterated integral. The order of integration can thus be changed so that

$$K(x)\ f(g(x),\ h(y)) \overset{\mathcal{H}_\lambda}{\circ\!-\!\circ} \int_0^\infty\int_0^\infty\left[\int_0^\infty\int_0^\infty e^{-pw}\sqrt{qz}\ J_\lambda(qz)\ f(p,q)\ dpdq\right]$$

$$\phi\ (u,v;w,z)\ dwdz\ .$$

Finally from (ii)

$$K(x)f(g(x),h(y)) \overset{\mathcal{H}_\lambda}{\circ\!-\!\circ} \int_0^\infty\int_0^\infty F(w,z)\ \phi\ (u,v;w,z)\ dwdz\ .$$

**Theorem : 2.2.2**

If (i) $A, k, g, h$ and $g^{-1}=G$, $h^{-1}=H$ are single valued analytic functions, real on $(o,\infty)$ and such that $g(o)=0$ and $g(\infty)=\infty$, $h(\infty)=\infty$ (or $g(o)=\infty$, $h(o)=\infty$ and $g(\infty)=0$, $h(\infty)=0$ ) ;

(ii) $A(x)\ f(x,y) \overset{\mathcal{L}\mathcal{H}_\lambda}{\text{O--O}} F(^{*}u,v)$

which converges for $Re u > o$ and $o < v < \infty$ ;

(iii) there exists a function $\phi^{*}(u,v;w,z)$,

$$\phi^{*}(u,v;w,z) \overset{\mathcal{L}\mathcal{H}_\lambda}{\text{O--O}} \bar{\phi}^{*}(u,v;p,q)$$

which converges for $Re p > o$ and $o < q < \infty$, and

$$\bar{\phi}^{*}(u,v;p,q) = e^{-uG(p)} \sqrt{vH(q)}\ J_\lambda(vH(q))\ K(G(p))\ |G'(p)|$$

$$|H'(q)|\ \left(A(p)\right)^{-1} ;$$

(iv) $\int_o^\infty \int_o^\infty \left[ \int_o^\infty \int_o^\infty e^{-pw} \sqrt{qz}\ J_\lambda(qz)\ \phi^{*}(u,v;w,z) f(p,q)\, dw\, dz \right] dp\, dq$

converges absolutely for $Re u > o$ and $o < v < \infty$ ;

then $K(x) f(g(x), h(y)) \overset{\mathcal{L}\mathcal{H}_\lambda}{\text{O--O}} \int_o^\infty \int_o^\infty F^{*}(w,z)\phi^{*}(u,v;w,z)\, dw\, dz$

which converges absolutely for $Re u > o$ and $o < v < \infty$ .

**Proof :**

Since from (iv) the iterated integral is absolutely convergent for $Re u > o$ and $o < v < \infty$, and since

$$\left| \int_o^\infty \int_o^\infty \left| \int_o^\infty \int_o^\infty e^{-pw} \sqrt{qz}\ J_\lambda(qz)\ \phi^{*}(u,v;w,z)\, dw\, dz \right| \left| A(p) f(p,q) \right| dp\, dq \right|$$

$$\int_0^\infty \int_0^\infty \left| \int_0^\infty \int_0^\infty e^{-pw} \sqrt{qz}\, J_\lambda(qz)\, \phi^*(u,v;w,z)\, A(p) f(p,q)\, dwdz \right| dpdq \,.$$

then by substitution from (iii) the first iterated integral becomes

$$\int_0^\infty \int_0^\infty \left| e^{-uG(p)} \sqrt{vH(q)}\, J_\lambda\,(vH(q)\, K\,(G(p)\,)\, |\,G'(P)|\, |H'(q)|\, (\,A(p)\,)^{-1} \right| |A(p)f(p,q)|\, dpdq \,.$$

Thus

$$\int_0^\infty \int_0^\infty e^{-uG(p)} \sqrt{vH(q)}\, J_\lambda\,(vH(q))K(G(p)\,)\, |G'(p)|\, |\,H'(q)|\, (A(p)\,)^{-1} A(p)\, f(p,q)\, dpdq$$

is absolutely convergent for $Re u > 0$ and $0 < v < \infty$ .

There are two cases to be considered.

<u>Case 1</u> :

If $g(0) = 0$, $h(0) = 0$ and $g(\infty) = \infty$ , $h(\infty) = \infty$ then $G'(p) \geqslant 0$, $H'(q)$ 0 so that if the substitutions $x = G(p)$, $y = H(q)$ are made, then

$$K(x)f(g(x),h(y)) \overset{\mathcal{B}\mathcal{H}_\lambda}{\circ\!-\!\circ} e^{-ux} \sqrt{vy}\, J_\lambda(vy)\, k(x)f(g(x),h(y)\,)dxdy$$

becomes

$$K(x)\; f(g(x),h(y)\,) \overset{\mathcal{B}\mathcal{H}_\lambda}{\circ\!-\!\circ} \int_0^\infty \int_0^\infty e^{-uG(p)} \sqrt{vH(q)}\; J_\lambda(vH(q))\; K(G(p)\,)\, |G'(p)|\; |H'(q)|\; A(p)(A(p))^{-1} f(p,q)dpdq$$

Case 2 :

If $g(0) = \infty$, $h(0) = \infty$ and $g(\infty) = 0$, $h(\infty) = 0$ then $G'(p) \leqslant 0$, $H'(q) \leqslant 0$ so that if the substitutions $x = G(p)$, $y = H(q)$ are made, then

$$K(x)f(g(x),h(y)) \;\overset{\mathcal{L}\mathcal{H}_\lambda}{0\!-\!0}\; \int_0^\infty \int_0^\infty e^{-ux} \sqrt{vy}\, J_\lambda(vy) k(x) f(g(x),h(y))\, dxdy$$

becomes

$$K(x)f(g(x),h(y)) \;\overset{\mathcal{L}\mathcal{H}_\lambda}{0\!-\!0}\; \int_\infty^0 \int_\infty^0 e^{-uG(p)} \sqrt{vH(q)}\, J_\lambda(vH(q))\, K(G(p))\, G'(p)\, H'(q)\, A(p)\, (A(p))^{-1} f(p,q)\, dpdq$$

which is equivalent to

$$K(x)f(g(x),h(y)) \;\overset{\mathcal{L}\mathcal{H}_\lambda}{0\!-\!0}\; \int_0^\infty \int_0^\infty e^{-uG(p)} \sqrt{vH(q)}\, J_\lambda(vH(q))\, K(G(p))\, |G'(p)|\, |H'(q)|\, A(p)\, (A(p))^{-1} f(p,q)\, dpdq.$$

Thus in either case $\mathcal{L}\mathcal{H}_\lambda \{k(x)\, f(g(x),h(y))\}$ is absolutely convergent for $Re\,u > 0$ and $0 < v < \infty$.

Now the substitution from (iii) is used in either of these cases; the result is

$$K(x)\, f(g(x),h(y)) \;\overset{\mathcal{L}\mathcal{H}_\lambda}{0\!-\!0}\; \int_0^\infty \int_0^\infty e^{-pw} \sqrt{qz}\, J_\lambda(qz)\, \phi^*(u,v;w,z)\, dwdz \mid A(p) f(p,q)\, dpdq\,.$$

which is an absolutely convergent iterated integral. The order of integration can thus be changed so that

$$K(x)\ f(g(x),h(y))\ \overset{\mathcal{L}\mathcal{H}_\lambda}{0\!-\!0}\ \int_0^\infty \int_0^\infty \left[\int_0^\infty \int_0^\infty e^{-pw} \sqrt{qz}\ J_\lambda(qz)\ A(p) f(p,q)\, dp\, dq\right] {}^{*}\phi(u,v;w,z)\, dw\, dz.$$

Finally from (ii)

$$K(x)\ f(g(x),h(y))\ \overset{\mathcal{L}\mathcal{H}_\lambda}{0\!-\!0}\ \int_0^\infty \int_0^\infty F^{*}(w,z)\ \phi^{*}(u,v;w,z)\ dw\, dz\ .$$

<u>Theorem</u> 2.2.3

Let $(\mathcal{L}\mathcal{H}_\lambda f)(u,v) = F(u,v)$ $\operatorname{Re} u > 0$ and $0 < v < \infty$,

Then

$$\left[\mathcal{L}\mathcal{H}_\lambda\right]^{-1} \left[K(u)\ F\ (g(u),\ h(v))\right] = \int_0^\infty \int_0^\infty f(x,y)\ \Theta\ (w,z;\ x,y)\ dx\,dy\ ;$$

where k, g, h are analytic functions and

$$\mathcal{L}\mathcal{H}_\lambda \left[\Theta\ (w,z;\ x,y)\right] = k(u)\ e^{-g(u)x} \sqrt{h(v)y}\ J_\lambda\ (h(v)y\ )\ .$$

<u>Proof</u> :

Since $(\mathcal{L}\mathcal{H}_\lambda f)(u,v) = F(u,v)$

$$F(u,v) = \int_0^\infty \int_0^\infty e^{-ux} \sqrt{vy}\ J_\lambda\ (vy)\ f(x,y)\ dx\,dy\ .$$

$$\therefore\ F(g(u),\ h(v)) = \int_0^\infty \int_0^\infty e^{-g(w)x} \sqrt{h(v)y}\ \ J_\lambda(h(v)y) f(x,y)\,dx\,dy.$$

$$\therefore\ K(u)\ F(g(u),h(v)) = \int_0^\infty \int_0^\infty K(u) e^{-g(u)x} \sqrt{h(v)y}\ J_\lambda(h(v)y)\ f(x,y)\ dx\,dy\ .$$

$$= \int_0^\infty \int_0^\infty \left[\int_0^\infty \int_0^\infty e^{-ux} \sqrt{vy}\ J_\lambda\ (vy)\ \phi\ (w,z;x,y)\ dx\,dy\right] f(x,y)\,dx\,dy\ .$$

$$= \int_0^\infty \int_0^\infty \left[ \int_0^\infty \int_0^\infty f(x,y)\, \phi(w,z;x,y)\, dxdy \right] e^{-ux} \sqrt{vy}\, J_\lambda(vy)\, dxdy .$$

$$\therefore (\mathcal{L}\mathcal{H}_\lambda)^{-1} \left[ K(u)\, F((g(u), h(v))) \right] = \int_0^\infty \int_0^\infty f(x,y)\phi(w,z;x,y)\, dxdy .$$

2.2.3 <u>Example</u> :

Let $K(x) = x^{-1}$, $g(x) = x^{-1}$, $h(y) = \frac{v}{4y}$ and $g^{-1} = G(p) = p^{-1}$, $h^{-1} = H(q) = \frac{1}{4} v^2 q^{-1}$. Then

$| G'(p) | = p^{-2}$ , $| H'(q) | = \frac{1}{4} vq^{-2}$

and

$$\Phi(u,v;p,q) = e^{-uG(p)} \sqrt{vH(q)}\, J_\lambda(v(h(q))\, |G'(p)|\, |H'(q)|\, K(G(p)))$$

$$= \frac{1}{8} e^{-u/p} v^2 q^{-3/2} J_\lambda\left(\frac{1}{4} v^2 q^{-1}\right) . p^{-2} p^{-\nu+1}$$

$$= \mathcal{L}\mathcal{H}_\lambda \left[ \left(\frac{w}{u}\right)^{\nu/2} J(2\sqrt{uw}) \frac{1}{2} z^{1/2} J_{2\lambda}(vz^{1/2}) \right]$$

(4.14 (30) Vol 1 [3] 8.12 (10) Vol 2 [4])

$$\therefore \phi(u,v;w,z) = \frac{1}{2} \left(\frac{w}{u}\right)^{\nu/2} J_\lambda(2\sqrt{uw})\, z^{1/2} J_{2\lambda}(vz^{1/2})$$

Consider

$$K(x)\, f(g(x),h(y)) \overset{\mathcal{L}\mathcal{H}_\lambda}{\longleftrightarrow} \int_0^\infty \int_0^\infty e^{-ux} \sqrt{vy}\, J_\lambda)\, K(x) f(g(x),h(y))\, dxdy.$$

Thus

$$\mathcal{L}\mathcal{H}_\lambda\ k(x) f(g(x), h(y)) = \int_0^\infty \int_0^\infty e^{-ux} \sqrt{vy}\, J_\lambda(vy)\, x^{\nu-1} f\left(x^{-1}, \frac{v}{4y}\right) dxdy$$

Put $x = G(p)$ and $y = H(q)$

$$\mathcal{L}\mathcal{H}_\lambda \{k(x)\, f(\, g(x),\, h(y)\,)\} = \int_0^\infty \int_0^\infty e^{-uG(p)} \sqrt{vH(q)}\, J_\lambda\,(vH(q)\,)$$

$$K(G(p)\,)\,|G'(p)|\,|H'(q)|$$

$$f(p,q)\,dpdq\,.$$

$$= \int_0^\infty \int_0^\infty 1/8\;\; p^{-\nu-1}\; e^{-u/p}(v^2)\;\; q^{-3/2}\; J_\lambda(\,\tfrac{1}{4}\, v^2 q^{-1})f(p,q)dpdq\,.$$

By theorem 2.1,

$$\int_0^\infty \int_0^\infty 1/8\;\; p^{-\nu-1}\; e^{-u/p}\;(v^2)\; q^{-3/2}\; J_\lambda\;(\,\tfrac{1}{4}\, v^2\; q^{-1})\; f(p,q)dpdq$$

$$= \int_0^\infty \int_0^\infty 1/2\; F(w,z)\;(w/u)^{\nu/2}\; J_\lambda\;(2\sqrt{uw})\; z^{1/2}\; J_{2\lambda}(vz^{1/2})dwdz\,.$$

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