CHAPTER - III

CHAPTER-III

THE CHARACTERIZATION OF DISTRIBUTIONAL FINITE FOURIER COSIN-SINE TRANSFORMATION AND APPLICATIONS

3.1 INTRODUCTION

The finite Fourier cosine sine transform of a function $f(x,y)$ is defined by the equation

$$F(m,n) = \int_0^a \int_0^b f(x,y) \operatorname{Cos}\left(\frac{m\pi x}{a}\right) \operatorname{Sin}\left(\frac{n\pi y}{b}\right) dx\,dy \qquad 3.1.1$$

In chapter II we have extended this transform to certain class $S'(I)$ of distributions through the definition

$$f'_{cs}\,[f(x,y)] = F(m,n) = \frac{\sqrt{ab}}{2} \langle f, \psi_{m,n} \rangle, \quad f \in S'(I)$$

$$m,n = 0,1,2,3,\ldots$$

where $\psi_{o,n}(x,y) = \frac{1}{\sqrt{ab}} \operatorname{Sin}\left(\frac{n\pi y}{b}\right)$, $n=1,2,3,\ldots\ldots$

and $\psi_{m,n}(x,y) = \frac{2}{\sqrt{ab}} \operatorname{Cos}\left(\frac{m\pi x}{a}\right) \operatorname{Sin}\left(\frac{n\pi y}{b}\right)$

$$m,n = 1,2,3,\ldots.$$

In this chapter we now turn to the problem of precisely characterizing the functions $F(m,n)$ that are generated by transformation f'_{cs}. In particular we shall prove that complex valued functions $F(m,n)$ defined on the set of all ordered pairs of positive integers is the Fourier cosine-sine transform of some member $f \in S'(I)$, if and only if the growth condition stated below is satisfied that is if and only if

$$\frac{4}{ab} \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} \frac{|F(m,n)|^2}{|\lambda_{m,n}^{2q}|} \quad \text{Converges},$$

where

$$\lambda_{m,n} = -\pi^2 \left(\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)\right)$$

This in turn will lead to a characterization of the members of $S'(I)$ in terms of certain combinations of generalized derivatives of the elements of $L_2(I)$. The notations and termilogy in this chapter follow that of Zemanian A.H.[37]. An operation transform formula is also established and is applied to solve two dimensional heat flow equation.

LEMMA 3.1.1

Let $a_{m,n}$ denote the complex number then the series $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n} \psi_{m,n}$ converges in $S(I)$ if and only if $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |\lambda_{m,n}|^{2k} |a_{m,m}|^2$ converges for every nonnegative integer k.

PROOF

We employ the fact that $\{\psi_{m,n}\}_{m,n=0}^{\infty}$ from an orthonormal set to write

$$\int_0^a \int_0^b |R^k \sum_{m=0}^{p} \sum_{n=0}^{l} a_{m,n} \psi_{m,n}|^2 \, dxdy$$

$$= \int_0^a \int_0^b | \sum_{m=0}^{p} \sum_{n=0}^{l} a_{m,n} \lambda_{m,n}^k \psi_{m,n} |^2 \, dxdy$$

$$= \int_0^a \int_0^b \sum_{m=0}^{p} \sum_{n=0}^{l} \sum_{r=0}^{p} \sum_{s=0}^{l} a_{m,n} \overline{a}_{r,s} \lambda_{m,n}^k \lambda_{r,s}^k \psi_{m,n} \overline{\psi}_{r,s} \, dxdy$$

$$= \sum_{m=0}^{p} \sum_{n=0}^{l} |\lambda_{m,n}|^{2k} |a_{m,n}|^2 \int_0^a \int_0^b \psi_{m,n} \overline{\psi}_{r,s} \, dxdy$$

$$= \sum_{m=0}^{p} \sum_{n=0}^{l} | \lambda_{m,n}|^{2k} |a_{m,n}|^2 .1$$

Our assersion follows directly from this equation.

THEOREM 3.1.1

Let F(m,n) be the complex valued functions defined on the set of all ordered pairs (m,n) of integers then

$$\frac{2}{\sqrt{ab}} \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} F(m,n) \; \psi_{m,n} \qquad 3.1.1$$

converges in S'(I) if and only if there exists a nonnegative integer q such that

$$\frac{4}{ab} \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} \frac{|F(m,n)|^2}{|\lambda_{m,n}|^{2q}} \qquad 3.1.2$$

converges Furthermore, if f denotes the sum in S'(I) of (3.1.1), then F(m,n) is the distributional Fourier cosine sine transform of f.

PROOF

First supose that (3.1.1) converges in S'(I) say to f, then since $\psi_{m,n} \in S(I)$ we have

$$(f, \psi_{m,n}) = \left(\frac{2}{\sqrt{ab}} \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} F(m,n) \; \psi_{m,n}, \psi_{m,n} \right)$$

$$= \frac{2}{\sqrt{ab}} \sum_{p=o}^{\infty} \sum_{l=o}^{\infty} F(p,l) \; (\psi_{p,l}, \psi_{m,n})$$

$$= \frac{2}{\sqrt{ab}} \; F(m,n)$$

Since $\{\psi_{m,n}\}_{m,n}^{\infty}$ is a complete orthonormal system of eigen functions of differential operator R therefore

$$(\psi_{p,l}, \psi_{m,n}) = 1 \; , \text{ if } (p,l) = (m,n)$$

$$= 0 \; , \text{ if } (p,l) \neq (m,n)$$

Thus $F(m,n) = \frac{\sqrt{ab}}{2} \; (f, \psi_{m,n})$.

Hence if the series (3.1.1) converges to f in S'(I) then F(m,n) is the distributional Fourier cosine sine transform of f.

Now we shall show that (3.1.2) converges for some q, for that we first prove that the sequence $\left\{ \frac{F(m,n)}{\lambda^{q}_{m,n}} \right\}^{\infty}_{m,n=0}$ is bounded for some value say q_0 of q. To prove this, assume that the above sequence is unbounded for every q=1,2,...

Hence there are increasing sequences $\{m_q\}$ and $\{n_q\}$ of positive integers such that

$$\left| \frac{F(m_q,n_q)}{\lambda^{q}_{m_q,n_q}} \right| \geq 1, \text{ for } q=1,2,\ldots.$$

Now for every q=1,2,3,...., let

$$a_{m,n} = \left| \lambda^{q}_{mq,nq} \, q \right|^{-1} \quad \text{if } m=m_q \text{ and } n=n_q$$

$$= 0 \quad \text{if } m \neq m_q \text{ or } n \neq n_q$$

Now , for any fixed non-nagative integer k,

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left| \lambda^{k}_{m,n} a_{m,n} \right|^2 = \sum_{q=0}^{\infty} \left[\left| \lambda^{k}_{mq,nq} \right| \left| \lambda^{q}_{mq,nq} q \right|^{-1} \right]^2$$

$$= \sum_{q=0}^{\infty} q^{-2} \left| \lambda_{mq,nq} \right|^{2k-2q}$$

But since $\left| \lambda_{mq,nq} \right|^{2k-2q}$ is bounded by 1 for all sufficiently large q

The series $\sum_{q=0}^{\infty} q^{-2} \left| \lambda_{mq,nq} \right|^{2k-2q}$ converges

Hence $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left| \lambda^{k}_{m,n} a_{m,n} \right|^2$ converges for every nonnegative integer k, but by lemme (3.1.1)

$$\sum_{m=o}^{p}\sum_{n=o}^{l}|\lambda^{k}_{m,n}a_{m,n}|^{2}=\int_{o}^{a}\int_{o}^{b}|R^{k}\sum_{m=o}^{p}\sum_{n=o}^{l}a_{m,n}\psi_{m,n}|^{2}dx\,dy \qquad 3.1.3$$

Hence the series $\sum_{m=o}^{\infty}\sum_{n=o}^{\infty}a_{m,n}\psi_{m,n}$ converges in S(I), say to ϕ.

Thus for this $\phi = \sum_{m=o}^{\infty}\sum_{n=o}^{\infty}a_{m,n}\psi_{m,n}$ in S(I)

$$\sum_{m=o}^{\infty}\sum_{n=o}^{\infty}|a_{m,n}F(m,n)|\geq\sum_{q=o}^{\infty}|a_{mq,nq}\lambda^{q}_{mq,nq}|$$

$$=\sum_{q=o}^{\infty}|\lambda^{q}_{mq,nq}q|^{-1}|\lambda^{q}_{mq,nq}|$$

$$=\sum_{q=o}^{\infty}q^{-1}$$

$$=0$$

This contradicts the condition (A) stated below.

For every $\phi = \sum_{m=o}^{\infty}\sum_{n=o}^{\infty}a_{m,n}\psi_{m,n}\ \varepsilon\ S(I)$

the series $\sum_{m=o}^{\infty}\sum_{n=o}^{\infty}a_{m,n}b_{m,n}$ converges, (A)

where $b_{m,n}=\sqrt{\frac{ab}{2}}(f,\psi_{m,n})$.

Hence our assumption must be wrong. Thus, the sequence $\left\{\frac{F(m,n)}{\lambda^{q}_{m,n}}\right\}$ is bounded for some value q_o of q.

Hence, since $|\lambda_{m,n}|\to\infty$ as $m\to\infty$, $n\to\infty$

we assume that $|\bar{\lambda}^{q}_{m,n}F(m,n)|\to 0$ as $m\to\infty$, $n\to\infty$

and for each $q > q_o$.

Now we shall prove that (3.1.2) converges for some $q > q_0$. Again to prove this assume that (3.1.1) diverges for every $q > q_0$. Hence there are increasing sequences $\{m_q\}$ and $\{n_q\}$ of positive integers such that

$$1 \leqslant \sum_{m=m_{q-1}}^{m_q-1} \sum_{n=n_{q-1}}^{n_q-1} | \lambda_{m,n}^{-q} F(m,n) |^2 \leqslant 2 \qquad 3.1.4$$

for $q = q_0+1, q_0+2, \ldots$.

Now in this case choose

$$a_{m,n} = |F(m,n) \lambda_{m,n}^{-2q} q^{-1}|$$

if $m_{q-1} \leqslant m < m_q$, $n_{q-1} \leqslant n < n_q$, $q < q_0$

Then for every nonnegative integer k

$$\sum_{m=m_{q-1}}^{m_q-1} \sum_{n=n_{q-1}}^{n_q-1} | \lambda_{m,n}^{k} a_{m,n} |^2 = \sum_{m=m_{q-1}}^{m_q-1} \sum_{n=n_{q-1}}^{n_q-1} |\lambda_{m,n}|^{2k-2q} | \lambda_{m,n}^{-q} F(m,n)|^2 q^{-2}$$

$$\leqslant 2q^{-2}$$

Thus, since $\lambda_{m,n} \to \infty$ as $m \to \infty$, $n \to \infty$ and by inequality (3.1.4)

$$\sum_{m=m_{q-1}}^{m_q-1} \sum_{n=n_{q-1}}^{n_q-1} | \lambda_{m,n}^{k} a_{m,n} |^2 < 2q^{-2}$$

for sufficiently large q.

This implies the series

$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} | \lambda_{m,n}^{k} a_{m,n} |^2$ converges for each k.

Thus again by lemma (3.1.1) the series

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n} \psi_{m,n}$$

converges in S(I) , say to ϕ .

on the other hand

$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left| a_{m,n} F(m,n) \right|$ diverges because

$$\sum_{m=m_{q-1}}^{m_q-1} \sum_{n=n_{q-1}}^{n_q-1} | a_{m,n} F(m,n)| = \sum_{m=m_{q-1}}^{m_q-1} \sum_{n=n_{q-1}}^{n_q-1} |(F(m,n)^2 \lambda_{m,n}^{-2q} q^{-1}|$$

$$= \sum_{m=m_{q-1}}^{m_q-1} \sum_{n=n_{q-1}}^{n_q-1} | \lambda^{-q} F(m,n)|^2 q^{-1}$$

$$\geqslant q^{-1}$$

by inequality (3.1.4)

Again, we have contradicted condition (A). This shows that (3.1.2) converges for $q > q_o$.

CONVERSLY

Assume that (3.1.2) converges for some $q \geqslant 0$, then we wish to show that for each $\phi \epsilon$ S(I) the series

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (F(m,n) \psi_{m,n}, \phi) \text{ converges}$$

This will confirm the convergence of (3.1.1) in S'(I). By using the Schwarz inequality for some real number. Widder [33,p.313]/ we may write

$$\sum_{m=o}^{\infty} \sum_{n=o}^{\infty} (F(m,n)\ \psi_{m,n}, \Phi) = \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} F(m,n)(\psi_{m,n}, \phi)$$

$$= \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} |\lambda_{m,n}^{-q} F(m,n)|\ |\lambda_{m,n}^{q} (\phi, \psi_{m,n})|$$

$$\lambda_{m,n} \neq 0.$$

$$\leq \{ \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} |\lambda_{m,n}^{-q} F(m,n)|^2 \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} |\lambda_{m,n}^{q} (\phi, \psi_{m,n})|^2 \}^{\frac{1}{2}}$$

The first series in right hand side converges by assumption Zemanian [37,p.254-255].

Also by λ for every $\Phi \in S(I)$ the series

$$\sum_{m=o}^{\infty} \sum_{n=o}^{\infty} (\phi, \psi_{m,n})\ \psi_{m,n}$$

converges in $S(I)$ to ϕ, and hence by (3.1.3) with $k=q$ the second series also converges. Thus the series (3.1.1) converges in $S'(I)$ Q.E.D.

THEOREM 3.1.2

A necessary and sufficient condition for f to be a member of $S'(I)$ is that there exists some nonnegative integer q and some $g \in L_2(I)$ such that

$$f = R^{q} g + \sum_{\lambda_{m,n}=0} \sum C_{m,n} \psi_{m,n}$$

where $C_{m,n}$ denotes the complex constants, and R^{q} is understood to be a generalized differential operator on S'(I). Also $\sum_{\lambda m,n=0}\sum$ denotes a summation on those m,n=0,1,2,...., for which $\lambda_{m,n}$ =0 there are only a finite number of such m,n...

PROOF

SUFFICIENCY

By [3, pp. 258 NOte III]

If $f = R^{q} g$ for some $g \varepsilon L_{2}(I)$ and some nonnegative integer q then $f \varepsilon S'(I)$. This follows directly from the fact that $L_{2}(I) \subset S'(I)$ and that R maps S'(I) into S'(I) and $S(I) \subset S'(I)$ and since each $\psi_{m,n} \varepsilon S(I)$ it follows that $f \varepsilon S'(I)$

NECESSITY

$$\text{Let } f = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} F(m,n) \psi_{m,n} \varepsilon S'(I)$$

set $G(m,n) = \lambda_{m,n}^{-q} F(m,n)$, whenever $\lambda_{m,n} \neq 0$

where $q > 0$ is such that

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |\lambda_{m,n} F(m,n)|^{2}, \lambda_{m,n} \neq 0 \text{ converges}$$

Also set $G(m,n) = 0$ when $\lambda_{m,n}=0$. Hence $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |G(m,n)|^{2}$ converges, and by the Riesz Fisher theorem there exists some $g \varepsilon L_{2}(I)$ such that $G(m,n) = (g, \psi_{m,n})$ moreover, since $\psi_{m,n} \varepsilon S(I)$ the the definition of R on S'(I) yields

$$(g, \lambda^q_{m,n} \psi_{m,n}) = (g, R^q \psi_{m,n}) = (R^q g, \psi_{m,n})$$

Altogether then, we may write

$$f = \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} F(m,n) \psi_{m,n} = \sum_{\lambda_{m,n} \neq 0} \sum \lambda^q_{m,n} G(m,n) \psi_{m,n}$$

$$+ \sum_{\lambda_{m,n} = o} \sum F(m,n) \psi_{m,n}$$

$$= \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} \lambda^q_{m,n}(g, \psi_{m,n}) + \sum\sum_{\lambda_{m,n}=0} F(m,n) \psi_{m,n}$$

$$= \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} (g, \lambda^q_{m,n} \psi_{m,n}) \psi_{m,n} + \sum\sum_{\lambda_{m,n}=o} F(m,n) \psi_{m,n}$$

$$= \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} (g, R^q \psi_{m,n}) \psi_{m,n} + \sum\sum_{\lambda_{m,n}=o} F(m,n) \psi_{m,n}$$

$$= \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} (R^q g, \psi_{m,n}) \psi_{m,n} + \sum\sum_{\lambda_{m,n}=o} F(m,n) \psi_{m,n}$$

$$= R^q g + \sum\sum_{\lambda_{m,n}=o} F(m,n) \psi_{m,n}$$

Q.E.D

OPERATION TRANSFORM FORMULA AND APPLICATION

3.2 INTRODUCTION

Our distributional finite Fourier cosine sine transformation generates an operation transform formula. this together with the inversion theorem proved in chapter II can be applied in solving certain differential equation involving initial conditions as distributions.

3.3 AN OPERATION CALCULUS FOR R

Since we have already indicated that the differential operator $R = \partial^2 / \partial x^2 + \partial^2 / \partial y^2$

is continuous and linear mapping of S'(I) into S'(I). Therefore we may write for every $f \in S'(I)$ and for each nonnegative integer k.

$$R^k f = \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} (f, \psi_{m,n}) R^k \psi_{m,n}$$

$$= \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} (f, \psi_{m,n}) \lambda_{m,n}^k \psi_{m,n} \qquad 3.3.1$$

We can use this fact to solve the differential equation

$$P(R)\, u\, (x,y) = g(x,y) , \qquad 3.3.2$$

Where P is a polynomial and given g(x,y) and unknown u(x,y) are required to be in S'(I). Now by applying the distributional finite Fourier cosine sine transformation f'_{cs} to (3.3.2) we obtain

$$P(\lambda_{m,n})\, U\, (m,n) = G(m,n) \qquad 3.3.3$$

where $U(m,n) = f'_{cs} [U(x,y) \; ; \; x \to m, \; y \to n]$

$G(m,n) = f'_{cs} [g(x,y); \; x \to m, \; y \to n]$

That is U(m,n) and G(m,n) are distributional finite Fourier cosine sine transformations of u(x,y) and g(x,y) respectively.

Case (i)

If $p(\lambda_{m,n}) = 0$ for every $m,n=0,1,2,\ldots$ we can divide equation (3.3.3) by $P(\lambda_{m,n})$ so that we have

$$U(m,n) = \frac{G(m,n)}{P(\lambda_{m,n})}$$

Then by applying inverse Fourier cosine sine transform (i.e. f'^{-1}_{cs}) we get

$$u(x,y) = \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} \frac{G(m,n)}{P(\lambda_{m,n})} \psi_{m,n} \qquad 3.3.4$$

then by theorem (2.3.2) and theorem (3.1.1) this solution exists and is unique in $S'(I)$

Case (ii)

If $P(\lambda_{m,n})=0$ for some $\lambda_{m,n}$ say $\lambda_{m_r n_s}$ $(r,s=1,2,\ldots p)$, then a solution exists in $S'(I)$ if and only if $G(m_r,n_s) = 0$ for $r,s,=1,2,\ldots.p$. In this case solution to equation (3.3.2) is given by

$$u(x,y) = \sum_{P(\lambda_{m,n}) \neq 0} \sum \frac{G(m,n)}{P(\lambda_{m,n})} \psi_{m,n} \qquad 3.3.5$$

But, it is no longer unique in $S'(I)$, and we may add to (3.3.5) any complementary solution

$$u_c(x,y) = \sum_{r=1}^{p} \sum_{s=1}^{p} a_{m_r n_s} \psi_{m_r n_s}$$

$$= \sum_{r=1}^{p} \sum_{s=1}^{p} a_{r,s} \psi_{mr,ns}$$

where $a_{r,s}$ are arbitrary numbers.

PROPERTIES

3.4 ~~PROPERTY~~ OF DISTRIBUTION FOURIER COSINE SINE TRANSFORMATION

a) If $f \in S'(I)$, then the distributional Fourier cosine sine transformation of R f(x,y) exists and is given by

$$f'_{cs}[R\ f(x,y)] = \lambda_{m,m} F(m,n) = \lambda_{m,n} f'_{cs}[f(x,y)] \qquad 3.4.1$$

PROOF

To prove this let $\Psi_{m,n} \in S(I)$ so that we have

$$f'_{cs}\,(]R\ f(x,y) = \frac{\sqrt{ab}}{2}\ (Rf(x,y),\ \Psi_{m,n}(x,y)$$

$$= \frac{\sqrt{ab}}{2}(f(x,y)\ ,R\ \psi_{m,n}(x,y)$$

$$= \frac{\sqrt{ab}}{2}(x,y)\ ,\ \lambda_{m,n}\psi_{m,n}(x,y)$$

$$= \frac{\sqrt{ab}}{2}\ \lambda_{m,n}\ (f(x,y)\ ,\ \psi_{m,n}\ (x,y)$$

$$= \lambda_{m,n}\ \frac{\sqrt{ab}}{2}\ (f(x,y)\ ,\ \Psi_{m,n}\ (x,y)$$

$$= \lambda_{m,n}\ F_{cs}(m,n)$$

$$= \lambda_{m,n}\ F(m,n)$$

This shows that f'_{cs} transforms differential operator R [f(x,y)] into an algebraic operator $\lambda_{m,n} F(m,n)$. Hence it can be applied to get the solution of boundary value problems.

NOTE

In general $f'_{cs}[R^k\ f(x,y) = \lambda^k_{m,n} f'_{cs}\ ]f(x,y)]$

$$= \lambda^k_{m,n}\ F(m,n) \qquad 3.4.2$$

b) If $f \in S'(I)$ and C is any arbitrary positive constant then

$$f'_{cs}[f(x,y) + C] = \frac{\sqrt{ab}}{2} < f(x,y) + C , \psi_{m,n} >$$

$$= \frac{\sqrt{ab}}{2}(f(x,y), \psi_{m,n}) + \frac{\sqrt{ab}}{2}(C, \psi_{m,n})$$

$$= F(m,n) + C \frac{\sqrt{ab}}{2} (1, \psi_{m,n})$$

$$= F(m,n) + C \int_0^a \int_0^b \text{Cos} \frac{m\pi x}{a} \text{Sin} \frac{n\pi y}{b} dxdy$$

3.6 APPLICATION TO HEAT FLOW EQUATION IN THIN RECTANGULAR PLATE

Our object is to solve the heat flow problem for temperature distribution in a thin rectangular plate whose surface is insulated. This plate has an arbitrary distribution of temperature. The edges of the rectangular plate be kept at zero temperature. And there is no heat source in this plate. Again radiation loss is negliigible.

We shall solve this problem by using distributional Fourier cosinesine transformation with initial and boundary conditions as a generalized functions.

THE STATEMENT OF THE PROBLEM

Find the conventional function $u(x,y,t)$ in the space $S'(I)$ on the domain I given by

$$I = \{ (x,y) : 0 < x < a \ , 0 < y < b \ , 0 < t < \infty \}$$

which satisfies the two dimensional heat flow equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{h^2} \frac{\partial u}{\partial t} , \quad u = u(x,y,t) \qquad 3.6.1$$

where h^2 is constant known as diffusivity

with initial and boundary conditions :

i) As $t \to o^+$, $u(x,y,t)$ converges to a certain generalized function $f(x,y)$ in $S'(I)$.

That is $u(x,y,o) = f(x,y)$ 3.6.2

ii) $u(x,y,t)=0$, along the boundaries of the open rectangle I

iii) $u(x,y,t)=0$ as $t \to \infty$

SOLUTION

To solve this problem we use the differential operator $R = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ as defined in (2.5.3) so that equation (3.6.1) will be written as

$$h^2 \, R(u) = \frac{\partial u}{\partial t} \qquad 3.6.3$$

Now by applying distribution Fourier cosine sine transformation f'_{cs} to above equation we get

$$h^2 \, f'_{cs}[R(u)] = \frac{\partial}{\partial t} \, f'_{cs}[u]$$

i.e. $h^2 \lambda_{m,n} U(m,n,t) = \frac{\partial}{\partial t} U(m,n,t)$,

where $\lambda_{m,n} = -\pi^2 \, (m^2/a^2 + n^2/b^2)$

i.e. $\frac{\partial}{\partial t} U(m,n,t) - h^2 \lambda_{m,n} U(m,n,t) = 0$ 3.6.4

whose solution is given by

$$U(m,n,t) = A e^{h^2 \lambda_{m,n} t}$$

But by initial condition (3.6.2) we have

$$U(m,n,o) = F(m,n) = A$$

Substituting value of A the above solution becomes

$$U(m,n,t) = F(m,n)\, e^{h^2 \lambda_{m,n} t},$$

where $F(m,n) = \frac{\sqrt{ab}}{2} (f, \psi_{m,n})$,

and $\psi_{m,n} = \frac{2}{\sqrt{ab}} \text{Cos} \frac{m\pi x}{a} \text{Sin} \frac{n\pi y}{b}$

So that by applying corresponding inversion theorem we shall get the formal solution

$$u(x,y,t) = \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} F(m,n)\, e^{h^2 \lambda_{m,n} t} \cdot \psi_{m,n} \qquad 3.6.5$$

u(x,y,t) is conventional function that satisfies differential equation (3.6.3) or (3.6.1)

Now we shall verify that (3.6.5) is the possible solution i.e. (3.6.5) satisfies initial and boundary conditions.

To verify the boundary condition (ii) we have to show that for each $\phi \in S(I)$

$$(u(x,y,t), \phi(x,y)) \to (f(x,y), \phi(x,y)), \text{ as } t \to o^{+} \qquad 3.6.6$$

Now for fixed $t > 0$ (3.6.5) converges in S(I) and therefore in $L_2(I)$, consequently we can take its inner product with ϕ term by term to write

$$(u(x,y,t),\ \phi(x,y)) = \sum_{m=o}^{\infty}\sum_{n=o}^{\infty} F(m,n)e^{h^2\lambda_{m,n}t}\ (\psi_{m,n},\phi) \qquad 3.6.7$$

But by Lemma 2.6.6 and 2.6.8 $(\psi_{m,n},\phi)$ is of rapid descent. It follows that series in (3.6.7) converges uniformly on $0 \leqslant t \leqslant \infty$. So that we may pass to the limit as $t \to o^+$ under the summation sign to get

$$(u(x,y,t),\ \phi(x,y)) \to \sum_{m=o}^{\infty}\sum_{n=o}^{\infty} F(m,n)(\psi_{m,n},\ \phi) \qquad 3.6.8$$

But sinc f ε S'(I) , the right hand side of (3.6.8) is equal to (f,ϕ). This proves (3.6.6)

Finally, for $T \leqslant t < \infty$ $(T > o)$ we have from (3.6.5) that

$$|u(x,y,t)| \leqslant \sum_{m,n}^{\infty}\sum_{n=o}^{\infty} |F(m,m)|\,|e^{h^2\lambda_{m,n}t}|\,|\psi_{m,n}| \qquad 3.6.9$$

And by our previous comments the series herein converges uniformly on I. So, we may take limit as $x \to o^+$ or $x \to a^-$ and $y \to o^+$ or $y \to b^-$ so that condition (ii) is also satisfied .

Now to verify boundary conditon (iii) in the same way

$$|u(x,y,t)| \leqslant \sum_{m=o}^{\infty}\sum_{n=o}^{\infty} |F(m,n|\,|e^{h^2\lambda_{m,n}t}|\,|\psi_{m,n}| \to 0$$

as $t \to \infty$ since $\lambda_{m,n} = -\pi^2(m^2/a^2 + n^2/b^2)$

then $e^{-h^2\pi^2(m^2/a^2+n^2/b^2)t} \to o$, as $t \to \infty$.

which verifies boundary conduition (iii) Q.E.D.