# CHAPTER - III

## SOME SUBSTITUTION THEOREMS FOR DISTRIBUTIONAL LAPLACE-HANKEL TRANSFORMATIONS

### 3.1 Introduction :

If $\phi(x,y)$ is suitably restricted function on $o < x < \infty$, $o < y < \infty$, then the Laplace-Hankel Transformation is defined by the integral

(3.1.1) $$\Phi(u,v) = (\mathcal{L}\mathcal{H}_\lambda \phi)(u,v) = \int_o^\infty \int_o^\infty e^{-ux} \sqrt{vy}\, J_\lambda(vy)\, \phi(x,y)\, dx\,dy$$

where $u = \sigma + i\omega$ is a restricted complex variable, $o < v < \infty$ and $J_\lambda(vy)$ is the Bessel function of first kind of order $\lambda$ with $\lambda$ real.

Let $a$ and $\lambda$ be any real numbers and $b$ any positive real number. For each triplet $\lambda$, $a$ and $b$, $B_{\lambda,a,b}$ is the space of all complex valued smooth functions $\phi(x,y)$ on $o < x < \infty$, $o < y < \infty$ such that

(3.1.2) $$\beta^{\lambda}_{a,b,k,k'}(\phi) = \sup_{\substack{o < x < \infty \\ o < y < \infty}} \left| e^{ax-by} y^{-\lambda-1/2} D_x^k S^{k'}_{\lambda,y} \phi(x,y) \right|$$

exists (i.e. finite) for all non-negative integers $k$ and $k'$.

$B_{\lambda,a,b}$ is a linear space under the pointwise addition of functions and their multiplication by complex numbers. Each $\beta^{\lambda}_{a,b,k,k'}$ is clearly a seminorm on $B_{\lambda,a,b}$. Since $\beta^{\lambda}_{a,b,0,0}$ is a norm the countable collection $\beta = \{\beta^{\lambda}_{a,b,k,k'}\}_{k,k'=0}^{\infty}$ of seminorms is a countable multinorm on $B_{\lambda,a,b}$. We assign to $B_{\lambda,a,b}$ the topology generated by the countable multinorm $\beta$, thereby making it a countably multinormed space. $B_{\lambda,a,b}$ is complete and therefore a Frechet space. $B_{\lambda,a,b}$ is a testing function space [2]. The dual space $B'_{\lambda,a,b}$ of $B_{\lambda,a,b}$ consists of all continuous linear functionals defined over $B_{\lambda,a,b}$. The dual space $B'_{\lambda,a,b}$ is a linear space to which we assign the weak topology generated by the multinorm $\{\xi_{\phi}\}_{\phi \in B_{\lambda,a,b}}$ where $\xi_{\phi}(f) = |\langle f,\phi\rangle|$, $\phi \in B_{\lambda,a,b}$.
Since $B_{\lambda,a,b}$ is a complete countably multinormed space, $B'_{\lambda,a,b}$ is also complete. The members of $B'_{\lambda,a,b}$ are the generalized functions.

Let $\lambda > -1/2$ and $\Omega$ be a subset of $C^2$ defined as follows

$$\Omega = \{(u,v) \in C^2 : \ \mathrm{Re}\,u > a,\ |\mathrm{Im}\,v| < b,\ v \neq 0 \text{ or a negative integer}\}$$

then for any $(u,v) \in \Omega$, $e^{-ux}\sqrt{vy}\,J_{\lambda}(vy)$ is a member of $B_{\lambda,a,b}$. The Laplace - Hankel transformation $\mathcal{LH}_{\lambda}$ is now defined on the dual space $B'_{\lambda,a,b}$ as follows :

Let $\lambda$ be restricted to $-1/2 \leq \lambda < \infty$. Then for $f \in B'_{\lambda,a,b}$

(3.1.3) $F(u,v) = [\mathscr{L}\mathscr{H}_\lambda(f)](u,v) = \langle f(x,y), \; e^{-ux}\sqrt{vy}\; J_\lambda(vy)\rangle$ ,

where $(u,v) \in \Omega$, [1] ,[2]

For each choice of a pair of real numbers $a$ and $\lambda$, let $LH_{a,\lambda}$ denote the set of all smooth functions $\phi(x,y)$ defined on $0<x<\infty$, $0<y<\infty$, and such that $\phi(0^+,y)$ on $0<y<\infty$ and for each pair triplet of non-negative integers m, k and k' ,

$$(3.1.4) \quad \gamma^{\lambda}_{a,m,k,k'}(\phi) = \sup_{\substack{0<x<\infty \\ 0<y<\infty}} \left| e^{ax} y^m D_x^k (y^{-1}D_y)^{k'} y^{-\lambda-1/2}\phi(x,y) \right|$$

assume finite values. It is easy to verify that $LH_{a,\lambda}$ is a linear space and each $\gamma^{\lambda}_{a,m,k,k'}$ is a seminorm. Also, since each $\gamma^{\lambda}_{a,m,0,0}$ is a norm the collection $S = \{\gamma^{\lambda}_{a,m,k,k'}\}_{k,k'=0}^{\infty}$ is a countable multinorm for $LH_{a,\lambda}$ . $LH_{a,\lambda}$ is complete. $LH_{a,\lambda}$ is a testing function space. Members of $LH_{a,\lambda}$ are of rapid descent [2] .

The space $LH_{a,\lambda}$ is a subspace of $B_{\lambda a,b}$ and topology of $LH_{a,\lambda}$ is stronger than the induced topology on it by $B_{\lambda a,b}$ [2].

In this chapter we shall extend the results obtained in Chapter II to distributional Laplace-Hankel transformations.

3.2 :

To extend the theorems obtained in Chapter II, we first establish the following lemma :

Lemma 3.2.1

Let $(\mathcal{L}\mathcal{H}_\lambda f) = F(u,v)$, $(u,v) \in \Omega_f$. Let $\sigma_f < a < \infty$ and $0 < b < \sigma_f$. Let $\phi(u,v;w,z) \in LH_{a,\lambda}$, then

$$(3.2.1) \quad \int_0^\infty \int_0^\infty \phi(u,v;w,z) \langle f(p,q),\, e^{-pw}\sqrt{qz}\, J_\lambda(qz) \rangle \, dwdz$$

$$= \langle f(p,q), \int_0^\infty \int_0^\infty e^{-pw}\sqrt{qz}\, J_\lambda(qz)\phi(u,v;w,z)\,dwdz \rangle$$

Proof :

If $\phi(u,v;w,z) = 0$ on $0 < w < \infty$, $0 < z < \infty$, (3.2.1) is satisfied. Hence assume $\phi(u,v;w,z) \neq 0$ on $0 < w < \infty$, $0 < z < \infty$

Let

$$(3.2.2) \quad V_R(u,v;p,q) = \int_0^R \int_0^R e^{-pw}\sqrt{qz}\, J_\lambda(qz)\phi(u,v;w,z)\,dwdz .$$

First we shall show that $V_R(u,v;p,q)$ $B_{\lambda,a,b}$. Consider,

$$(3.2.3) \quad e^{ap-bq}\, q^{-\lambda-1/2}\, D_p^k\, S_{\lambda,q}^{k'}\, V_R(u,v;p,q)$$

$$= e^{ap-bq}\, q^{-\lambda-1/2} D_p^k\, S_{\lambda,q}^{k'} \int_0^R \int_0^R e^{-pw}\sqrt{qz}\, J_\lambda(qz)\, \phi(u,v;w,z)\,dwdz .$$

By smoothness of the integrand, we may carry the operator $D_p^k\, S_{\lambda,q}^{k'}$ under the integral sign in the equation (3.2.3) to write

$$e^{ap-bq}\, q^{-\lambda-1/2}\, D_p^k\, S_{\lambda,q}^{k'}\, V_R\,(u,v;p,q)$$

$$= (-1)^{k+k'} \int_0^R \int_0^R \phi\,(u,v;w,z)\, w^k\, e^{-(w-a)p}\, z^{2k'+\lambda+1/2}$$

$$e^{-bq}\,(qz)^{-\lambda}\, J_\lambda\,(qz)\,dwdz\ .$$

Hence

$$\left|\ e^{ap-bq}\, q^{-\lambda-1/2} D_p^k\, S_{\lambda,q}^{k'}\ \ V_R\,(u,v;p,q)\ \right|$$

$$\leq C_\lambda \int_0^R\ z^{2k'+\lambda+1/2}dz \int_0^R \left| w^k\phi(u,v;w,z)\right|\ dwdz < \infty$$

where $C_\lambda$ is the bound of $e^{-(\sigma-a)b}\, e^{-bq}\,(qz)^{-\lambda}J_\lambda\,(qz)$ for $o<p<\infty$, $o<q<\infty$, $\sigma-iR < \sigma + iw < \sigma + iR$, $o<w<R$. This shows that $V_R\,(u,v;\,p,q) \in B_{\lambda,a,b}$ .

Now we shall show that

(3.2.4) $$\int_0^R \int_0^R \phi\,(u,v;w,z)\langle f(p,q),\ e^{-pw}\,\sqrt{qz}\ J_\lambda\,(qz)\rangle\,dwdz$$

$$= \langle f(p,q),\ \ V_R\,(u,v;\,p,q)\rangle$$

Since $F(u,v)$ is analytic on $\Omega_f$ and $\phi\,(u,v;w,z)$ is entire, the integral of the left-hand side of the equation (3.2.4) exists, and since $V_R\,(u,v;p,q)\in B_{\lambda,a,b}$, the right hand side of equation (3.2.4) has sense.

Now, consider the Riemann sum

(3.2.5) $$\textcircled{R}_{M,N}(u,v;p,q) = \sum_{i=1}^{M} \sum_{j=1}^{N} e^{-w_i p}\,\sqrt{qz_j}\ \ J_\lambda\,(qz_j)$$

$$\phi\,(u,v;w_i,z_j)\ \frac{2RR}{MN}\ .$$

By applying $f(p,q)$ to (3.2.5) term by term, we get

(3.2.6) $\langle f(p,q), \text{Ⓗ}_{M,N}(u,v;p,q)$

$$= \sum_{i=1}^{M} \sum_{j=1}^{N} \langle f(p,q), e^{w_i p} \sqrt{qz_j}\, J_\lambda(qz_j) \rangle$$

$$\phi(u,v;w_i,z_j) \frac{2RR}{MN} .$$

Since $\langle f(p,q), e^{-w_i p} \sqrt{qz_j}\ J_\lambda(qz_j) \rangle \phi(u,v;w_i,z_j)$ is a continuous function on $o < w < R$, $o < z < R$, the sum in the right hand side of the equation (3.2.6) tends to

$$\int_o^R \int_o^R \langle f(p,q), e^{-pw} \sqrt{qz}\, J_\lambda(qz) \rangle \phi(u,v;w,z)\, dwdz \quad \text{as}$$

$M \longrightarrow \infty$, $N \longrightarrow \infty$.

Since $f \in B'_{\lambda,a,b}$ our proof will be complete if we show that $\text{Ⓗ}_{M,N}(u,v;p,q)$ converges in $B_{\lambda,a,b}$ to

$$\int_o^R \int_o^R e^{-pw} \sqrt{qz}\ J_\lambda(qz)\ \phi(u,v;w,z)\, dwdz$$

as $M \longrightarrow \infty$, $N \longrightarrow \infty$. For this purpose we shall show that

$$\left[ \text{Ⓗ}_{M,N}(u,v;p,q) - V_R(u,v;p,q) \right] \longrightarrow 0$$

uniformly on $o < p < \infty$, $o < q < \infty$.

Consider.

(3.2.7) $A_{M,N}(u,v;p,q) = e^{ap-bq}\, q^{-\lambda-1/2}\, D_p^k\, S_{\lambda,q}^{k'}$

$$\left| \text{Ⓗ}_{M,N}(u,v;p,q) - V_R(u,v;p,q) \right| .$$

$$= e^{ap-bq} \sum_{i=1}^{M} \sum_{j=1}^{N} (-1)^{k+k'} w_i^k z_j^{2k+\lambda+1/2} e^{-pw_i}$$

$$(qz_j) J_\lambda(qz_j)\ \phi(u,v;w,z).\ \frac{RR}{MN}$$

$$- (-1)^{k+k'} \int_0^R \int_0^R \phi(u,v;w,z)\, w^k z^{2k'+\lambda+1/2} e^{-(w-a)p-bq} (qz)^{-\lambda} J_\lambda(qz)\, dwdz$$

Now $| e^{-(w-a)p-bq} (qz)^{-\lambda} J_\lambda(qz) | \longrightarrow 0$ as $p \longrightarrow \infty$, $q \longrightarrow \infty$

For $w > a$, $b > 0$ and $(qz)^{-1} J_\lambda(qz)$ is bounded on $0\ q\ \infty$. So given $\epsilon > 0$, there exists $p > p_1$, $q > q_1$, and $0 < w \leq R$, $0 < z \leq R$.

$$\left| e^{-(w-a)p-bq} (qz)^{-\lambda} J_\lambda(qz) \right| < \epsilon/3 \left[ \int_0^R \int_0^R \left| w^k z^{2k'+\lambda+1/2} \phi(u,v;w,z) \right| dwdz \right]^{-1}$$

Since $\phi(u,v;w,z) \neq 0$ on $0 < w < \infty$, $0 < z < \infty$, the right hand side is finite. Hence

$$\sup_{\substack{p_1 < p < \infty \\ q_1 < q < \infty}} \left| e^{ap-bq} q^{-\lambda-1/2} D_p^k S^{k'}_{\lambda,q} V_R(u,v;p,q) \right| < \epsilon/3$$

Also, for $p > p_1$, $q > q_1$ and for all $M$ and $N$

$$\sup_{\substack{p_1 < p < \infty \\ q_1 < q < \infty}} \left| e^{ap-bq} q^{-\lambda-1/2} D_p^k S^{k'}_{\lambda,q} M_{M,N}(u,v;p,q) \right|$$

$$< \epsilon/3 \left[ \int_0^R \int_0^R \left| w^k z^{2k'+\lambda+1/2} \phi(u,v;w,z) \right| dwdz \right]^{-1} \times \sum_{i=1}^{M} \sum_{j=1}^{N} \left| w_i^k z_j^{2k'+\lambda+1/2} \phi(u,v;w_i,z_j) \right| \frac{2RR}{MN}$$

We can choose $M_0$, $N_0$ so large that for all $M > M_0$, $N > N_0$, the last expression is less than $2\epsilon/3$. Therefore for $p > p_1$, $q > q_1$

$\left| A_{M,N} (u,v;p,q) \right| < \epsilon$ .

Next on the domain $o \leq p \leq p_1$ , $o < q < q_1$ , $o < w \leq R$ , $o < z \leq R$ ,

$$w^{k} z^{2k'+\lambda+1/2} e^{-(w-a)p} e^{-bq} (qz)^{-\lambda} J_\lambda(qz) \phi (u,v;w,z)$$

is uniformly continuous function of $(p,q;w,z)$. Therefore, in view of (3.2.7), there exists $M_1$ and $N_1$ such that for all $M > M_1$ , $N > N_1$ ,

$$\left| A_{M,N} (u,v;p,q) \right| < \epsilon \quad \text{on} \quad o < p < \infty, \quad o < q < \infty .$$

Since $\epsilon$ is arbitrary we conclude that $A_{M,N} (u,v;p,q)$ converges uniformly to zero on $o < p < \infty$, $o < q < \infty$. Thus we have proved equation (3.2.4).

Since $\phi (u,v;w,z)$ as a function of $(w,z)$ is of rapid descent and

$$\left| \langle f(p,q), \ e^{-pw} \sqrt{qz} \, J_\lambda(qz) \rangle \right| < |z|^{\lambda+1/2} P ( |wz| )$$

where $P ( |wz| )$ is a polynomial [ ], the integral of the left-hand side of (3.2.4) converges to the left-hand side of (3.2.1) as $R \longrightarrow \infty$ .

Finally we shall show that $V_R (u,v;p,q)$ converges to

$$\bar{\Phi} (u,v;p,q) = \int_0^\infty \int_0^\infty e^{-pw} \sqrt{qz} \ J_\lambda (qz) \ \phi (u,v;w,z) dwdz$$

in $B_{\lambda,a,b}$ as $R \longrightarrow \infty$ .

Consider,

$$\Phi (u,v;p,q) = V_R (u,v;p,q) + V_{1R} (u,v;p,q)$$

$$+ V_{2R}(u,v;p,q) + V_{3R} (u,v;p,q)$$

where,

$$V_{1R}(u,v;p,q) = \int_R^\infty \int_R^\infty e^{-pw} \sqrt{qz}\, J_\lambda(qz)\, \phi (u,v;w,z) dwdz ,$$

$$V_{2R}(u,v;p,q) = \int_0^R \int_R^\infty e^{-pw} \sqrt{qz}\, J_\lambda (qz)\, \phi (u,v;w,z) dwdz ,$$

and

$$V_{3R}(u,v;p,q) = \int_R^\infty \int_0^R e^{-pw} \sqrt{qz}\, J_\lambda (qz)\, \phi (u,v;w,z) dwdz .$$

Clearly $V_{iR} (u,v;p,q)$ ( $i = 1,2,3$) are members of $B_{\lambda,a,b}$ . we shall show that $V_{iR} (u,v;p,q)$ converges to zero in $B_{\lambda,a,b}$ as $R \longrightarrow \infty$. ( $i = 1,2,3$).

Consider,

$$\left| e^{ap-bq} q^{-\lambda-1/2} D_p^k S_{\lambda,q}^{k'} \quad V_{1R} (u,v;p,q) \right|$$

$$= \Big| \int_R^\infty \int_R^\infty \phi (u,v;wz)\, w^k e^{-(w-a)p} z^{2k'+\lambda+1/2}$$

$$e^{-bq} J_\lambda(qz)\, (qz)^{-1}\, dwdz \Big|$$

$$< B_{k,k'} \int_R^\infty \int_R^\infty \left| \phi (u,v; w,z)\, w^k z^{2k'+\lambda+1/2} \right| dwdz$$

where $B_{k,k'}$ are constants, and since as a function of $w,z$ $\phi (u,v;w,z)$ is of rapid descent, the last integral which is independent of $u,v$ vanishes as $R \to \infty$. Then $V_R(u,v;p,q)$ converges to zero in $B_{\lambda,a,b}$ as $R \to \infty$. Similarly we can prove that ~~$V_{1R}(u,v;p,q)$~~, $V_{2R}(u,v;p,q)$ and $V_{3R}(u,v;p,q)$ converges to zero in $B_{\lambda,a,b}$ as $R \to \infty$.

Therefore $V_R(u,v;p,q)$ converges to $\Phi(u,v;p,q)$ in $B_{\lambda,a,b}$ ~~to zero~~ as $R \to \infty$. This completes the proof of the lemma.

**Theorem : 3.2.1**

If (i) $k$, $g$, $h$ and $g^{-1} = G$, $h^{-1} = H$ are single valued analytic functions, real on $(o, \infty)$ and such that $g(o) = 0, h(o)=0$ and $g(\infty) = \infty$, $h(\infty) = \infty$, (or $g(o) = \infty$, $h(o) = \infty$ and $g(\infty) = 0$, $h(\infty) = 0$ ) ;

(ii) $(\mathcal{L}\mathcal{H}_\lambda f)(u,v) = F(u,v)$, $(u,v) \in \Omega_f$

(iii) there exists a function $\phi(u,v;w,z)$ in $LH_{a,\lambda}$ such that

$(\mathcal{L}\mathcal{H}_\lambda \phi)(u,v;p,q) = \Phi(u,v;p,q) \qquad (u,v) \in \Omega_f$

where $\sigma_f < a < \infty$ and

$$\Phi(u,v;p,q) = e^{-uG(p)} \sqrt{vH(q)}\; J_\lambda(vH(q))\; K(G(p))\; |G'(p)|\; |H'(q)|$$

then

$$\langle k(x)\, f(g(x), h(y)),\; e^{-ux}\sqrt{vy}\, J_\lambda(vy) \rangle = \int_o^\infty \int_o^\infty F(w,z)\, \phi(u,v;w,z)\, dw\, dz .$$



**Proof :**

Choose $b$ and $c$ such that $o < b < \sigma_f$ and $a < c < \infty$. Let $\psi(x,y)$ be arbitrary member of $B_{\lambda,a,b}$. Let for each non-negative integer k, $(1+x^2)^{-Nk} D^k K(x)$ is bounded on $o < x < \infty$.

Then $\phi \to K\phi$ is a continuous linear mapping of $B_{\lambda,a,b}$ into $B_{\lambda,c,b}$. To show this

$$e^{ax-by} y^{-\lambda-1/2} D_x^{k'} S_{\lambda,y}^{k'} K(x)\psi(x,y) = e^{ax-by} y^{-\lambda-1/2} D_x^{k} K(x) S_{\lambda,y}^{k'} \phi(x,y) .$$

$$= \sum_{\nu=o}^{k} \binom{k}{\nu} \frac{e^{ax}}{e^{cx}} D_x^{k-\nu} K(x)\, e^{cx-by-\lambda-1/2} D_x^{\nu} S_{\lambda,y}^{k'} \phi(x,y)$$

$$\beta^{\lambda}_{a,b,k,k'}(k\phi) = B_k \sum_{\nu=o}^{k} \binom{k}{\nu} B^{\lambda}_{c,b,\ ,k'}(\phi) < \infty$$

where $B_k$ are constants.

$\therefore$ $K\phi$ is member of $B_{\lambda,c,b}$ whenever $\phi$ is a member of $B_{\lambda,a,b}$.

Let $T_1 : \phi \to K\phi$ of $B_{\lambda,a,b}$ into $B_{\lambda,c,b}$ and $T_2 : \phi \to k^{-1}\phi$ of $B_{\lambda,c,b}$ into $B_{\lambda,a,b}$. Then

$$T_1(\phi) = K\phi \quad \text{and} \quad T_2(\phi) = K^{-1}\phi$$

$$\therefore (T_1 \circ T_2)(\phi) = T_1 \circ [T_2(\phi)] = T_1(K^{-1}\phi).$$
$$= K(K^{-1}\phi)$$
$$= \phi$$

Similarly,

$$(T_2 \circ T_1)(\phi) = T_2 \circ \{T_1(\phi)\} = T_2(K\phi)$$
$$= K^{-1}(K\phi)$$
$$= \phi .$$

$\therefore$ $T_1 o T_2 = T_2 o T_1 =$ identity.

This implies that $T_1 = T_2^{-1}$ and $T_2 = T_1^{-1}$ .

Thus if $\phi \rightarrow K \phi$ is a continuous linear mapping of $B_{\lambda,a,b}$ into $B_{\lambda,c,b}$, then its inverse mapping is $\phi \rightarrow K^{-1}\phi$ of $B_{\lambda,c,b}$ into $B_{\lambda,a,b}$ .

Thus $\phi \rightarrow K \phi$ is an isomorphism of $B_{\lambda,a,b}$ onto $B_{\lambda,c,b}$, then the mapping $f \rightarrow K f$ of $B'_{\lambda,c,b}$ onto $B'_{\lambda,a,b}$ is and isomorphism; and we write,

$$\langle K(x)\, f(x,y),\ \phi\,(x,y)\rangle = \langle f(x,y),\ K(x)\,\phi\,(x,y)\rangle$$

Therefore if

$$(\mathcal{L}\mathcal{H}_\lambda\, f)\,(u,v) = F(u,v) \quad \mathrm{Re}\,u > e, \quad o < v < \infty ,$$

the equation

$$\langle K(x)\ f(x,y),\ e^{-ux}\sqrt{vy}\ J_\lambda(vy)\rangle = \langle f(x,y), K(x) e^{-ux}\sqrt{vy}\ J_\lambda(vy)\rangle$$

has sense. Indeed we have $f(x,y) \in B'_{\lambda,c,b}$ , $k(x)\ f(x,y) \in B'_{\lambda,a,b}$, $e^{-ux}\sqrt{vy}\ J_\lambda\ (vy) \in B_{\lambda,a,b}$ and $K(x)\ e^{-ux}\sqrt{vy}\ J_\lambda\ (vy) \in B_{\lambda,c,b}$ .

If $\mathcal{X}\,(x,y) = f[\,g(x),\ h(y)\,] \in B'_{\lambda,c,b}$

then $\mathcal{X}(x,y) \rightarrow K(x)\mathcal{X}\,(x,y)$ is an isomorphism from $B'_{\lambda,c,b}$ onto $B'_{\lambda,a,b}$ and we can write

$$\langle K(x)\ f\,(\,g(x),\ h(y)\,),\ \ e^{-ux}\ \sqrt{vy}\ \ J_\lambda\ (vy)\rangle$$

$$= \langle f\,(\,g(x),\ h(y)\,),\ K(x)\ e^{-ux}\sqrt{vy}\ J_\lambda\ (vy)\rangle \ .$$

Here $f\,[g(x), h(y)] \in B'_{\lambda,c,b}$ $K(x)\, f[g(x),h(y)] \in B'_{\lambda,a,b}$,
$e^{-ux}\sqrt{vy}\; J_\lambda(vy) \in B_{\lambda,a,b}$ and $K(x)\; e^{-ux}\sqrt{vy}\; J_\lambda(vy) \in B_{\lambda,c,b}$ .

Let $K(x)\,\psi(x,y) = \eta(x,y)$ be an arbitrary number of $B_{\lambda,c,b}$; choose a real number $d < c$ such that
$\eta\,(G(p), H(q))\; |G'(p)|\; |H'(q)| \in B_{\lambda,d,b}$ .
Then the mapping

$$\eta\,(x,y) \longrightarrow \eta\,(G(p), H(q))\; |G'(p)|\; |H'(q)|$$ is a

continuous linear mapping of $B_{\lambda,c,b}$ into $B_{\lambda,d,b}$ . The unique inverse mapping is

$$\eta\,(p,q) \longrightarrow \eta\,(g(x), h(y))$$

and it mapps all of $B_{\lambda,d,b}$ onto $B_{\lambda,c,b}$ . Hence

$$\eta\,(x,y) \longrightarrow \eta\,(G(p), H(q))\; |G'(p)|\; |H'(q)|$$

is an isomorphism from $B_{\lambda,c,b}$ into $B_{\lambda,d,b}$. We denote the adjoint of the mapping

$$\eta\,(x,y) \longrightarrow \eta\,(G(p), H(q))\; |G'(p)|\; |H'(q)|$$ by

$f(p,q) \longrightarrow f\,(g(x), h(y))$ . Since this is what we would have if $f$ were a conventional function, and we write

$$\langle f(g(x), h(y)), \eta(x,y)\rangle = \langle f(p,q), \eta(G(p), H(q))\; |G'(p)|\; |H'(q)|\rangle .$$

Thus $f(p,q) \longrightarrow f(g(x), h(y))$ is an isomorphism from $B'_{\lambda,d,b}$ to $B'_{\lambda,c,b}$ .

Therefore, if $(\mathcal{L}\mathcal{H}_\lambda\, f)\,(u,v) = F\,(u,v)$

$\mathrm{Re}\,u > 0$ and $0 < v < \infty$, the equation,

$$\langle f(\,g(x),\ h(y)\,),\ K(x)\, e^{-ux}\sqrt{vy}\ J_\lambda\,(vy)\rangle$$

$$= \langle f(p,q),\ K(G(p)\,)e^{-uG(p)}\sqrt{vH(q)}\ J_\lambda(vH(q)|G'(p)||H'(q)|\rangle$$

has sense. Indeed we have $f(p,q) \in B'_{\lambda,d,b}$, $f(g(x),h(y)\,) \in B'_{\lambda,c,b}$, $K(x)\ e^{-ux}\sqrt{vy}\ J_\lambda\ (vy) \in B_{\lambda,c,b}$ and $K\,(\,G(p)\,)\ e^{-uG(p)}\sqrt{vH(q)}\ J_\lambda\ (vH(q))|G'(p)|\ |H'(q)| \in B_{\lambda,d,b}$.

Thus we conclude that

$f(p,q) \longrightarrow K(x)\ f(\,g(x),\ h(y)\,)$ is an isomorphism from $B'_{\lambda,d,b}$ onto $B'_{\lambda,c,b}$, where $d < c$, and we write

$$\langle K(x)\ f(g(x),\ h(y)\,),\ e^{-ux}\sqrt{vy}\,J_\lambda\ (vy)\rangle$$

$$= \langle f(p,q),\ K(G\,(p)\,)\ e^{-uG(p)}\sqrt{vH(q)}\ J_\lambda(vH(q)|G'(p)||H'(q)|\rangle$$

where $f(p,q) \in B'_{\lambda,d,b}$, $K(x)f(\,g(x),\ h(y)\,) \in B'_{\lambda,c,b}$, $e^{-ux}\sqrt{vy}\,J_\lambda\ (vy) \in B_{\lambda,c,b}$ and $K\ (G(p)\,)\ e^{-uG(p)}\sqrt{vH(q)}\ J_\lambda\ (vH(q)\,)\ |G'(p)|\ |H'(q)| \in B_{\lambda,d,b}$.

The above equation further can be written as

$$(\,\mathcal{L}\mathcal{H}_\lambda)\ [\,K(x)\ f(\,g(x),h(y)\,)\,] = \langle f(p,q),\ \Phi\,(u,v;\ p,q)\rangle$$

$$= \langle f(p,q),\ \int_0^\infty\int_0^\infty \phi(u,v;w,z)e^{-pw}\sqrt{qz}\ J_\lambda(qz)dwdz\rangle\ .$$

$$= \int_0^\infty\int_0^\infty \langle f(p,q),\ e^{-pw}\sqrt{qz}\ J_\lambda(qz)\rangle\ \phi(u,v;w,z)\ dwdz\ .$$

( see lemma 3.2.1 )

$$= \int_0^\infty \int_0^\infty F(w,z)\, \phi\, (u,v;w,z)\, dwdz \ .$$

This completes the proof.

__Theorem__ : 3.2.2

If (i) A, k, g, h and $g^{-1} = G$, $h^{-1} = H$ are single valued analytic functions real on $(o, \infty)$ and such that $g(o) = 0$, $h(o) = 0$ and $g(\infty) = \infty$, $h(\infty) = \infty$, (or $g(o) = \infty$, $h(o) = \infty$ and $g(\infty) = 0$, $h(\infty) = 0$) ;

(ii) Let

$$(\mathcal{B}\mathcal{H}_\lambda Af)(u,v) = F^*(u,v), \quad (u,v) \in \Omega_f$$

(iii) there exists a function $\phi^*(u,v;w,z)$ in $LH_{a,\lambda}$ such that $(\mathcal{B}\mathcal{H}_\lambda \phi^*)(u,v;p,q) = \Phi^*(u,v;p,q)$ $(u,v) \in \Omega_f$ , where $\sigma_f < a < \infty$ and ;

$$\Phi^*(u,v;p,q) = K\big(G(p)\big)\, e^{-uG(p)} \sqrt{vH(q)}\, (A(p))^{-1}\, |G'(p)|\, |H'(q)|$$

then

$$K(x)\, f(g(x), h(y)),\ e^{-ux}\sqrt{vy}\, J_\lambda(vy)$$

$$= \int_0^\infty \int_0^\infty F^*(w,z)\phi^*(u,v;w,z)\, dwdz \ .$$

__Proof__ :

Choose b and c such that $o < b < \sigma_f$ and $a < c < \infty$. Let $\psi(x,y)$ be any arbitrary member of $B_{\lambda,a,b}$ . Let for each non-negative integer k, $(1+x^2)^{-Nk} D^k K(x)$ is bounded on $o < x < \infty$

Then $\psi \rightarrow K\psi$ is a continuous linear mapping of $B_{\lambda,a,b}$ into $B_{\lambda,c,b}$ . To show this

$$e^{ax-by}\, y^{-\lambda-1/2}\, D_x^k S_{\lambda,y}^{k'}\, K(x)\psi(x,y) = e^{ax-by} y^{-\lambda-1/2}\, D_x^k\, K(x)\, S_{\lambda,y}^{k'}\, \psi(x,y)$$

$$= \sum_{\nu=0}^{\infty} \binom{k}{\nu} \frac{e^{ax}}{e^{cx}}\, D_x^{k-\nu} K(x)\; e^{cx-by-\lambda-1/2}\, D_x^{\nu}\, S_{\lambda,y}^{k'}\, \psi(x,y).$$

$\therefore\ \gamma^{\lambda}_{a,b,k,k'}(K\psi) = B_k \sum_{\nu=0}^{\infty} \binom{k}{\nu}\, B^{\lambda}_{a,b,\ ,k'}(\psi) < \infty$ ,

where $B_k$ are constants.

$\therefore$ $K\psi$ is a member $B_{\lambda,c,b}$ whenever $\psi$ is a member $B_{\lambda,a,b}$.

Let $T_1 : \psi \rightarrow K\psi$ of $B_{\lambda,a,b}$ into $B_{\lambda,c,b}$ and

$T_2 : \psi \rightarrow K^{-1}\psi$ of $B_{\lambda,c,b}$ into $B_{\lambda,a,b}$ .

Then, $T_1(\psi) = K\psi$ and $T_2(\psi) = K^{-1}\psi$ .

$$\therefore\ (T_1 o T_2)(\psi) = T_1 o \left( T_2(4) \right) = T_1 \left( K^{-1}\psi \right) = K\,(k^{-1}\psi) = \psi .$$

Similarly, $(T_2 o T_1)\,\psi = T_2 o \left( T_1(\psi) \right) = T_2\,(k\psi) = K^{-1}\,(K\psi) = \psi .$

$\therefore\ T_1 o T_2 \equiv T_2 o T_1$ = identity. This implies that $T_1 = T_2^{-1}$ and $T_2 = T_1^{-1}$ . Thus if $\psi \rightarrow K\psi$ is a continuous linear mapping of $B_{\lambda,a,b}$ into $B_{\lambda,c,b}$, then its inverse mapping is $\psi \rightarrow K^{-1}\psi$ of $B_{\lambda,c,b}$ into $B_{\lambda,a,b}$.

Thus $\psi \rightarrow K\psi$ is an isomorphism of $B_{\lambda,a,b}$ onto $B_{\lambda,c,b}$, then, the mapping $f \longrightarrow Kf$ of $B'_{\lambda,c,b}$ into $B'_{\lambda,a,b}$ is an isomorphism; and we write,

$$\langle K(x)\ f(x,y),\ \phi\ (x,y)\rangle = \langle f(x,y),\ k(x)\ \phi\ (x,y)\rangle$$

Therefore if

$$(\ \mathscr{H}_\lambda)\ (Af)\ (u,v)\ =\ F^{*}\ (u,v)$$

$Re u > 0$ , $0 < v < \infty$, the equation

$$\langle K(x)\ f\ (x,y),\ \ e^{-ux}\sqrt{vy}\ J_\lambda\ (vy)\rangle$$

$$= \langle f(x,y),\ K(x)\ e^{-ux}\sqrt{vy}\ \ J_\lambda\ (vy)\rangle$$

has sense. Indeed $K(x)\ f(x,y) \in B'_{\lambda,a,b}$, $f(x,y) \in B'_{\lambda,c,b}$, $e^{-ux}\sqrt{vy}\ \ J_\lambda\ (vy) \in B_{\lambda,a,b}$ and $K(x)\ e^{-ux}\sqrt{vy}\ J_\lambda\ (vy) \in B_{\lambda,c,b}$.

If $\mathfrak{X}(x,y)\ =\ f\ (\ g(x),\ h(y)\ ) \in B'_{\lambda,c,b}$,
then

$\mathfrak{X}(x,y) \longrightarrow K(x)\ \mathfrak{X}(x,y)$ is an isomorphism from $B'_{\lambda,c,b}$ onto $B'_{\lambda,a,b}$ and we can write

$$\langle K(x)\ f(g(x),\ h(y)\ ),\ \ e^{-ux}\sqrt{vy}\ \ J_\lambda\ (vy)\rangle$$

$$= \langle f(\ g(x),\ h(y)\ ),\ \ K(x)\ e^{-ux}\sqrt{vy}\ J_\lambda\ (vy)\rangle$$

Here $K(x)\ f(\ g(x),\ h(y)\ ) \in B'_{\lambda,a,b}$, $f(\ g(x),\ h(y)\ ) \in B'_{\lambda,c,b}$ $e^{-ux}\sqrt{vy}\ J_\lambda(vy) \in B_{\lambda,a,b}$ and $K(x)\ e^{-ux}\sqrt{vy}\ J_\lambda(vy) \in B_{\lambda,c,b}$

Let $K(x)\ \phi(x,y) = \psi\ (x,y)$ be arbitrary member of $B_{\lambda,c,b}$, choose a real number $d$ such that $d < c$ such that $\psi\ |\ G(p),\ H(q)\ |\ |G'(p)\ |\ |H'(q) \in B_{\lambda,d,b}$. Then the mapping $\psi\ (x,y) \longrightarrow \psi\ (G\ (p),\ H(q)\ )\ |\ G'(p)|\ |H'\ (q)\ |$ is a continuous linear mapping of $B_{\lambda,c,b}$ into $B_{\lambda,d,b}$. The unique inverse mapping is

$$\varphi(p,q) \longrightarrow \varphi(\, g(x),\, h(y)\, )$$

and its mapps all of $B_{\lambda,d,b}$ onto $B_{\lambda,c,b}$. Hence

$$\varphi(x,y) \longrightarrow \varphi(\, G(p),\, H(q)\, ) \;|\, G'(p) \,|\, |H'(q)\,|$$

is an isomorphism from $B_{\lambda,c,b}$ into $B_{\lambda,d,b}$. We denote the adjoint of the mapping

$$\varphi(x,y) \longrightarrow \varphi(\, G(p),\, H(q)\, ) \;|\, G'(p) \,|\; |\, H'(q) \,|$$

by $F(p,q) \longrightarrow f(g(x),\, h(y)\, )$. Since this is what we would have if $f$ were a conventional function, and we write

$$\langle f(g(x),h(y)),\ \varphi(x,y)\rangle = \langle f(p,q);\ \varphi(G(p),\, H(q)\,|\; |\, g'(p)|\ |H'(q)|\,\rangle \, .$$

Thus $f(p,q) \longrightarrow f(\, g(x),\, h(y)\, )$ is an isomorphism from $B'_{\lambda,d,b}$ to $B'_{\lambda,c,b}$.

Therefore if $(\, \mathcal{B}\mathcal{H}_\lambda)\,(Af) = F^*(u,v)$ $\operatorname{Re} u > 0$ and $0 < v < \infty$, the equation,

$$\langle f\,\big(\, g(x),\, h(y)\,\big),\ K(x)\, e^{-ux}\sqrt{vy}\, J_\lambda(vy)\, A(g(x))\, \big(A(g(x))\big)^{-1}\rangle$$

$$= \langle f(p,q),\ K(G(p)\, e^{-uG(p)}\sqrt{vH(q)}\, J_\lambda(vH(q))\, (A(p)\, (A(p)\,)^{-1}\ |\, G'(p)\,|\;|\, H'(q)\,|\, \rangle$$

has sense indeed, $f(p,q) \in B'_{\lambda,d,b}$.

$K\,\big(\, G(p)\,\big)\, e^{-uG(p)}\sqrt{vH(q)}\, J_\lambda\,(vH(q))\, A(p)(A(p))^{-1}|G'(p)||H'(q)|$ $\in B_{\lambda,d,b}$, $f\big(\, g(x),h(y)\,\big) \in B'_{\lambda,c,b}$ and $K(x)\, e^{-ux}\sqrt{vy}\, J_\lambda\,(vy)$ $A\,(\, g(x)\,)\,\big(\, A(g(x)\,)\,\big)^{-1} \in B_{\lambda,c,b}$.

Thus we conclude that $f(p,q) \rightarrow K(x)\, f\,(\, g(x),\, h(y)\,)$ is an isomorphism from $B'_{\lambda,d,b}$ onto $B'_{\lambda,c,b}$, where $d < c$, and we write

$$\langle K(x)\, f\,(g(x),\, h(y)\,),\, e^{-ux}\sqrt{vy}\, J_\lambda(vy)\, A(g(x)\,)\,(\, A(\, g(x)\,)\,)^{-1}\rangle$$

$$= \langle f(p,q),\, K\, G(p)\, e^{-uG(p)}\sqrt{vH(q)}\, J_\lambda(vH(q))\, |\, G'(p)|\, |\, H'(q)\, |\, A(p)\, |\, A(p)\, |^{-1}\rangle,$$

where $f(p,q) \in B'_{\lambda,d,b}$, $k(x)\, f\,(\, g(x),h(y)\,) \in B'_{\lambda,c,b}$,

$e^{-ux}\,\sqrt{vy}\, J_\lambda\,(vy)\, A\,(\, g(x)\,)\,(\, A\,(\, g(x)\,)^{-1} \in B_{\lambda,c,b}$ and

$K\,(G(p)\,)\, e^{-uG(p)}\sqrt{vH(q)}\, J_\lambda\,(vH(q))\, |\, G'(p)\, |\, |\, H'(q)\, |\, A(p)\, |\, A(p)\, |^{-1} \in B_{\lambda,d,b}$ .

The above equation further can be written as

$$(\, \mathcal{L}\mathcal{H}_\lambda)\,[\,K(x)f(g(x),h(y)\,] = \langle f(p,q),\, \Phi^*(u,v;p,q)\, A(p)$$

$$= \langle A(p)\, f(p,q),\, \phi^*(u,v;p,q)\rangle$$

$$= \langle A(p)f(p,q),\, \int_0^\infty \int_0^\infty e^{-pw}\sqrt{qz}\, J_\lambda(qz)\phi^*(u,v;w,z)\, dwdz$$

$$= \int_0^\infty \int_0^\infty \langle A(p)f(p,q),\, e^{-pw}\sqrt{qz}\, J_\lambda(qz)\rangle\, \phi^*(u,v;w,z)\, dwdz$$

( See lemma 3.2.1 )

$$= \int_0^\infty \int_0^\infty F^*(w,z)\, \phi^*\,(u,v;w,z)\, dwdz\ .$$

This completes the proof .

**Theorem : 3.3.3**

Let $(\mathcal{L}\mathcal{H}_\lambda) f = F(u,v)$ $\operatorname{Re} u > 0$, $0 < v < \infty$,

then

$$(\mathcal{L}\mathcal{H}_\lambda)^{-1}\left[K(u)\, f!\; g(u), h(v)\right] = \int_0^\infty \int_0^\infty f(x,y)\, \Theta(w,z;x,y)\,dxdy \; ,$$

where k, g, h are analytic functions on $(0, \infty)$ and

$$(\mathcal{L}\mathcal{H}_\lambda)\left[\Theta(w,z;x,y)\right] = K(u)\; e^{-g(u)x}\sqrt{h(v)}y\; J_\lambda(h(v)y) \; .$$

**Proof :**

Now

$$F(u,v) = \left\langle f(x,y),\; e^{-ux}\sqrt{vy}\; J_\lambda(vy)\right\rangle$$

$$F(g(u),h(v)) = \left\langle f(x,y),\; e^{-g(u)x}\sqrt{h(v)}y\; J_\lambda(h(v)\,y)\right\rangle$$

$$\therefore K(u)\, F(g(u),h(v)) = \left\langle f(x,y),\; K(u)e^{-g(u)x}\sqrt{h(v)y}\; J_\lambda(h(v)y)\right\rangle$$

$$= \left\langle f(x,y),\; \int_0^\infty \int_0^\infty \Theta(w;z;x,y)\; e^{-uw}\sqrt{vz}\; J_\lambda(vz)\; dwdz \right\rangle$$

$$= \int_0^\infty \int_0^\infty f(x,y)\; \Theta(w,z;x,y)\; dxdy, \quad e^{-uw}\sqrt{vz}\; J_\lambda(vz)$$

( see lemma 3.2.1 )

$$= \mathcal{L}\mathcal{H}_\lambda\left[\int_0^\infty \int_0^\infty f(x,y)\; \Theta(w,z,x,y)\; dxdy\right]$$

$$\therefore (\mathcal{L}\mathcal{H}_\lambda)^{-1}\left[K(u)f!\; g(u),h(v)\right]! = \int_0^\infty \int_0^\infty f(x,y)\Theta(w,z;x,y)\; dxdy \; .$$

3.4 **Example** :

Let $(\mathcal{A}\mathcal{H}_\lambda F)(u,v) = F(u,v)$ $\text{Re}\,u > 0$ and $0 < v < \infty$ .

$f \in B'_{\lambda,a,b}$. Let $K(x) = x^{\nu-1}$, $g(x) = x^{-1}$, $h(y) = \frac{v}{4y}$

for this $G(p) = p^{-1}$, $H(q) = \frac{1}{4}\, q^{-1}\, v$

$\therefore$ $|G'(p)| = p^{-2}$ and $|H'(q) = \frac{1}{4}\, v q^{-2}$ and $K(G(p)) = p^{-\nu+1}$

The mapping $f(p,q) \longrightarrow K(x)\, f(g(x), h(y))$

i.e. $f(p,q) \longrightarrow x^{-\nu-1}\, f\left(x^{-1}, \frac{v}{4y}\right)$

is an isomorphism from $B'_{\lambda,d,b}$ onto $B'_{\lambda,c,b}$ .

where $d < c$ , and we write

$$\left\langle x^{\nu-1}\, f\left(x^{-1}, \frac{v}{4y}\right),\; e^{-ux}\sqrt{vy}\, J_\lambda(vy) \right\rangle$$

$$= \left\langle f(p,q),\; p^{-\nu-1}\, e^{-u/p}\, \frac{1}{8}\, v^2\, q^{-5/2}\, J_\lambda\left(\frac{1}{4} v^2 q^{-1}\right) p^{-2} \right.$$

$$= \left\langle f(p,q),\; \Phi(u,v;\, p,q) \right\rangle$$

$$= \left\langle f(p,q),\; \int_0^\infty \int_0^\infty e^{-pw}\sqrt{qz}\, \phi(u,v;w,z)\, dw\, dz \right\rangle$$

$$= \int_0^\infty \int_0^\infty \left\langle f(p,q),\; e^{-pw}\sqrt{qz}\, J_\lambda(qz) \right\rangle \phi(u,v;w,z)\, dw\, dz$$

$$= \int_0^\infty \int_0^\infty F(w,z)\, \phi(u,v;w,z)\, dw\, dz \, ,$$

where, $\phi(u,v;w,z) = \frac{1}{2}\left[(w/u)^{\nu/2} J_\nu(2\sqrt{uw})\, z^{1/2}\, J_{2\lambda}(vz)^{1/2}\right]$.

# REFERENCES

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2.2 **Theorem** : 2.2.1

If (i) k, g, h and $g^{-1} = G$, $h^{-1} = H$ are single valued analytic functions, real on $(0, \infty)$ and such that $g(0) = 0$, $h(0) = 0$ and $g(\infty) = \infty$, $h(\infty) = \infty$ (or $g(0) = \infty$, $h(0) = \infty$ and $g(\infty) = 0$, $h(\infty) = 0$ ) ;

(ii) $f(x,y) \overset{\mathcal{L}\mathcal{H}_\lambda}{\text{O—O}} F(u,v)$

which converges for $\mathrm{Re}\, u > 0$ and $0 < v < \infty$ ;

(iii) there exists a function $\phi$ (u,v; w,z),

$$\phi(u,v;w,z) \overset{\mathcal{L}\mathcal{H}_\lambda}{\text{O—O}} \Phi(u,v;p,q)$$

which converges for $\mathrm{Re}\, p > 0$ and $0 < q < \infty$, and

$$\Phi(u,v;p,q) = e^{-uG(p)} \sqrt{vH(q)}\; J_\lambda(vH(q))\; k\left(G(p)\right)\; |G'(p)|\; |H'(q)| \;;$$

(iv) $\int_0^\infty \int_0^\infty \left[ \int_0^\infty \int_0^\infty e^{-pw} \sqrt{qz}\; J_\lambda(qz)\, \phi(u,v;w,z) f(p,q)\, dw dz \right] dp dq$

converges absolutely for $\mathrm{Re}\, u > 0$ and $0 < v < \infty$ ;

then $k(x) f(g(x), h(y)) \overset{\mathcal{L}\mathcal{H}_\lambda}{\text{O—O}} \int_0^\infty \int_0^\infty F(w,z) \phi(u,v;w,z)\, dw dz$

which converges for $\mathrm{Re}\, u > 0$ and $0 < v < \infty$

**Proof** :

Since from (iv) the iterated integral is absolutely convergent for $\mathrm{Re}\, u > 0$ and $0 < v < \infty$, and since

$$\left| \int_0^\infty \int_0^\infty \left[ \int_0^\infty \int_0^\infty e^{-pw} \sqrt{qz}\; J_\lambda(qz) \phi(u,v;w,z)\, dw dz \right] f(p,q)\, dp dq \right|$$