CHAPTER - I

SOME DEFINITIONS

ABSTRACT

In the present chapter we give in detail the definitions and statements of known results which we are making use, in course of our research. The appropriate references are cited at the end of the chapter.

DEFINITIONS AND TERMINOLOGY

In this first chapter we present some definitions concerning with univalent functions and some terminology which we are going to use in this context. Here we shall state some statements of the known results.

Definition :-

A complex valued function $f(z)$ is said to be holomorphic in a domain D in the complex plane, if it has a uniquely determined derivative at each point of D. Equivalently it can also be defined in the following way.

A holomorphic function is a meromorphic function without poles [6, pp. 51 - 52]

Definition :-

Let $E = \{ z : z \text{ is a complex number and } |z| < 1 \}$

Definition :-

A holomorphic function $f(z)$ in some domain D is said to be univalent in D if $f(z_1) = f(z_2)$ implies that $z_1 = z_2$ for all z_1, z_2 in D.

Remark :-

The terms "Conformal mapping", "injective meromorphic function", and "Univalent holomorphic (meromorphic) function" all have same meaning. Sometimes we follow time honored practice and say "Univalent" instead of "Univalent holomorphic". [6, pp 51 - 52]

Definition :-

Let S denote the class of all normalised univalent functions defined on E and having the Taylor series expansion of the form,

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n , \; z \text{ in } E.$$

By normalised conditions on f(z) we mean that $f(0) = 0$ and $f'(0) = 1$.

Definition :-

A domain containing the origin is starlike with respect to the origin if it is intersected by any straight line through the origin in a linear segment. We note that starlike with respect to the origin will be referred to as simply starlike.

Definition :-

Let S^* denote the subclass of S, whose members transform every disc $|z| \leqslant \rho$, $0 < \rho < 1$, onto a starlike domain.

The analytic description of starlike functions is given by the following statement.

Statement :-

Let f be holomorphic in the domain $D \subset \mathbb{C}$ ($\mathbb{C}$ denoting the complex plane) with $f(0) = 0 = f'(0) - 1$. Then $f \in S^*$ if and only if $zf'(z)/ f(z) \in P$, P denoting the class of all functions ψ , which are holomorphic and having positive real part in D with $\psi(0)=1$.

Definition : -

Let f(z) be holomorphic at z=0 and satisfying the conditions of normalisations. Then we define the radius of univalence to be the largest value of r such that f(z) is holomorphic and univalent for $|z| < r$.

Definition :-

Let f(z) be normalised holomorphic function at z = 0, let λ be a real number such that $0 \leq \lambda < 1$. We define the radius of starlikeness of order λ , denoted by S_λ , to be the largest value of r such that f(z) is holomorphic and

$$\operatorname{Re}\left[\frac{z f'(z)}{f(z)}\right] > \lambda, \quad \text{for } |z| < r.$$

Definition :-

Let K denote the subclass of S whose members map every disc $|z| \leq \rho$, $0 < \rho < 1$, onto a convex domain. Convex functions can be defined in the following way :

Definition :-

The set E is said to be convex if it is starlike with respect to each of its points, that is, if the linear segment joining any two points of E lies entirely in E . Hence a convex function is one which maps the unit disc conformally onto a convex domain.
The convex function can be described in the following way :

Statement :-

Let f be holomorphic in the domain D , with the conditions of normalisations. Then $f \in K$, if and only if $(1+zf''(z)/f'(z)) \in P$, P having the same significance as defined in case of starlike functions.

A close analytic connection between convex and starlike transformations was first observed by Alexander [1] in 1915. We shall merely state his observation in the following theorem.

Theorem :-

Let f be holomorphic in D, with $f(0) = 0$ and $f'(0) = 1$. Then $f \in K$ if and only if $z f'(z) \in \Xi^*$.

Definition :-

Let f(z) be holomorphic at z = 0 and satisfy f(0)=0 and $f'(0) \neq 0$ there, let λ be a real number satisfying $0 \leq \lambda < 1$. We denote the radius of convexity of order λ by K_λ and be defined as the largest value of r such that f(z) is holomorphic and satisfying

$$\mathrm{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > \lambda \text{, for } |z| < r.$$

Definition :-

A function $f(z) \in S$ is said to be close - to - convex with respect to the convex function $e^{i\alpha}g(z)$, where $g(z) \in K$ and $0 \leq \alpha < 2\pi$ if

$$\mathrm{Re}\left[\frac{f'(z)}{e^{i\alpha}\, g'(z)}\right] > 0$$

for $z \in E$. We denote this class by C .

Definition :-

Let f(z) be holomorphic at z = 0 and satisfy the conditions of normalisations there. Then the radius of close-to-convexity is defined to be the largest value of r such that f(z) is holomorphic and close-to-convex for $|z| < r$.

This class of close-to-convex functions was introduced by Kaplan [6] in 1952.

In the above definition of close-to-convex function we note that f is not required a priori to be univalent, also we note that the correlated function g need not be normalised.

In the context of the above definitions of subclasses of univalent functions, we would like to pass the following remark.

Remark :-

Every convex function is obviously close-to-convex. More generally, every starlike function is close-to-convex. [4, pp. 47] .

Close-to-convex functions can be stamped by a geometric condition somewhat similar to the defining properties of convex and starlike functions. Let f be holomorphic in D and let C_r denote the image of the unit circle $|z| = r$, under the mapping f , lying between 0 and 1 . Then roughly speaking f is close-to-convex if and only if none of the curves C_r makes a "reverse hairpin turn". In this connection Kaplan [6] has stated the following definition of close-to-convex function which is known as Kaplan's Theorem.

Kaplan's Theorem :

Let f be analytic and locally univalent in D. Then f is close-to-convex if and only if

$$\int_{\theta_1}^{\theta_2} \text{Re} \left[1 + \frac{zf''(z)}{f'(z)} \right] d\theta > -\pi, \quad z = re^{i\theta}$$

for each r , $0 < r < 1$ and for each pair of real numbers θ_1 and θ_2 , with $\theta_1 < \theta_2$.

Lastly, we assert the following inclusion relations to summerise normalised classes of functions K, S^*, C, S, as

$$K \subset S^* \subset C \subset S.$$

We humbly state that the spiral- like functions, Typically real functions, Bazilvic functions are also the subclasses of univalent functions S, but Since we are concentrated only on starlike, convex and close-to-convex subfamilies, we are not rushed to define them also.

We note that the close-to-convex functions, starlike functions and convex functions are all univalent functions.

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5. Kaplan, W :

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