C H A P T E R - II

N-IDEALS AND N-FILTERS

INTRODUCTION :

If I is an ideal of the Boolean algebra N of normal elements of a pseudocomplemented semilattice S, the extension I_E of I is defined to be the least ideal in S containing it. Venkatnarasimhan [12] has proved that the extension of an ideal I of N is the least ideal of S meeting N in I i.e. $I_E \cap N = I$. Note that, I_E coincides with I_e, the notation used by Venkatnarasimhan. But $(I \cap N)_E \neq I$ for every ideal I in S.

For example

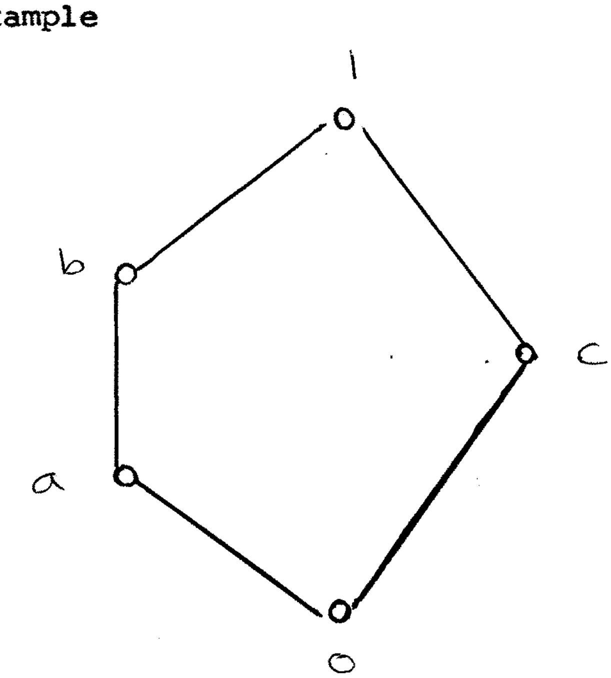

Fig.1

Here pseudocomplements of 0, a, b, c, 1 are 1, c, c, b, 0 respectively and $N = \{0, b, c, 1\}$.

Let $A = \{a, 0\}$. Then $A \cap N = \{0\}$ and hence $(A \cap N)_E = \{0\}' \neq A$.

Let $I = \{0, a, b\}$ then $I \cap N = \{0, b\}$ and hence $(I \cap N)_E = \{0, a, b\} = I$.

This motivates us to study those ideals I in S for which $(I \cap N)_E = I$.

R. Cignoli [2] has defined K-ideals (K-filters), conjuctively K-regular and K-normal in a distributive Lattice L, where K is a sublattice of L and characterized conjuctively K-regular and K-normal lattices as

Result 1 : L is conjuctively K-regular if and only if any K-prime ideal is a prime ideal of L

Result 2 : L is K-normal if and only if any K-prime filter is contained in a unique maximal filter.

These definitions and results are generalized to a pseudocomplemented semilattice.

Let S be pseudocomplemented semilattice and N be the Boolean algebra of normal elements of S. Guided by definitions of R Cignoli [2] we study N-ideals, conjuctively N-regular and N-normal semilattices.

The section 1, contains the characterization of conjuctively N-regular semilattices as

<u>Theorem</u> : The following three statements are equivalent

(1) S is conjuctively N-regular

(2) The minimal prime ideals are exactly the N-prime ideals in S.

(3) Any N-prime ideal is a prime ideal in S.

A relation between conjuctively N-regular and N-normal semilattice is established in Section 2.

In Section 3, we change our base of consideration from pseudocomplemented semilattices to S-lattices i.e. a pseudocomplemented lattice L in which $a^* \vee a^{**} = 1$ for all a in L. We characterize S-lattices as

"The pseudocomplemented lattice is an S-lattice if and only if it is N-normal".

§ 2.1 Conjuctively N-regular Semilattices :

We begin with the following :

2.1.1 Definition : Let $\bar{S}$ denotes a subsemilattice of S. An ideal I of S is called a $\bar{S}$-ideal if for any x in I, there is an element n in $I \cap \bar{S}$ such that $x \leq n$.

2.1.2 Definition : Let $\bar{S}$ denotes a subsemilattice of S. An ideal I of S is called as a $\bar{S}$-prime ideal if I is a $\bar{S}$-ideal and $I \cap \bar{S}$ is a prime ideal of $\bar{S}$.

In a semilattice S represented in Fig.1 if $\bar{S} = \{0, c, b, f\}$ then $I_1 = \{0, a, b\}$ is a $\bar{S}$-prime ideal and $I_2 = \{0, a\}$ is not a $\bar{S}$-ideal

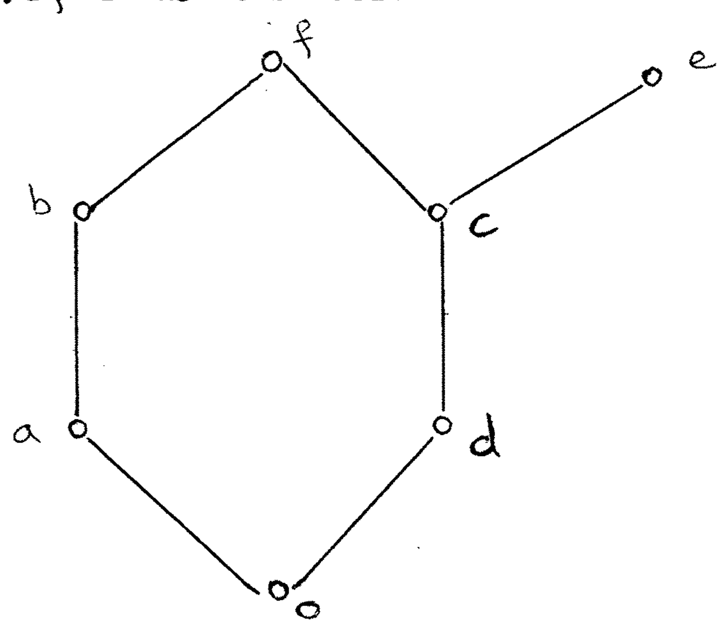

Fig.2

Dually we can define $\bar{S}$-filter and $\bar{S}$-prime filter.

2.1.3 Definition : Let $\bar{S}$ denote a subsemilattice of S. S is called conjuctively $\bar{S}$-regular if given x, y in S and a in $\bar{S}$ such that $x \wedge y \leq a$ there are elements b and c in $\bar{S}$ such that $x \leq b$, $y \leq c$ and $b \wedge c \leq a$.

The semilattice represented by the diagram of Fig.2 is not conjuctively $\bar{S}$-regular if $\bar{S} = \{o, a, c, e\}$

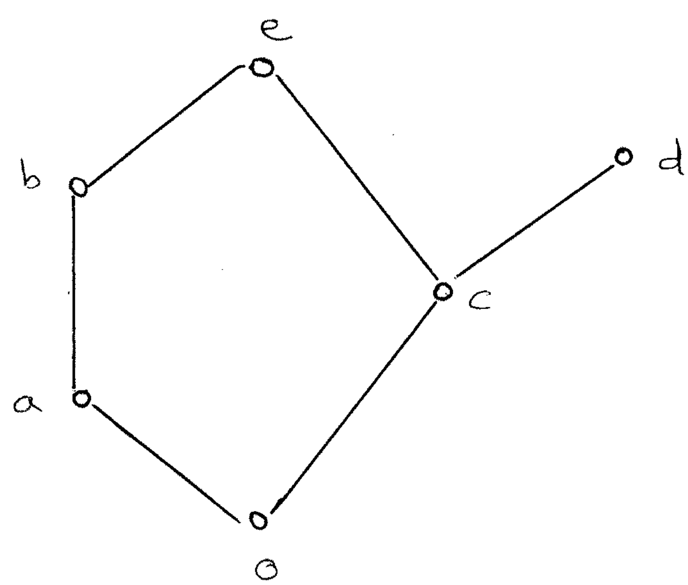

Fig.3

Now we turn our attention to a very particular case where S is a pseudocomplemented semilattice and $\bar{S} = N$ is a Boolean algebra of normal elements of S.

In this case $\bar{S}$-ideal, $\bar{S}$-prime ideal and conjuctively. $\bar{S}$-regular are called as N-ideal, N-prime ideal and conjuctively N-regular.

A property of N-prime ideals of S stated in the following :

<u>2.1.4 Theorem</u> : The minimal prime ideals in S are exactly the N-prime ideals in S if and only if any N-prime ideal is a prime ideal in S.

<u>Proof</u> : 'Only if part' being obvious we prove 'if part' only.

Let P be any minimal prime ideal in S Define $I = (P \cap N)_E$. Then $(I \cap N)_E = ((P \cap N)_E \cap N)_E = (P \cap N)_E = I$ by result (9). Thus $(I \cap N)_E = I$, proves that I is a N-ideal. I is a N-prime ideal because $P \cap N$ is a prime ideal in N and $I \cap N = (P \cap N)_E \cap N = P \cap N$. Hence by assumption I is a prime ideal in S. Again $P \cap N \subseteq P$ implies $(P \cap N)_E \subseteq P$ i.e. $I \subseteq P$ which in turn implies $I = P$ and hence P is a N-prime ideal.

Now, let P be a N-prime ideal in S then $P \cap N$ is prime in N. As N is a Boolean algebra, $P \cap N$ is a maximal ideal in N. Let there exist a prime ideal Q in S such that $Q \subseteq P$.

As $Q \cap N$ is prime and hence maximal in N, $Q \cap N \subseteq P \cap N$ implies $Q \cap N = P \cap N$. Therefore $(Q \cap N)_E = (P \cap N)_E$. But P being N-ideal $(P \cap N)_E = P$ and hence $(Q \cap N)_E \subseteq Q$ implies $P \subseteq Q$. Thus $P = Q$ which shows P must be a minimal prime ideal in S.

As a characterization of conjuctively N-regular semilattice we have.

<u>2.1.5 Theorem</u> : S is conjuctively N-regular if and only if N-prime ideal is a prime ideal in S.

<u>Proof</u> : <u>If part</u> : Suppose $x \wedge y \leq a$ for x, y in S and a is in N. Assume that there exist no b,c in N such that $b \wedge c \leq a$ for $x \leq b$ and $y \leq c$.

Define $F_x = \{ z \in N : z \geq x \}$ and $F_y = \{ z \in N : z \geq y \}$ Let F denotes the filter $[(F_x \cup F_y)) \cap N$. If $a \in F$ then $b \wedge c \leq a$ for some $b \in F_x$ and $c \in F_y$. But $b \wedge c \nleq a$. Hence $a \notin F$. As N is distributive by Stone's theorem there exists a prime ideal P' in N such that $a \in P'$ and $P' \cap F = \emptyset$ (see [5] p 119). Let $P = (P')_E$. Then obviously P is prime ideal and hence by assumption it is prime. If $x \in P$ then $x \in (P')_E$ implies $x \leq q$ for some $q \in P'$. But then $q \in F_x \cap P' \subseteq F \cap P' = \emptyset$; which is impossible. Hence $x \notin P$. Similarly we can prove $y \notin P$; contradicting the primeness of P. Hence there exist b,c in N such that

$b \wedge c \leq a$ and $x \leq b$, $y \leq c$; proving that S is conjuctively N-regular.

<u>Only if part</u> : Let I be any N-prime ideal Let $x \wedge y \in I$. Then $x \wedge y \in (I \cap N)_E = I$ implies that $x \wedge y \leq a$ for some $a \in I \cap N$. As S is conjuctively N-regular, there exist b,c in N such that $x \leq b$, $y \leq c$ and $b \wedge c \leq a$. But then $b \wedge c \in I \cap N$ implies $b \in I$ or $c \in I$; $I \cap N$ being prime in N. Thus $x \in I$ or $y \in I$. This proves that I is a prime ideal.

Summing up above results we get

<u>2.1.6 Corollary</u> : The following three statements are equivalent :

(a) S is conjuctively N-regular

(b) The minimal prime ideals are exactly the N-prime ideals in S.

(c) Any N-prime ideal is a prime ideal in S.

§ 2.2. N-normal semilattices :

Now we define $\bar{S}$-normal semilattice in a bounded meet semilattice as follows.

2.2.1 Definition : Let $\bar{S}$ denotes bounded subsemilattice of S. S is called $\bar{S}$-normal semilattice if given x, y in S such that $x \wedge y = 0$ there exist a, b $\in \bar{S}$ such that $x \wedge b = 0 = y \wedge a$ and $a \vee b$ exists and is equal to 1 (i.e. 1 is the only upper bound of a and b in S).

The $\bar{S}$ - normal semilattices are said to be normal semilattices if $\bar{S} = S$.

The semilattice with the diagram sketched in Fig.3, is conjuctively $\bar{S}$-regular but not $\bar{S}$-normal semilattice where $\bar{S} = \{0, a, b, 1\}$.

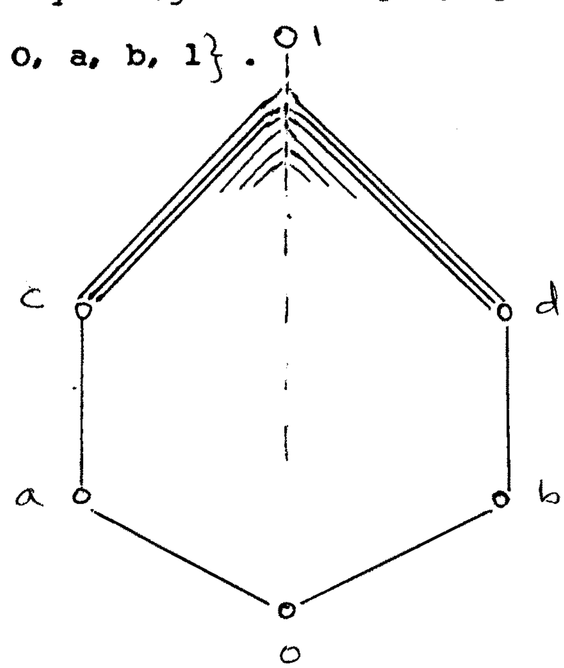

Fig.4

The semilattice represented in Fig.4 is the example of $\bar{S}$-normal semilattice. Here $\bar{S} = \{0, b, c, 1\}$

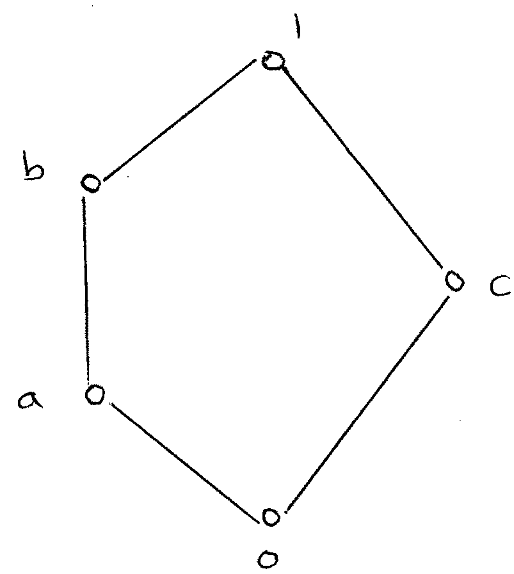

Fig.5

Now we concentrate to a very particular case where S is a pseudocomplemented semilattice and $\bar{S}$=N is a Boolean algebra of normal elements of S. In this case the $\bar{S}$ - normal semilattice is called as N-normal semilattice.

An interesting property of N-normal semilattice is proved in the following :

<u>2.2.2 Theorem :</u> If S is N-normal then any N-prime filter is contained in a unique maximal filter in S.

<u>Proof</u> : Let S be N-normal and let I be any N-prime filter in S. $I \cap N$ being a proper in N, it follows that I is a proper filter in S. As $0 \in S$, I is contained in a maximal filter in S by Result (1). Let if possible $I \subseteq M_1$ and $I \subseteq M_2$ where M_1 and M_2 are any two distinct maximal filters in S. Since $M_1 \neq M_2$ there exist $m_1 \in M_1$ and $m_2 \in M_2$ such that $m_1 \wedge m_2 = 0$ by Result (3). As $m_1 \wedge m_2 = 0$ and S is N-normal there exist b, c in N such that $m_1 \wedge c = 0 = m_2 \wedge b$ and $b \vee c$ exists and is equal to 1. Hence $c \notin M_1$ and $b \notin M_2$. $I \subseteq M_1$ and $I \subseteq M_2$ imply $b \notin I$ and $c \notin I$. But $b \vee c = 1 \in I \cap N$ and $I \cap N$ is prime in N will give $b \in I \cap N$ or $c \in I \cap N$. Thus $b \in I \subseteq M_1$ and c $c \in I \subseteq M_2$; a contradiction. Hence $M_1 = M_2$.

In the following theorem we establish a relation between conjuctively N-regular and N-normal semilattices.

<u>2.2.3 Theorem</u> : If S is N-normal then S is conjuctively N-regular.

<u>Proof</u> : Suppose $x \wedge y \leq a$ for x, y in S and $a \in N$. Define $x_1 = x \wedge a^*$ and $y_1 = y \wedge a^*$. Then $x_1 \wedge y_1 = x \wedge y \wedge a^* \leq a \wedge a^* = 0$ i.e. $x_1 \wedge y_1 = 0$. As S is N-normal there exist b, c in N such that $x_1 \wedge c = 0 = y_1 \wedge b$ and $b \vee c$ exists and is equal to 1. Define $b_1 = a \underline{\vee} b^*$ and $c_1 = a \underline{\vee} c^*$. Then obviously b_1 and c_1 are elements of N.

Further $x_1 \wedge c = 0$ implies $x \wedge a^* \wedge c = 0$ and hence $x \leq (a^* \wedge c)^* = a \underline{\vee} c^* = c_1$. Thus $x \leq c_1$ similarly we get $y \leq b_1$. Now $b_1 \wedge c_1 = (a \underline{\vee} b^*) \wedge (a \underline{\vee} c^*) = a \underline{\vee} (b^* \wedge c^*)$ $= a \underline{\vee} 0 = a$, since $0 = 1^* = (b \underline{\vee} c)^* = b^* \wedge c^*$. Thus $b_1 \wedge c_1 = a$, it follows that S is conjuctively N-regular.

Converse of this theorem need not be true. For this consider the semilattice represented in the following diagram.

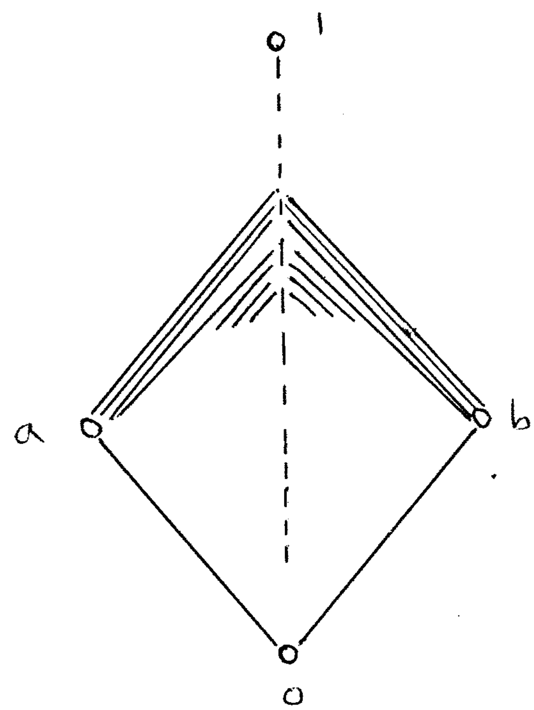

Fig.6

This semilattice is conjuctively N-regular but it is not N-normal.

<u>2.2.4 Corollary</u> : In a N-normal semilattice the following statements are true

(a) Any N-prime ideal is a prime ideal in S.

(b) S is conjuctively N-regular.

(c) The minimal prime ideals are exactly the N-prime ideals in S.

(d) Any N-prime filter is contained in a unique maximal filter in S.

As is well known that the set of uniquely complemented elements form a Boolean algebra in a very weakly distributive (distributive) semilattice [6], with corresponding modifications in the proof of the above theorems we have

<u>2.2.5 Theorem</u> : Let S be any very weakly distributive (distributive) semilattice and B be the Boolean algebra of all complemented elements of S. Then the first three of the following statements are equivalent and each of these is implied by the fourth.

(a) The minimal prime ideals are exactly the B-prime ideals in S.

(b) Any B-prime ideal is a prime ideal in S

(c) S is conjuctively B-regular

(d) S is B-normal.

2.2.6 Corollary : Let B be the Boolean algebra of all complemented elements of a very weakly distributive (distributive) semilattice S. If S is B-normal then any B-prime filter is contained in a unique maximal filter in S.

§ 2.3 S-lattices

Throughout this section L denotes pseudocomplemented lattice and N denotes the Boolean algebra of normal elements of L.

R.Cignoli [2] has proved in a distributive lattice L if K is a sublattice of L then L is K-normal iff any K-prime filter is contained in a unique maximal filter. As an extension of this result to the pseudocomplemented lattice we prove

2.3.1 Theorem : L is N-normal if and only if any N-prime filter is contained in a unique maximal filter.

Proof : As 'if part' follows from Theorem 2.2.2, we prove 'only if part'. Let $x \wedge y = 0$ for some x, y in L. Then $I_x = \{ z \in N : z \wedge x = 0 \}$ and $I_y = \{ z \in N : z \wedge y = 0 \}$ denote the ideals in N, N being a distributive lattice. Denote by I, the ideal generated by I_x and I_y in N. Assume that L is not a N-normal i.e. for no $a \in I_x$ and $b \in I_y$ $a \vee b = 1$. Hence $1 \notin I$. N being a distributive lattice, by Stone's theorem there exists a prime filter P' in N which is disjoint with I. If $P = (P')_E$ then P is a N-prime filter. If $x \in P$ then $x \geq K$ for some $K \in P'$. But then $K \wedge y = 0$ and hence $K \in I_y \cap P' \subseteq I \cap P' = \emptyset$;

which is impossible. Hence $x \notin P$. Similarly $y \notin P$. Define $P_x = [P \cup \{x\})$ and $P_y = [P \cup \{y\})$. We claim $P_x \neq L$ for, if $0 \in P_x$ then $0 = p \wedge x$ for some $p \in P = (P')_E$ implies that $p \geq t$ for some $t \in P'$. But then $t \wedge x = 0$ implies $t \in I_x \cap P' \subseteq I \cap P' = \emptyset$, a contradiction. Hence P_x is proper filter in L. Similarly we can prove P_y is proper filter in L. As P_x and P_y are proper filters in L, $P_x \subseteq M_1$ and $P_y \subseteq M_2$ for some maximal filters M_1 and M_2 in L by Result (1). As $x \in P_x \subseteq M_1$ and $x \wedge y = 0$ we get $y \notin M_1$. Similarly $x \notin M_2$; proving that $M_1 \neq M_2$. This shows that the N-prime filter P is contained in two distinct maximal filters M_1 and M_2 in L.

Now, we characterize S-lattices as

<u>2.3.2 Theorem</u> : L is an S-lattice if and only if L is N-normal.

<u>Proof</u> : Let L be N-normal and x be any element in L. As $x^* \wedge x^{**} = 0$ there exist b, c in N such that $a \wedge x^* = 0 = b \wedge x^{**}$ and $a \vee b = 1$. But then $a \leq x^{**}$ and $b \leq x^*$. Hence $a \vee b \leq x^* \vee x^{**}$ implies that $x^* \vee x^{**} = 1$ i.e. L is an S-lattice.

Conversely, let L be an S-lattice. If $x \wedge y = 0$ for some x, y in L then $y \leq x^*$ implies $y \wedge x^{**} = 0$. Thus $x \wedge x^* = 0 = y \wedge x^{**}$ where x^*, x^{**} are in N with $x^* \vee x^{**} = 1$. This proves that L is N-normal. ∎

2.3.3 Corollary : The following statements are equivalent

(a) L is an S-lattice

(b) Every N-prime filter in S is contained in a unique maximal filter in S.

(c) L is a N-normal lattice.

Summing up above results we get

<u>2.3.4 Corollary</u> : In a S-lattice, the following statements are true

(a) L is conjuctively N-regular

(b) Any N-prime ideal is a prime ideal of L

(c) The minimal prime ideals of L are exactly the N-prime ideals of L.

Converse of this corollary need not be true. For this, consider the lattice shown in the following diagram.

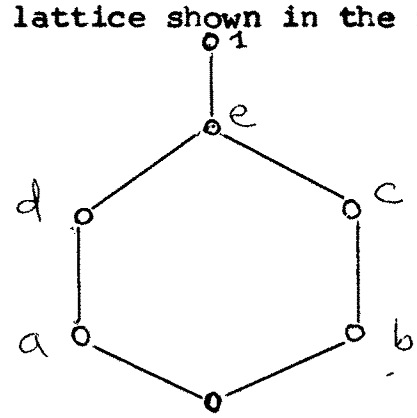

Fig.7

This lattice is conjuctively N-regular but it is not an S-lattice.