CHAPTER - I

CHAPTER-I

SURVEY OF LITERATURE AND PRE-REQUISITES

1.1 INTRODUCTION

Many of the phenomena of classical physics may be described by partial differential aquation. And the oldest systematic technique for the solution of partial differential equation of mathematical physics is the method of separation of variables introduced by D.Alembert, Daniel Bernoulli and Euler in the middle of the eighteenth century. It remains a method of great value today and lies at the heart of the use of INTEGRAL TRANSFORMS in the solution of problems in applied mathematics.

The theory of Integral Transformations is used for a systematic study of certain type of boundary value problems depending upon initial and boundary conditions. A variety of integral transforms [3]Laplace, Fourier, Mellin, Hankel etc.[2] suitable to the need of the physical problems have been constructed by choosing proper kernels and applied in solving boundary value problems.

A function is a potent notion in mathematics. Some manipulations like Dirac Delta function introduced by Dirac [10] in 1947. $\delta(x)=0$ for $x \neq 0$

$$\int_a^b \delta(x)\, dx = 1 \quad \text{for } x = 0, \quad \infty < a < b < \infty$$

in technical literature have motivated the mathematicians to re-examine the concept of function not by its value but by its behaviour as a functional on some space of testing functions. This is the new concept. This new mode of thinking gave birth to the theory of generalized functions.

The impact of generalized functions on the integral transforms has recently revolutionalised the theory of generalized integral transformations.

In this chapter we give a brief account of the elementary concepts that are required for the development of the work of dissertation.

1.2 LINEAR SPACE

A collection V of elements ϕ , ψ , θ .....is said to be a linear space if the following axioms are satisfied

1) There is an operation +, called "addition" by which any pair of elements ϕ and ψ can be combined to yield a unique element $\phi + \psi$ in V. moreover, + has the following properties

1a) $\phi + \psi = \psi + \phi$ (Commutativity)

1b) $(\phi + \psi) + \theta = \phi + (\psi + \theta)$ (associiativity)

1c) There exists a unique element 0 in V such that $\phi + 0 = \phi$ for every $\phi \in V$

1d) For every $\phi \in V$ there exists a unique element $-\phi$ in V such that $\phi + (-\phi) = 0$

2) There is an operation, called "multiplication by a

complex number ", by which any complex number α and $\phi \varepsilon V$ can be combined to yield a unique element $\alpha\phi$ in V. Moreover, the following properties are satisfied for every choice of $\phi \varepsilon V$ and complex numbers α and β.

2a $\alpha(\beta\phi) = (\alpha\beta)\phi$

2b) $1\phi = \phi$ (1 denotes the number one)

3 In addition, the following distributive laws must be fulfilled.

3a $\alpha(\phi + \psi) = \alpha\phi + \alpha\psi$

3b $(\alpha + \beta)\phi = \alpha\phi + \beta\phi$

Subspace

A subset U of linear space V is called a linear subspace (or simply a subspace) of V if for every ϕ and ψ in U and for every complex number α, $\phi + \psi$ and $\alpha\phi$ are both in U.

1.3 TOPOLOGICAL VECTOR SPACE

Suppose T is topology on a vector space χ such that

a) Every point of χ is closed set

b) The vector space operations are continuous with respect to T.

Under these conditions, T is said to be vector topology on χ and X is called topological vector space.

Condition (a) and (b) together imply that T is a Hausdorff topology so that every topological vector space is Hausdorff space. Also is locally convex if there is a local base B whose members are convex. [A set $C \subset X$ is said to be convex if $tc + (1-t)C \subset C$, $\forall\ 0 \leqslant t \leqslant 1$)

1.4 COMPACT SET AND SUPPORT

A set k in R^n or C^n is said to be compact if every open covering of k contains a finite subcollection which also converse K. A subset of R^n or C^n is compact if and only if it is closed.

The support of a continuous function f(t) defined on some open set I in R is the closure with respect to I of the set of points t, where $f(t) \neq 0$.

1.5 CONVENTIONAL FUNCTION

By a coventional function we mean a function whose domain is contained in R^n or C^n and whose range is either R^1 or C^1 not necessarily respectively

1.6 SMOOTH FUNCTION

A conventional function is said to be smooth (or infinitely smooth) if all its derivatives of all orders exist and are continuous at all points of its domain.

1.7 LOCALLY AND QUADRATICALLY INTEGRABLE FUNCTION

Let I be open set in R^n .By locally integrable function on I we mean a conventional function that is Lebesgue integrable on every open set J in R^n whose closure $\bar{J}$ is a compact subset of I. $L_p(I)$ denotes the collection of all locally integrable function f on I satisfying

$$\int_I |f(t)|^p \, dt < \infty \quad , \; 1 \leqslant P < \infty$$

If P = 2, $f \in L_2$ (I) is called quadratically integrable function on I.

1.8 SECTIONALLY OR PIECEWISE CONTINUOUS FUNCTION

A function $f(x,y)$ is said to be piecewise or sectionally continuous in domain D. If the domain D can be partitioned into finite number of nonintersecting subdomains $D_1, D_2, \ldots D_n$. In each of which the function $f(x,y)$ is continuous and has finit limits as (x,y) approches the boundaries of each subdomain.

1.9 INTEGRAL TRANSFORMS

A function $F(s)$, where s is real or complex, expressed in the form of the convergent integrals

$$F(s) = \int_{-\infty}^{\infty} K(s,t)\, f(t)\, dt < \infty$$

is called integral transformation of function $f(t)$. The function $K(s,t)$ in the integrand is called Kernel of the transformation. Different forms of kernel $k(s,t)$ and the range of integration give rise to different integral transforms; such as Fourier, Laplace, Mellin, Hankel etc transforms. The theory of integral transformation is used for a systematic study of certain types of boundary value problems depending upon the initial and boundary conditions. A variewty of integral transforms suitable to the needs of the physical problem have been constructed by choosing proper kernels and applied in solving the boundary value problems.

$$\text{If } k(s,t) = 0 \quad , t < 0$$
$$= e^{-st} \quad t \geqslant 0$$

we get what is called the Laplace transform of the function f(t) given by

$$F(s) = \int_0^{\infty} e^{-st} f(t)\,dt$$

If $K(s,t) = 1/\sqrt{2\pi}\ e^{ist}$, $-\infty < t < \infty$

we get Fourier transform given by

e^{ist}

$$F(s) = 1/\sqrt{2\pi} \int_{-\infty}^{\infty} e^{-st} f(t)\,dt \qquad \ldots..(1.9.2)$$

If $K(s,t) = 0$, $t < 0$

$= t^{s-1}$, $t \geqslant 0$

we get Mellin transform given by

$$F(s) = \int_0^{\infty} t^{s-1} f(t)\,dt \qquad \ldots.(1.9.3)$$

the choice of kernel is largely determined by the form of the differential equation whose solution is sought. The extension of integral transform tofinite intervals have been first suggested by Doetsch (1935) [26,p.424] in the case of trigemetric kernels. Defining the finite Fourier sine transform of the function f(x), $0 < x < \pi$ by the equation

$$F_s(n) = \bar{f}_s(n) = \int_0^{\pi} f(x) \sin(n x)\,dx \qquad \ldots.(1.9.4)$$

Doetsch pointed out that the ivnersion formula

$$f(x) = 2/\pi \sum_{n=1}^{\infty} \bar{f}_s(n) \sin(nx) \qquad \ldots.(1.9.5)$$

is an immediate consequence of well-known theorem in the theory of Fourier series, Extending these ideas to two dimensions the finite double Fourier sine transforms of f(x,y) becomes

$$F_{ss}\ (m,n) = \overline{f}_{ss}\ (m,n) = \int_0^{\pi}\int_0^{\pi} f(x,y)\ \mathrm{Sin}(mx)\ \mathrm{Sin}(ny)\ dxdy \qquad \ldots.(1.9.6)$$

And the inversion is given by the double Fourier series

$$f(x,y) = 4/\pi^2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} F_{ss}(m,n)\mathrm{Sin}(mx)\mathrm{Sin}(ny) \qquad \ldots.(1.9.7)$$

As has been stated earlier, there are several problems which need repeated as well as different types of transforms for their solutions, one such transform is the finite Fourier cosine sine transform which is givne by

$$f_{cs}\,[f(x,y);\ (x,y)\text{---}\ (m,n)] = \overline{f}_{cs}\,(m,n) = F_{cs}\,(m,n)$$

$$= \int_0^{\pi}\int_0^{\pi} f(x,y)\ \mathrm{Cos}\ (mx)\ \mathrm{Sin}\ (ny)dxdy \qquad \ldots.(1.9.8)$$

And inversion formula is given by

$$f(x,y) = 2/\pi^2 \sum_{n=1}^{\infty} F_{cs}(o,n)\mathrm{Sin}(ny) + 4/\pi^2 \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} F_{cs}\,(m,n)\mathrm{Cos}(mx)\mathrm{Sin}(ny)$$

$$\ldots.(1.9.9)$$

The history of transform technqiue begins with the last century. The direct application of the transforms replaces the earlier symbolic method known as Heaviside's Operational calculus which goes by he name of the English electrical engineer (1850-1915). Laplace (1749-1827) and Cauchy (1789-1857) were two of the earlier contributers to this subject ; latter studies by Bromwich, Carson and Vanderpol placed the Heaviside calculus on on a sound footing. G.Doetsch (1935) unified the work of the above mathematicians. However, the problems involving several variables can not be solved by the use of only one transform. Therefore the generalization of one transform and of other transforms have been made and applied successfully by many mathematicians, some applications of the repeated use of the transforms are given by Sneddon [26]. There are several problems which are solved by the repeated applications of the same transforms one such transformation is the finite double Fourier transforms. In majority of the books which deal with the Fourier transforms, the transform is derived by first setting up the Fourier series and then showing how it can be extended so that it goes over to the iintegral form of Fourier transform.

1.10 SEMINORM AND MULTINORM

Let V be a linear space. A seminorm on V is a rule that assigns a real number $\gamma(\phi)$ to each $\phi \in V$ and that satisfies the following axioms :

i) $\gamma(\alpha\phi) = |\alpha| \gamma(\phi)$

ii) $\gamma(\phi+\psi) \leqslant \phi(\phi) + \gamma(\psi)$

γ

where ϕ and ψ are arbitrary elements of V and α is any complex number.

If in addition $\gamma(\phi) = 0 \Rightarrow \phi = 0$ (i.e. ϕ is zero element of V) then γ is called norm on V.

The collection $S = \{\gamma_\nu\}_{\nu \varepsilon A}$ of seminorms (Here A denotes any finite or infinite index set) on linear space V is said to be separating (or S separates V), if for every $\phi \neq 0$ in V, there is atleast on γ_ν such that $\gamma_\nu(\phi) \neq 0$. In other words S is separating if only the zero element in V has the number zero assigned to it by every seminorm in S.

A separating collection of seminorms is called multinomr. Obviously sufficient condition for S to be multinorm is that at least one of the seminorm is norm.

If S is a countable separating collection of seminorms it is caled countable multinorm.

1.11 COUNTABLE MULTINORMED SPACE

Let $S = \{\gamma_\nu\}_{\nu \varepsilon A}$ be a set seminorms on V, which need not separate V. Given any nonvoid finite subset $\{\gamma_{\nu_k}\}_{k=1}^{n}$ of S and arbitrary positive numbers $\varepsilon_1 \varepsilon, \varepsilon_2 \ldots\ldots \varepsilon_n$, a ballon centered at ψ, where ψ is fixed point in V, is defined as the set of all $\phi \varepsilon$ V such that

$$\gamma_{\nu_k}(\phi - \psi) < \varepsilon_k \ , \ k=1,2,\ldots\ldots n$$

Clearly, the intersection of two balloons centered at the same point ψ is also a balloon centered at ψ .

A neighborhood in V is any set in V that contains a balloon, and a neighbourhood of $\psi \varepsilon V$ is any set that contains a balloon centered at ψ .A neighbourhood of the origin 0 is called the neighborhood of zero. We shall consider the topology generated by the collection of all neighborhoods in V as the topology of V.

A multinormed space V is a linear space having topology generated by a multinorm S (i.e. by a separating collection of seminorm); If S is countable, V is called countably multinormed space.

Let V be countably multinormed space with countable multinorm S.A sequence $\{ \phi_\nu \}^{\infty}_{\nu=1}$ in V is said to converge to a limit ϕ in V, if and only if

$\gamma (\phi_\nu - \phi) \longrightarrow 0$ as $\nu \longrightarrow \infty$, for every $\gamma \varepsilon S$

A sequence $\{ \phi_\nu \}^{\infty}_{\nu=1}$ in V is said to be Cauchy sequence in V if and only if for each $\gamma \varepsilon S$, $\gamma(\phi_\nu - \phi_\mu) \rightarrow 0$, as ν and μ tends to infinity independently.

Every convergent sequence in countably multinormed space V is a Cauchy sequence, but the converse is not necessarily true. But when it is true (i.e. when every Cauchy sequence in V is convergent) V is called a complete space. A complete countably multinormed space is called Fréchet space.

Let M be a subset of multinormed space V; $\phi \in V$ is called a contact point of M if every neighborhood of intersects M. The set obtained by addition of M all of its contact points that are not already in M is called the closure of M and is denoted by $\bar{M}$. When $\bar{M} = V$,M is said to be dense in V. Obviously, M is dense in V if, for each $\phi \in V$, there exists a sequence $\{\phi_\nu\}_{\nu=1}^{\infty}$ of elements in M which converges in V to ϕ. The converse of this assertion is not true in general.

In the linear space V let T_1 and T_2 denote two topologies generated by respectively two different multinorms $R = \{\rho_\mu\}_{\mu \in A}$ and $S = \{\gamma_\mu\}_{\mu \in B}$. The topology T_1 is said to weaker than the topology T_2 , and T_2 is said to be stronger than T_1 , if every T_1 neighborhood is also T_2 neighborhood. We write $T_1 \subset T_2$.

1.12 COUNTABLE UNION SPACE

Let U be a linear subspace of multinormed space V. Also let S denotes the multinorm on V. Clearly, S is also a multinorm on U. The topology generated in U by S is called the induced topology in U by V.

Let $\{V_m\}_{m=1}^{\infty}$ be a sequence of countably multinormed spaces such that $V_1 \subset V_2 \subset V_3 \subset \ldots$.Furthermore, assume that the topology of V_m is stronger than the topology induced on it by V_{m+1} Let V denote the union of these spaces:$V = \bigcup_{m=1}^{\infty} V_m$ The V is clearly linear space.

A sequence $\{\phi_\nu\}_{\nu=1}^{\infty}$ is said to converge in V to ϕ if all the ϕ_ν and ϕ belong to some particular V_m and $\{\phi_\nu\}_{\nu=1}^{\infty}$ converges to ϕ in V_m -(And therefore in V_{m+1}, V_{m+2}, ...as well). Under these circumstances V is called countable union space. Spaces of this type were introduced by Gelfand and Shilov [13,14].

The countable union space $V = \bigcup_{m=1}^{\infty} V_m$ is complete whenever all the V_m are complete countably multinormed spaces.

A countable union space $V = \bigcup_{m=1}^{\infty} V_m$ will be called a strict countable union space if for each m the topology of V_m is identical to the topology induced on V_m by V_{m+1}. Obviously V will certainly be a strict countable union space if the following condition is satisfied : If , for each $\{\gamma_{m,k}\}_{k=1}^{\infty}$ denotes the multinorm for V_m then

k.

$$\gamma_{m,k}(\phi) = \gamma_{m+1,k}(\phi) \text{ for all } \phi \in V_m \text{ , all } m \text{ and } k.$$

1.13 CONTINUOUS LINEAR FUNCTIONAL

Let V be a countably multinormed space. A rule that assigns a unique complex number to each $\phi \in V$ is called a functional. This complex number is denoted by $\langle f, \phi \rangle$

The functional f is said to be linear if for each $\phi, \psi \in V$ and every complex number α and β,

$$< f, \alpha\phi + \beta\psi > = \alpha < f, \phi > + \beta < f , \psi >$$

The functional f is said to be continuous at $\phi \in V$ if for each $\varepsilon > 0$, there exists a neighbourhood Ω of ϕ in V such that

$| < f, \psi > - < f, \phi > | < \varepsilon$, whenever $\psi \in \Omega$ and f is said to be simply continuous if and only if it is continuous at each and every point $\phi \in V$.

A functional f is continuous on a countable union space $V = \bigcup_{m=1}^{\infty} V_m$ if it is continuous at every V_m .

1.14 THE DUAL SPACE

The collection of all continuous linear functionals on a countably multinormed space or countable union space V is called the dual space of V and is denoted by V'.

Two members of f and g of V' are said to be equal in V', if for every $\phi \in V$, $< f \phi > = < g, \phi >$.

Also addition of two continuous linear functionals and multiplication by a complex number is defiend in V' by the following relation

a) $< f + g, \phi > = < f, \phi > + < g, \phi >$

b) $< \alpha f, \phi > = \alpha < f, \phi >$

where ϕ is in V. With these definitions V' becomes a linear space, the zero element in V' being the functional that assigns the number zero to every $\phi \in V$.

?

Assume that U and V are countably multinormed spaces with U being subspace of V. by the restriction of $f \in V'$ to U we mean that unique functional g on U defined by

$\langle f, \phi \rangle = \langle g, \phi \rangle$, for every $\phi \in U$. Clearly g is a continuous linear functional on U. Hence the restriction of $f \in V'$ to U is a member of U'.

The dueal space V' of countably multinormed space V is assigned the topology generated by the multinorm $\{\xi_\phi\}_{\phi \in V}$, where ξ_ϕ in V' is defined by $\xi_\phi(f) = |\langle f, \phi\rangle|$. This topology is he weak topology on V'.

A sequence $\{f_\nu\}_{\nu=1}^{\infty}$ in a dual space V' of countably multinormed space V is convergent if and only if there is $f \in V'$ such that for every $\phi \in V$ $\langle f_\nu - f, \phi\rangle \longrightarrow 0$ as $\nu \longrightarrow \infty$

A sequence $\{f_\nu\}_{\nu=1}^{\infty}$ is called Cauchy sequence in V' if and only if for each $\phi \in V$ $\langle f_\nu - f_\mu, \phi \rangle \longrightarrow 0$, as ν and μ tends to infinity independently.

It V is a complete space, then V' is also a complete space. In case of countable union space $V = \bigcup_{m=1}^{\infty} V_m$, if all V_m are complete countably multinormed spaces, then V is so, and therefore V' is also complete.

1.15 TESTING FUNCTION SPACE AND GENERALIZED FUNCTIONS

Let I be an open subset of R^n or C^n ; where C^n is the complex n-dimensional Educlidean space. A set V(I) is said to be a testing function space on I if the following conditions are satisfied.

i) V(I) consists entirely of smooth complex valued functions defined on I

ii) V(I) is either a complete countably multinormed space or a complete countable union space.

iii) If sequence $\partial \{\phi_\nu\}_{\nu=1}^{\infty}$ converges in V(I) to zero then for every non-negative integer k in R^n, $\{D^k \phi_\nu\}_{\nu=1}^{\infty}$ converges to the zero function uniformly on every compat subset of I

2

where $D^k = \dfrac{\partial^{k_1+k_2+\ldots+k_n}}{\partial x_1^{k_1}\, \partial x_2^{k_2} \ldots \partial x_n^{k_n}}$, $|k|=k_1+k_2+\ldots k_n$

A generalized function on I is any continuous linear functional on any testing space on I. Thus f is called a generalized function, if it is a member of the dual space V'(I) of some testing function space V(I).

1.16 OPERATORS ON TESTING FUNCTION SPACES AND THEIR DUALS :

Let U and V be both countably multinomred or countable union spaces. A mapping R from U to V is called an operator.

Thus an operator from U to V is a rule R that assigns precisely one element in V to each element in some subset of U.

An operator R from U into V will be linear if

$R(\alpha\phi + \beta\psi) = \alpha R\phi + \beta R\psi$, $\forall \phi, \psi \in U$ and $\forall \alpha, \beta \in C$

A linear operator R from U to V is continuous if and only if $R\,\phi_\nu \to 0$ in V, whenever $\phi_\nu \to 0$ in U as $\nu \to \infty$.

Let U' and V' be both dual spaces of countably multinormed or countably union spaces U and V respectively. A mapping R' from dual space V' into the dual space U' is called linear if for every f, g ε V' and for every complex number α , β

$$< R'(\alpha f + \beta g), \phi > = \alpha < R'f, \phi > + \beta < R'g, \phi > \text{ ,for all } \phi \in U.$$

from A linear mapping R' and V' into U' is continuous if and only if $R'f_\nu \to 0$ in U', whenever $f_\nu \to 0$ in V' as $\nu \to \infty$

The adjoint operator R' from the dual space V' into U' is defined by

$$< R'f\ \phi > = < f, R\ \phi >, \text{ for all } \phi \in U,\ f \in V'$$

Here R'f is that functional on U which assigns to each $\phi \in U$ the same numbers that $f \in V'$ assigns to $R\,\phi \in V$.

The adjoint operator R' is continuous linear mapping of V' into U', if R is a continuous linear mapping of U into V .

1.17 DISTRIBUTIONS

Let I be a nonempty open set in R^n and K be a compact subset of I. D_k (I) is the set of all complex vaued smooth functions defined on I which vanish at those points of I, that are not in $K.D_k$ (I) is a linear space under the usual definition of addition of functions and their multiplication by complex number. The zero element in D_k (I) is the identically zero function on I.

For each non-negative integer k in R^n we define seminorm γ_k on D_k (I) by

$$\gamma_k(\phi) = \sup_{t \in I} | D^k \phi(t) \; ; \; \phi \in D_k(I)$$

Then γ_0 will be norm on D_k (I) so that $\{\gamma_k\}_{k=0}^{\infty}$ is a countable multinorm on $D_k(I)$. We assign to $D_k(I)$ the topology generated by $\{\gamma_k\}_{k=0}^{\infty}$ and thus D_k (I) is a countably multinormed space. Moreover D_k (I) is complete and hence a Fréchet space.

Let $\{k_m\}_{m=1}^{\infty}$ be a sequence of compact subsets of I with properties

i) $k_1 \subset k_2 \subset k_3 \subset \ldots$

ii) Each compact subset of I is contained in one of the k_m

Then $I = \bigcup_{m=1}^{\infty} k_m$ nd D_{k_m} (I) $\subset D_{k_{m+1}}$ (I) and topology of D_{k_m} (I) is stronger than the topology induced on it by $D_{k_{m+1}}$ (I). Now the strict countable union space D(I) is defined by

$D(I) = \bigcup_{m=1}^{\infty} D_{k_m}$ (I) and its dueal space is denoted by D'(I). The members of D'(I) are called distributions on I. Thus distribution is a continuous linear functional on D(I) space.

Every distribution is generalized function but not conversely. Generalized functions were introduced in science in 1927 as a result of Dirac researches into quantum mechanics. It was introduced in Mathematics by Sobolev [28] in 1936. While he was studying the uniqueness of the solution of Cauchy problem for linear hyperbolic equation. The theory of generalized functions also known as distribution theory was widely propogated by the work of L.Schwartz from 1944 onwards. Schwartz's [24] work on the extension of the Fourier transformation to distribution proved to be remarkably powerful tool for solving different types of problems in physics and engineering. Main advantage of generalized functions and distributions is that by widening the class of functions, many theorems and operations are freed from tedious restrictions.

1.18 THE SPACE E(I) AND ITS DUAL

E(I) is the space of all complex valued smooth functions on nonempty open set I of R^n. For each compact subset k of I and a non-negative integer $k \in R^n$ the seminorm $\gamma_{K,k}(\phi)$ on E(I) is defined by

$$\gamma_{K,k}(\phi) = \sup_{t \in K} |D^k \phi|, \text{ for all } \phi \in E(I).$$

Then, clearly E(I) is a multinormed space with the topology generated by the multinorm $\{\gamma_{K,k}\}$ where K traverses through the set of all compact subsets of I and k=0,1,2...in R^n. E(I) is also testing function space. The members of the dual space E'(I) of E (I) are called distributions with compact support.

By the definition of testing function space $V(I)$ it is clear that $D(I) \subset V(I) \subset F(I)$. Since $D(I)$ is dense in $E(I)$, $V(I)$ is also dense in $E(I)$. The convergence in $V(I)$ to some limit implies the convergence in $E(I)$ to the same limit. Further $E'(I) \subset V'(I) \subset D'(I)$. That is $E'(I)$ can be defined as subspace of $V'(I)$.

1.19 HILBERT SPACE

Let X be a linear space over the complex field C. An inner product on X is a function $(\cdot\,,\cdot)\, X \times X \longrightarrow C$ which satisfies the following conditions

i) $(x,y)=\overline{(y,x)}$, for all $x,y \in \chi$

ii) $(\alpha x + \beta y, z) = \alpha(x,z) + \beta(y,z)$, for all $x,y,z \in \chi$ and $\alpha, \beta \in C$

iii) $(x,x) \geqslant 0$; $(x,x)=0$ if and only if $x=0$

A complete normed linear space is called Banach space. Hilbert space is a complex Banach space whose norm arises from an inner product.

1.20 THE SPACE $L_2(I)$

Let I denote any open interval $a < x < b$ on the real line. Here $a= -\infty$ and $b= \infty$ are permitted. A function $f(x)$ is said to be quadraticaly integrable on I if it is a locally integrable function on I such that

$$\alpha_o(f) = \left[\int_a^b |f(x)|^2 \, dx\right]^{\frac{1}{2}} < \infty$$

The set of quadratically integrable functions can be partictioned into equivalence classes of such functions by stipulating that two such functions f and g are in the same class if and only if $\alpha_o(f-g)=0$. This is the case of and only if f=g almost everywnere on I. The resulting space of equivalence classes is denoted by L_2 (I). Moreover, the functional α_o is also defined on L_2(I)....

That is, the number that α_o assigns to any equivalence class is defined as the number that α_o assigns to any one for
of its member, this number being the same foor all members of a given class.

L_2(I) is a linear space whose zero element is the class of all functions that are equal to zero almost everywhere on I. Moreover, α_o is a norm on L_2(I) and is therefore a special case of a countable multinomr (the multinorm has only one element). L_2 (I) is assigned the topology generated by α_o. It turns out that L_2(I) is a complete space.

An inner product, which is arule assigning a complex number (f,g) to each ordered pair is defined by

$$(f,g)=\int_a^b f(x)\overline{g(x)}\quad dx \qquad f,g \in L_2(I)$$

where $\overline{g(x)}$ denotes the complex conjugate of g(x). Furthermore , the inner product is continuous with respect to each of its arguments; that is, if $f_m \longrightarrow f$ in L_2(I) as $m \longrightarrow \infty$,then

$(f_m ,g) \rightarrow (f,g)$ and $(g,f_m) \longrightarrow (g,f)$.

The dual of $L_2(I)$ is $L_2(I)$. That is for each continuous linear functional H on $L_2(I)$ there exists a unique member h of $L_2(I)$ such that $H(f)=(h,f)$, for every $f \in L_2(I)$; here $H(f)$ denotes the number that H assigns to f.

Moreover, we assume that there exists a sequence $\{\lambda_n\}_{n=0}^{\infty}$ of real numbers called eigen function of operator R and a sequence $\{\psi_n\}_{n=0}^{\infty}$ of smooth functions in $L_2(I)$ called eigen functions of R, such that $|\lambda_n| \to \infty$ as $n \to \infty$ and $R\psi_n = \lambda_n \psi_n$ $n=0,1\ldots$ We also assume that $\{\psi_n\}_{n=0}^{\infty}$ form a complete orthonormal system in $L_2(I)$. By orthonormality of the sequence $\{\psi_n\}_{n=0}^{\infty}$ one means that

$$(\psi_n \psi_m) \neq 0 \text{ if } n \neq m$$
$$= 1 \text{ if } n=m$$

The completeness of $\{\psi_n\}$ means that every $f \in L_2(I)$ can be expanded into the series

$$f = \sum_{n=0}^{\infty} (f, \psi_n)\psi_n \qquad (*)$$

which converges in $L_2(I)$, that is

$$\alpha_o \left[f - \sum_{n=0}^{N} (f, \psi_n)\psi_n\right] \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

we call (*) as the orthonormal series expansion of f with respect to $\{\psi_n\}_{n=0}^{\infty}$

1.21 GENERALIZED INTEGRAL TRANSFORMATIONS

Generalized Integral Transformation is the study of the combination of two mathematical branches. The theory of integral transforms and the theory of generalized functions. the extension of the Fourier transformation to the generalized function is the first in the line of extension of the classical transformations. The Fourier transform of a function of rapid growth was first defined as generalized function by L.Schwartz [24]. In 1966, A.H.Zemanian [36] extended Laplace transform to generalized function. He defined the Laplace transform F(s) of a generalized function f(t) to some class of functions directly as the application of the function f(t) to the Kernel e^{-st} viz.

$$F(s) = \langle f(t), e^{-st} \rangle$$

A brief history and recent developments of the theory of generalized integral transformations can be seen in the presidential address delivered by Prof.K.M.Saxena at the I.S.C.A. Ahmeadabad in 1978.

Many researchers like Bremermann [4] ,Erdélyi [12], Gelfand and Shilov [13,14], Kanwal [17], Lighthill [18], Temple [29] and Zemanian [34,35,36,37,38] have studied the extension of various types of integral transformations to generalized functions.

Chaudhary M.S. [6] extended finite Fourier double sine transform to generalized functions. Reddy B.P. and Chaudhary M.S. [23] extended Mathiew transform to generalized functions.

Following are the main approaches by which a classical transform is extended to a class of generalized functions or distributions.

i) Direct or Kernel Method

In this method a testing function space is constructed, Say $V(I)$ containing the Kernel $K(s,t)$ considered as a function of t defined on domain I. Then $V'(I)$ is the space of all continuous linear functionals on $V(I)$. I being the domain for functions $\phi(t) \in V(I)$. The members of $V'(I)$ are generalized functions. From the definition of generalized function, $f \in V'(I)$ assigns some fixed number to each $\phi \in V(I)$. This number is the complex number generally denoted by $\langle f, \phi \rangle$. As $k(s,t) \in V(I)$, $\langle f(t), K(s,t) \rangle$ possesses sense as an application of $f(t) \in V'(I)$ to $k(s,t) \in V(I)$. Here we have attached the argument 't' with f only to mean that $k(s,t) \in V(I)$ when it is considered as a function of t. The value $f(t)$,$k(s,t)$ then is a function of s say $F(s)$ so we write $F(s) = \langle f(t), k(s,t) \rangle$. This can be viewed as a generalized integral transformation because if f is a generallized function then in the Lebesgue sense.

$$F(s) = \int_I k(s,t)\, f(t)\, dt$$

This method of extension of an integral transforms to generalized function is very handy. Almost all the classical integral transforms are now-a-days extended to generalized functions by this approach.

ii) **Adjoint Method**

The second approach for extending integral transform to generalized functions is as follows.

A testing function space V(I) is constructed that does not necessarily contain the Kernel K(s,t); but is closed under the conventional integral transform. Then, the integral transform of generalized function belonging to V'(I) denoted by Vf say, where V'(I) is the dual of V(I) is defined by the Parseval type relation

$$< Vf ,\phi > = < f ,V \phi > \text{ fopr all } \phi \varepsilon\ V(I)$$

Schwartz used this method for extending the Fourier transform of a distribution of slow growth and Zemanian used this method for extending the Hankel transform to generalized function. More recently Chaudhary and Bhonsle [4] used both these approaches to extend the Laplace Hankel transformation to generalized function. Chaudhary M.S. in his Ph.D. thesis [5] extended the classical transform which have their Kernel as the product of the Kernel of different transforms to generalized function.

iii) Reduction of Kernel Method

In this method, first we reduce the integral transformation with Kernel K, to another transform with Kernel K_1 ,by suitable change of variable. Then it can be generalized by the first method. It's properties can be studied with the help of the corresponding study of transform with Kernel K_1 for generalized function.

iv) Transformations Arising from Orthonormal Series Expansions :

This method is of a somewhat different character than are the previous ones. This method is related to Hilbert-space techniques. And its prototype is the Fourier series expansion of a periodic distribution (Zammanian [36] Chapter IX). A procedure will be developed for expanding a generalized function f into a series of the form

$$f = \sum_{n=0}^{\infty} F(n)\psi_n \qquad (*)$$

where the ψ_n constitute a complete system of orthonormalfunctions and the F(n) are the corresponding Fourier coefficients of f.

This procedure leads to a whole new class of generalized integral transformations. The basic idea is to view the mapping $f \longrightarrow F(n)$ as a transformation μ from a certain class of generalized functions f into the space of functions F(n) mapping the integers into the complex plane. Then (*)

defines the inverse transformation; of course, the convergence of the series (*) must be interpreted in a generalized sense. ?
Moreover, the permissible orthonormal functions $_n$ will be eigenfunctions of a certain type of self-adjoint differential operator R. particular generalized integral transformations that are encompassed by this technique are the finite Fourier transformation.

In this dissertation we have extended finite Fourier cosine-sine transformation to distributions by using the above technique.