# CHAPTER - II

## CHAPTER-II

## DISTRIBUTIONAL FIITE FOURIER COSINE SINE TRANSFORM

### 2.1 INTRODUCTION

The concept of orthonormal series related to generalized functions is some what different than other usual concept of integral transformation of gneralized function. Expansion of certain schwartz distribution in the series of orthonormal functions were given by Zemanian A.H. and thereby he extended number of integral transformations to distributions [37]. By using the technique of Zemanian we have extended Finite Fourier Cosine-sine transform to distribution.

The method involved in this work is very much related to Hilbert space technique, and its prototype is the Fourier series expansion of a periodic distribution (generalized function) with finite period. A procedure will be developed for expanding a generalized function f into a series of the form

$$f = \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} F(m,n)\, \psi_{m,n} \qquad 2.1.1$$

where the $\psi_{m,n}$ constitute a complete system of Orthonormal functions. And F(m,n) are the corresponding Fourier coefficients of f. This procedure will leads to a whole new class of generalized integral transformations. The basic idea is to view the linear mapping $f \longrightarrow F(m,n)$ as a transformation $f_{cs}$ from certain class of generalized functions f into the space of functions F(m,n), mapping the integers into the complex plane. Then (2.1.1) defines the inverse transformation; of course; the convergence of series (2.1.1) must be interpreted in a generalized sense. Moreover, the permissible orthonormal functions $\psi_{m,n}$ will be eigen functions of self adjoint operator R. As a result, the corresponding transformation $f_{cs}$ wil' generate an operational calculus for solving differential equations involving the operator R.

We have extended Finite Fourier cosine sine transformation of classical functions f(x,y) to a class of generalized functions and proved inversion and uniqueness theorems.

- We shall define the Finite Fourier Cosine Sine Transformation of classical functions f(x,y) over the open rectangle I as follows :

2.2 DEFINITION

Let f(x,y) be a function of two independent variables x and y, which is defined on open rectangle I such that f(x,y) is continuous on I and its first order partial derivative is piecewise (sectionally) continuous on I where I is given by

$$I = \{ (x,y) : 0 < x < a, \quad 0 < y < b \}$$

Then, here we may treat f(x,y) as a function of y alone (i.e. considering temperarily f(x,y) as function of y) so that f(x,y) will posssess finite Fourier sine transform (with respect to y) which we may denoote by $F_s(x,n)$ defined by the equation

$f_s[f(x,y)\,;\, y \to n\}$.  ?

$$F_s(x,n) = f_s(x,y)\ ;\ y \to n\ ]$$

$$= \int_0 f(x,y)\ \text{Sin}\left(\frac{n\pi y}{b}\right)dy \qquad 2.2.1$$

Now $F_s(x,n)$ will itself have a finite Fourier cosine transform with respect to x which we denote by $F_{cs}(m,n)$ and is defined by the equation

$$F_{cs}(m,n) = f_c[f_s\ [f(x,y);\ y \to n];\ x \to m]$$

$$= f_c[F_s(x,n)\ ;\ x \to m\ ]$$

$$= f_c[f_s[f(x,y);\ y \to n\ ];\ x \to m] \qquad 2.2.2$$

$$= f_{cs}[f(x,y);(x,y) \to (m,n)]$$

$$= \int_0^a \int_0^b f(x,y)\ \text{Cos}\left(\frac{m\pi x}{a}\right)\text{Sin}\left(\frac{n\pi y}{b}\right)dxdy \qquad 2.2.3$$

Equation (2.2.3) defines the Finite Fourier cosine-sine transform of f(x,y).

### 2.2.3 INVERSION FORMULA

By applying inversion theorem for finite Fourier sine transformation to equsation (2.2.1) we obtain the relation

$$f(x,y) = 2/b \sum_{n=1}^{\infty} F_s(x,n)\ \text{Sin}\left(\frac{n\pi y}{b}\right) \qquad 2.3.1$$

Similarly by applying inversion theorem for finite Fourier cosine transformation to equation (2.2.2) we shall get the relation [27]

$$F_s(x,n) = 1/a \quad F_{cs}(0,n) + 1/a \sum_{m=1}^{\infty} F_{cs}(m,n) \operatorname{Cos}\left(\frac{m\pi x}{a}\right) \qquad 2.3.2$$

Now by using (2.3.2) in (2.3.1) we obtain

$$f(x,y) = 2/b \sum_{n=1}^{\infty} [1/a \ F_{cs}(0,n) + 2/a \sum_{m=1}^{\infty} F_{cs}(m,n) \operatorname{Cos}\left(\frac{m\pi x}{a}\right) \operatorname{Sin}\left(\frac{n\pi y}{b}\right)$$

That is

$$f(x,y) = 2/ab \sum_{n=1}^{\infty} F_{cs}(o,n) \operatorname{Sin}\left(\frac{n\pi y}{b}\right) + 4/ab \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} F_{cs}(m,n) \operatorname{Cos}\left(\frac{m\pi x}{a}\right) \operatorname{Sin}\left(\frac{n\pi y}{b}\right) \qquad 2.3.3$$

## 2.4 FINITE FOURIER COSINE SINE TRANSFORM OF DERIVATIVES

The finite Fourier sine transformation of derivative of $f(x)$ i.e. $\partial f/\partial x$ will be given by

$$f_s\left[\frac{\partial f}{\partial x}; x \rightarrow n\right] = \int_0^a \frac{\partial f}{\partial x} \operatorname{Sin}\left(\frac{n\pi x}{a}\right) dx$$

$$= \left[f(x) \quad \operatorname{Sin}\left(\frac{n\pi x}{a}\right)\right]_0^a - \int_0^a f(x) \quad \left(\frac{n\pi}{a}\right) \quad \operatorname{Cos}\left(\frac{n\pi x}{a}\right) dx$$

$$= [0-0] - \frac{n\pi}{a} F_c(n) . \qquad 2.4.1$$

Similarly finite Fourier cosine transformation of $\frac{\partial f}{\partial x}$ will be given by

$$f_c\left(\frac{\partial f}{\partial x}; x \rightarrow n\right] = \int_0^a \frac{\partial f}{\partial x} \operatorname{Cos}\left(\frac{n\pi x}{a}\right) dx$$

$$= \left[f(x) \operatorname{Cos}\left(\frac{n\pi x}{a}\right)\right]_0^a + \frac{n\pi}{a} \int_0^a f(x) \operatorname{Sin}\left(\frac{n\pi x}{a}\right) dx$$

$$= (-1)^n f(a) - f(0) + \frac{n\pi}{a} F_s(n) \qquad 2.4.2$$

If we assume at the boundary of $f(x)$ , $f(a)=f(a)=0$ then (2.4.2) gives

$$f_c\left[\frac{\partial f}{\partial x};\ x\to n\right] = \frac{n\pi}{a} F_s(n) \qquad 2.4.2a$$

The result of higher order derivatives may be established by repeated use of fundamental results (2.4.1) and (2.4.2) viz.

$$f_s\left[\frac{\partial^2 f}{\partial x^2};\ x \to n\right] = \int_0^a \frac{\partial^2 f}{\partial x^2}\operatorname{Sin}\left(\frac{n\pi x}{a}\right)dx$$

$$= \left[\frac{f}{x}\operatorname{Sin}\left(\frac{n\pi x}{a}\right)\right]_0^a - \frac{n\pi}{a}\int_0^a \frac{\partial f}{\partial x}\operatorname{Cos}\left(\frac{n\pi x}{a}\right)dx$$

$$= 0 - \frac{n}{a} f_c\left[\frac{\partial f}{\partial x};x\to n\right]$$

$$= -\left(\frac{n\pi}{a}\right)^2 F_s(n) + \frac{n\pi}{a}\left[(-1)^n f(a)+f(0)\right] \qquad 2.4.3$$

Similarly

$$f_c\left[\frac{\partial^2 f}{\partial x^2};\ x\to n\right] = \int_0^a \frac{\partial^2 f}{\partial x^2}\operatorname{Cos}\left(\frac{n\pi x}{a}\right)dx$$

$$= \left[\frac{\partial f}{\partial x}\operatorname{Cos}\left(\frac{n\pi x}{a}\right)\right]_0^a + \frac{n\pi}{a}\int_0^a \frac{\partial f}{\partial x}\operatorname{Sin}\left(\frac{n\pi x}{a}\right)dx$$

$$= (-1)^n f'(a) - f'(0) + \frac{n\pi}{a} f_s\left[\frac{\partial f}{\partial x};x\to n\right]$$

$$= (-1)^n f'(a) - f'(0) + \frac{n\pi}{a}\left[-\frac{n\pi}{a}F_c(n)\right]$$

$$= (-1)^n f'(a) - f'(0) - \left(\frac{n\pi}{a}\right)^2 F_c(n) \qquad 2.4.4$$

Now if we regard the function f as function of two independent variable x and y defined on the domain I i.e. open rectangle $I = \{ (x,y);\ 0 < x < a,\ 0 < y < b \}$. then the double transform of partial derivatives of a function f(x,y) may readily be written down as

$$f\left[\frac{\partial^2 f(x,y)}{\partial x^2};\ x \to m\right] = \frac{m\pi}{a}[f(0,y) + (-1)^{m+1} f(a,y)] - \left(\frac{m\pi}{a}\right)^2 F_s(m,y).$$

$$= \frac{m\pi}{a}[f(0,y) + (-1)^{m+1} f(a,y)] - \left(\frac{m\pi}{a}\right) f_s[f(x,y);\ x \to m$$

Now by taking Fourier cosine transform of above equation obtain

$$f_{cs}\left[\frac{\partial^2 f}{\partial x^2};(x,y)\ \theta_1(m,n)\right] = (m,n) - \left(\frac{m\pi}{a}\right)^2 F_{cs}(m,n) \qquad 2.4.5$$

where $\theta_1(m,n) = \frac{m\pi}{a}[f_c[f(0,y);y \to n] + (-1)^{m+1} f_c(f)(a,y);\ y \to n]]$

in particular if $f(0,y) = f(a,y) = 0,\ 0 < y < b$ then

$$f\left[\frac{\partial^2 f}{\partial x^2};\ (x,y) \to (m,n)\right] = -\frac{m^2\pi^2}{a^2} F_{cs}(m,n) \qquad 2.4.5a$$

Similar result hold for Fourier cosine sine transform of $\frac{\partial^2 f(x,y)}{\partial y^2}$ that is

$$\text{Since } f_s\left[\frac{\partial^2 f}{\partial y^2};\ y \to n\right] = \frac{n\pi}{b}[f(x,0) + (-1)^{n+1} f(x,b)]$$

$$-\frac{n^2\pi^2}{b^2} f_s[f(x,y);\ y \to n]$$

So that by taking Fourier cosine transform of above equation we have

$$f_{cs}\left[\frac{\partial^2 f}{\partial y^2} ; (x,y) \to (m,n)\right] = \theta_2(m,n) - \frac{n^2\pi^2}{b^2} F_{cs}(m,n) \qquad 2.4.6$$

$\theta_2$ where $\theta_2(m,n) = \frac{n\pi}{b}\left[f_c[f(x,0); x \to m] + (-1)^{n+1} f_c[f(x,b); x \to m]\right]$

In particular if $f(x,0)=f(x,b)=0$, $0 < x < a$ then

$$f_{cs}\left[\frac{\partial^2 f}{\partial y^2} ; (x,y) \to (m,n)\right] = -\frac{n^2\pi^2}{b^2} F_{cs}(m,n) \qquad 2.4.6a$$

From result (2.4.5a) and (2.4.6a) we deduce that

$$f_{cs}\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f(x,y) ; (x,y) \to (m,n)\right] = \theta_1(m,n) + \theta_2(m,n) - \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) F_{cs}(m,n) \qquad 2.4.7$$

In particular if $f(x,y)$ vanish along the boundaries of the open rectangle $I = \{(x,y): 0 < x < a \;,\; 0 < y < b\}$ we have

$$f_{cs}\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f(x,y) ; (x,y) \to (m,n)\right] = -\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) F_{cs}(m,n) \qquad 2.4.8$$

## 2.5 THE SPACE $L_2(I)$

Let I denotes the open rectangel given by $I = \{(x,y): 0 < x < a , 0 < y < b\}$ . A function $f(x,y)$ is said to be quadratically integrable on I if it is a locally integrable function on I such that

$$\alpha_o(f) = \left[\int_o^a \int_o^b |f(x,y)|^2 \, dxdy\right]^{\frac{1}{2}} < \infty \qquad 2.5.1$$

The set of all quadratically integrable functions can be partitioned into equivalence classes of such functions by stipulating that two such functions f and g are in the same class if and only if $\alpha_o(f-g)=0$. This is the case if and only if f=g almost everywhere on I. The resulting space of equivalance classes is denoted by $L_2(I)$. It is customary to speak of $L_2(I)$ as space of functions, even though it is actually a space of equivalence classes.

The functional $\alpha_o$ is also defined by (2.5.1) on $L_2(I)$; that is, the number that $\alpha_o$ assigns to any equivalence class is defined as the number that $\alpha_o$ assigns to any one of its members, this number being the same for all members of a given class.

$L_2(I)$ is a linear space whose zero element is the class of all functions that are equal to zero almost everywhere on I. Moreover, $\alpha_o$ is a norm on $L_2(I)$ and is therefore a special case of a countable multinorm (the multinorm has only one element). $L_2(I)$ is assigned the topology generated by $\alpha_o$. It turns out that $L_2(I)$ is a complete space.

An inner product, which is a rule assigning a complex number $(\phi, \psi)$ to each ordered pair $\phi$, $\psi$ of elements in $L_2(I)$, is defined by

$$(\phi, \psi) = \int_0^a \int_0^b \phi(x,y)\,\overline{\psi(x,y)}\;dxdy \qquad 2.5.2$$

where $\overline{\psi(x,y)}$ denotes the complex conjugate of $\psi(x,y)$. It possesses the following properties

i) $(f+h,g) = (f,g) + (h,g) \quad f,g,h \in L_2(I)$

ii) $(\beta f,g) = (f,\beta g) = \beta(f,g)$, here $\beta$ denotes complex number

iii) $(f,g) = \overline{(g,f)}$

iv) $(f,f) = [\alpha_o(f)]^2 \geqslant 0$

And if $(f,f) = 0$, then f is the zero element in $L_2(I)$.

of

Furthermore, the inner product is continuous with respect to each off its arguments, that is if $f_m \to f$ in $L_2(I)$ as $m \to \infty$ then

$m \to \infty$

$(f_m, g) \to (f,g)$ and $(g, f_m) \to (g,f)$

Also Schwarz inequality hold in $L_2(I)$. That is $|(f,g)| \leqslant \alpha_o(f)\,\alpha_o(g)$ The dueal of $L_2(I)$ is $L_2(I)$. That is for each continuous linear functional H on $L_2(I)$ there exists a unique member h of $L_2(I)$ such that $H(f)=(h,f)$ for every $f \in L_2(I)$ where $H(f)$ denotes the number that H assigns to f.

Let R denote the differential operator given by

$$R = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \qquad 2.5.3$$

Also let $\psi_{m,n}(x,y)$ be defined by

$$\psi_{o,n}(x,y) = \frac{1}{\sqrt{ab}} \operatorname{Sin}\left(\frac{n\pi y}{b}\right)$$

$$n = 1,2,3,.... \qquad 2.5.4$$

$$\psi_{m,n}(x,y) = \frac{2}{\sqrt{ab}} \operatorname{Cos}\left(\frac{m\pi x}{a}\right) \operatorname{Sin}\left(\frac{n\pi y}{b}\right)$$

$$m,n = 1,2,3,....$$

Then there exists a sequence $\{ \psi_{m,n} \}_{m,n=0}^{\infty}$ of real numbers called eigen values of differential operator R. And a sequence $\{\psi_{m,n}\}_{m,n=0}^{\infty}$ of smooth functions in $L_2(I)$ called eigen fnctions of R, such that $|\psi_{m,n}| \to \infty$ as m and $n \to \infty$.

And

$$R\,\psi_{m,n} = \lambda_{m,n}\,\psi_{m,n}$$

$$m,n=0,1,2,\ldots \qquad 2.5.5$$

where

$$\lambda_{m,n} = -\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) \qquad 2.5.6$$

**LEMMA 2.5.1**

$\{ \lambda_{m,n} \}_{m,n=0}^{\infty}$ is an orthonormal system of eigen functions of operator R.

**Proof**

To prove the lemma we shall show that,

$$(\psi_{m,n}, \psi_{p,q}) = 1 \quad \text{if } (m,n) = (p,q)$$

$$= 0 \text{ if } (m,n) \neq (p,q)$$

**Case(i)**

If $(m,n) = (p,q)$ then the inner product gives

$$(\psi_{m,n}, \psi_{p,q}) = (\psi_{m,n}\,\psi_{m,n}) = \int_0^a \int_0^b \psi_{m,n}\,\psi_{m,n}\,dxdy$$

$$= \int_0^a \int_0^b \frac{2}{\sqrt{ab}} \cos\frac{m\pi}{a}x \sin\frac{n\pi y}{b} \frac{2}{\sqrt{ab}} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\,dxdy.$$

$$= \frac{4}{ab}\int_0^a \int_0^b \cos^2\frac{m\pi x}{a}\,\sin^2\frac{n\pi y}{b}\,dx\,dy$$

$$= \frac{4}{ab} \int_0^b \int_b^a 1/2[1+ \cos \frac{2m\pi x}{a}]\, dx[1/2[1- \cos \frac{n\pi y}{b}]dy$$

$$= \frac{1}{ab} \int_0^b [x + \frac{\frac{2m\pi x}{a}}{\frac{2m\pi}{a}}]_0^a [1- \cos \frac{2n\pi y}{b}]\, dy$$

$$= \frac{1}{ab} \int_0^b [a-0]\ [1- \cos \frac{2n\pi y}{b}]\ dy$$

$$= \frac{1}{ab} [y - \frac{\sin \frac{2n\pi y}{b}}{\frac{2n\pi}{b}}]_0^b$$

$$= \frac{a}{ab} [b - 0]$$

$$= \frac{ab}{ab} = 1$$

**Case (ii)**

If $(m, n) \neq (p,q)$ then

$$(\psi_{m,n}, \psi_{p,q}) + \frac{4}{ab} \int_0^a \int_0^b [\cos \frac{m\pi x}{a} \cos \frac{p\pi x}{a}]$$

$$[\sin \frac{n\pi y}{b} \sin \frac{q\pi y}{b} (\sin \frac{q\pi y}{b}]\, dxdy$$

$\pi$

$$= \frac{4}{ab} [\int_b^a 1/2(\cos (\frac{m+p}{a}) x + \cos (\frac{m-p}{a}) x]dx] \quad \pi$$

$$[\sin \frac{n\pi y}{b} \sin \frac{q\pi y}{b}]\ dy$$

$$= 2/ab \int_0^b \left[ \frac{\sin (\frac{m+p}{a})\pi x}{(\frac{m+p}{a})\pi} + \frac{\sin (\frac{m-p}{a})\pi x}{(\frac{m-p}{a})} \right]_0^a \sin \frac{n\pi y}{b}$$

$$\sin \frac{q\pi y}{b}\ dy$$

$$= 2/ab \int_0^b [0-0] \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{q\pi y}{b}\right) dy$$

$$= 2/ab \int_0^b [\,0\,] \, dy$$

$$= 0$$

Hence by definition of orthonormality our assertion follows

Again by completeness of $\{\psi_{m,n}\}_{m,n=0}^{\infty}$ means that every $f \in L_2(I)$ can be expanded into the series

$$f = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (f, \psi_{m,n}) \, \psi_{m,n} \qquad 2.5.7$$

which converges in $L_2(I)$ , that is

$$\alpha_o \left[f - \sum_{m=o}^{N} \sum_{n=o}^{N} (f, \psi_{m,n}) \, \psi_{m,n}\right] \to 0 \text{ as } N \to \infty \qquad 2.5.8$$

we call (2.5.7) the orthonormal expansion of f with respect to $\{\psi_{m,n}\}_{m,n=0}^{\infty}$

Moreover, for given operator R there may be more than one complete orthonormal system of given functions.

An important classical result states that $\{\psi_{m,n}\}$ is complete if and only if, for every $f \in L_2(I)$, the coefficients $(f, \psi_{m,n})$ satisfy Parseval's equation.

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |(f, \psi_{m,n})|^2 = \int_o^a \int_o^b |f,(x,y)|^2 dxdy = [\alpha_o(f)]^2$$

2.5.9

RIESZ FISHER THEOREM

Let $\{\psi_{m,n}\}$ be a complete orthonormal system, and let $\{C_{m,n}\}_{m,n=0}^{\infty}$ be a sequence of complex numbers such that $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |C_{m,n}|^2$ converges, then there exists a unique $f \in L_2(I)$ such that $C_{m,n} = (f, \psi_{m,n})$ consequently

$$f = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{m,n} \psi_{m,n}.$$

## 2.6 THE TESTING FUNCTION SPACE S(I)

Now we shall construct a testing function space S(I) which depend upon the choice of the domain I, the differential operator R and complete orthonormal system $\{\psi_{m,n}\}_{m,n=0}^{\infty}$ of eigen functions. It's dual is a space of generalized functions, each of which can be expanded into a series of the eigen functions $\psi_{m,n}$.

S(I) consists of all functions $\phi(x,y)$ that possesses the following properties.

i) $\phi(x,y)$ is defined, complex valued and smooth on I

ii) For each nonnegative integer k

$$\alpha_k(\phi) = \alpha_o(R^k \phi) = \left[\int_o^a \int_o^b |R^k \phi(\vec{x},y)|^2 dx\,dy\right]^{\frac{1}{2}} < \infty \qquad 2.6.1$$

iii) For nonnegative integers m,n and k

$$(R^k \phi, \psi_{m,n}) = (\phi, R^k \psi_{m,n}) \qquad 2.6.2$$

where R denotes the differential operator

$$R = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \qquad 2.6.3$$

And

$$\psi_{m,n}(x,y) = \frac{2}{\sqrt{ab}} \operatorname{Cos}\left(\frac{m\pi x}{a}\right) \operatorname{Sin}\left(\frac{n\pi y}{b}\right)$$

for each $m,n = 1,2,3$ 2.6.4

$$\psi_{o\,n}(x,y) = \frac{1}{\sqrt{ab}} \operatorname{Sin}\left(\frac{n\pi y}{b}\right)$$

for each $n=1,2,3,\ldots$

And

$(R^k\phi, \psi_{m,n})$ is an inner product in $L_2(I)$

Then both sides of equation (2.6.2) exists because in view of Lemma 2.5.1 and by Schwartz inequality we have

For all $\phi\ (x,y) \in S(I)$

$$\int_o^a \int_o^b \left| \psi_{m,n} R^k\phi \right| dxdy \leqslant \left[\int_o^a \int_o^b \psi^2_{m,n} dxdy\right]^{\frac{1}{2}} \left[\int_o^a \int_o^b \left|R^k\phi\right|^2 dxdy\right]^{\frac{1}{2}}$$

$$\leqslant 1.\ \alpha_k(\phi)$$

$$< \infty \qquad 2.6.5$$

And

$$\int_o^a \int_o^b \left| \phi R^k \psi_{m,n} \right| dxdy = \left| \lambda^k_{m,n} \right| \int_o^a \int_o^b \left|\phi\psi_{m,n}\right| dxdy$$

$$< \left|\lambda^k_{m,n}\right| \alpha_o(\phi)$$

$$< \infty \qquad 2.6.6$$

Moreover, under the pointwise addition of functions and their multiplication by complex number S(I) is a linear space.

**LEMMA 2.6.1**

Every $\psi_{m,n}(x,y)$ is a member of S(I) for each pair of nonnegative integers (m,n),

for all $m,n = 0,1,2\ldots.,$ and $(x,y) \in I$

PROOF

Since each $\psi_{m,n}(x,y)$ is complex valued smooth function on I and $R^k\psi_{m,n} = \lambda^k_{m,n}\ \psi_{m,n}$

we have

$$[\alpha_k(\psi_{m,n})]^2 = \int_0^a \int_0^b |R^k \psi_{m,n}|^2 dxdy$$

$$= \lambda^{2k}_{m,n} \int_0^a \int_0^b |\psi_{m,n}|^2 dxdy$$

$$= \lambda^{2k}_{m,n} .1 \qquad \text{(Since } \{\psi_{m,n}\} \text{ is orthonormal)}$$

$$= \lambda^{2k}_{m,n} < \infty$$

i.e. $\alpha_k(\psi_{m,n})$ exists for all m,n =0,1,2....

Finally for each nonnegative integers m,n and k if $(m,n)\neq(p,q)$,we have

$$(R^k\psi_{m,n}, \psi_{p,q}) = \lambda^k_{m,n}(\psi_{m,n}, \psi_{p,q}) = 0$$

$$= \lambda^k_{p,q}(\psi_{m,n}\ \psi_{p,q})$$

$$= (\psi_{m,n}, \lambda_{p,q}\ \psi_{p,q})$$

$$= (\psi_{m,n}, R^k\ \psi_{p,q})$$

And for $(m,n) = (p,q)$

$$(R^k\psi_{m,n}, \psi_{p,q}) = (\lambda^K_{m,n}, \psi_{m,n})$$

$$= (\psi_{m,n}, \lambda^K_{m,n}\ \psi_{m,n})$$

$$= (\psi_{m,n}, R^k\ \psi_{m,n})$$

Hence $\psi_{m,n}$ is member of S(I) for all m,n= 0,1,2,...

LEMMA 2.6.2

For every nonnegative integer k, $\{\alpha_k\}_{k=0}^{\infty}$ is multinorm on S(I).

PROOF

First we shall show that $\alpha_k$ is seminorm.

$\varphi(x,y) \in$

Now, for any S(I) and any constant $\beta \varepsilon C$ by equation (2.6.1) we have

i) $$\alpha_k(\beta\phi) = \left[\int_0^a \int_0^b | R^k \beta\phi \ (x,y)|^2 \, dx\, dy\right]^{\frac{1}{2}}$$

$$= |\beta| \left[\int_0^a \int_0^b | R^k \phi\ (x,y)|^2 \, dx\, dy\right]^{\frac{1}{2}}$$

$$= |\beta| \alpha_k(\phi)$$

ii) Again for any $\phi_1(x,y)$, $\phi_2(x,y)$ in S(I) we have

$$\alpha_k(\phi_1 + \phi_2) = \left[\int_0^a \int_0^b |R^k(\phi_1(x,y) + \phi_2(x,y)|^2 dxdy\right]^{\frac{1}{2}}$$

$$\leq \left[\int_0^a \int_0^b |R^k \phi_1(x,y)|^2 dxdy\right]^{\frac{1}{2}} +$$

$$+ \left[\int_0^a \int_0^b |R^k \phi_2(x,y)|^2 dxdy\right]^{\frac{1}{2}}$$

(By Minkowskis inequality)

$$= \alpha_k(\phi_1) + \alpha_k(\phi_2)$$

Thus $\alpha_k(\phi_1 + \phi_2) \leq \alpha_k(\phi_1) + \alpha_k(\phi_2)$, $\forall k, \phi_1, \phi_2$ S(I)

Thus (i) and (ii) imply that $\alpha_k$ is seminorm on S(I)

Now let $\{\alpha_k\}_{k=0}^{\infty}$ be a collection of seminorm on S(I). Then for every $\phi(x,y) \neq 0$ in S(I) there exists at least one $\alpha_k$ such that $\alpha_k(\phi) \neq 0$

In particular

$$\alpha_0(\phi) = \left[\int_0^a \int_0^b |\phi(x,y)|^2 \, dxdy\right]^{\frac{1}{2}} \neq 0$$

Then clearly $\alpha_0(\phi) = 0 \Rightarrow \phi = 0$ that is $\alpha_0$ is norm on S(I).

Thus at least one of the seminorm is norm the collection $m_{k=0}$ is separating therefore $\{\alpha_k\}_{k=0}^{\infty}$ is multinorm on S(I).

Furthermore, since $\{\alpha_k\}_{k=0}^{\infty}$ is countable separating collection of seminorm. Therefore $\{\alpha_k\}_{k=0}^{\infty}$ is countable multinorm on S(I) Thus S(I) is countably multinormed space.

LEMMA 2.6.3

S(I) is subspace of $L_2(I)$

PROOF

Since $L_2(I)$ is the class of quadratically integrable functions on I such that,

$$\alpha_0(f) = \left[\int_0^a \int_0^b |f(x,y)|^2 \, dxdy\right]^{\frac{1}{2}} < \infty \qquad (*)$$

Then clearly $L_2(I)$ is a linear space.

Moreover, $\alpha_0$ is norm on $L_2(I)$ and is therefore a special case of countable multinorm (i.e multinorm as only one element). Therefore

$L_2(I)$ is countably multinormed space. And we have shown in previous Lemma that S(I) is countably multinormed space.

Then to prove our assersion, we have to prove that $S(I) \subset L_2(I)$

Now by (2.6.1) for all $\phi(x,y) \varepsilon S(I)$ we have

$$\alpha_k(\phi) = \alpha_o(R^k\phi) = \left[\int_o^a \int_o^b |R^k\phi(x,y)|^2 \, dxdy\right]^{\frac{1}{2}} < \infty$$

Then from (*) it follows that,

$$R^k\phi \varepsilon L_2(I) \text{ for each } k \geqslant 0$$

In particular if $k=0$ then $R^k \phi = \phi$

So that $\phi(x,y) \varepsilon S(I) \Rightarrow \phi(x,y) \varepsilon L_2(I)$

That is $S(I) \subset L_2(I)$

Here S(I) is subspace of $L_2(I)$, when we identify each function in S(I) with the corresponding equivalence class in $L_2(I)$

**CONVERGENCE IN S(I)**

A sequence $\{\phi_\nu\}_{\nu=1}^{\infty}$ is said to converge in S(I) to $\phi$, if and only if for each nonnegative integer k,

$$\alpha_k(\phi_\nu - \phi) \to 0, \text{ as } \nu \to \infty$$

And a sequence $\{\phi_\nu\}_{\nu=1}^{\infty}$ is called a Cauchy sequence in S(I), if and only if for each nonnegative integer k,

$$\alpha_k(\phi_\nu - \phi_\mu) \to 0.$$

as $\nu$ and $\mu$ tends to infinity independently.

In the preceding lemma we shall show that every Cauchy sequence in S(I) is convergent which will imply that S(I) is complete. So that S(I) becomes complete countably multinormed space and hence a Frechet space.

**LEMMA 2.6.4**

S(I) is complete

**PROOF**

To prove this lemma we have to show that every Cauchy sequence in S(I) is convergent. Let R be the self adjoint operator given by $R = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

(see lemma 2.6.7. [16,p.187)

Then R is closed. It is convenient to regard S(I) as a countably Hilbert space with scalar product

$$(\phi\ \psi)_k = \sum_{h=0}^{k} (R^h \phi\ ,\ R^h \psi)$$

giving rise to the increasing set of seminorms.

$$||\phi||_k = \sqrt{(\phi\ ,\phi)_k}\ ,\ \text{on } S(I),$$

equivalent to the multinomr $\{\alpha_k\}_{k=0}^{\infty}$ on S(I)(see lemma 2.6.2) .

Indeed,

$$\alpha_k(\phi) \leq ||\phi||_k \leq k \max\{\alpha_1(\phi)\ ,\ \alpha_2(\phi)\ldots\ \alpha_k(\phi)\}$$

Therefore $\phi_m \to \phi$ in S(I) as $m \to \infty$ means that

$$||\phi - \phi_m||_k \to 0, \text{ as } m \to \infty \text{ and for each } k, k=0,1,,2\ldots$$

Thus if $\{\phi_m\}$ is Cauchy sequence in S(I) then for each k, $\{R^k \phi_m\}$ is a Cauchy sequence in $L_2(I)$. Then by sequential completeness of $L_2(I)$, there exists a function $\chi_k$ in $L_2(I)$ such that $R^k \phi_m \to \chi_k$ .

Similarly $R^{k+1} \phi_m \rightarrow \chi_{K+1}$, for all k since R is closed.

Then successive application of this result, for each $k=0,1,2,\ldots$, shows the existence of a function $\chi$ in S(I) such that $\phi_m \to \chi$ in S(I). To complete the proof we have to show that

$$(R^k \chi_o, \psi_{m,n}) = (\chi_o, R^k \psi_{m,n})$$

Indeed, $R^k \chi_o$ and $\chi_k$ are in the same equivalence class in $L_2(I)$. Again since the inner product is continuous with respect to the convergence in $L_2(I)$ of one of its arguments.

$$\begin{aligned}(R^k \chi_o, \psi_{m,n}) &= (\chi_k, \psi_{m,n}) \\ &= \lim_{m\to\infty} (R^k \phi_m, \psi_{m,n}) \\ &= \lim_{m\to\infty} (\phi_m, R^k \psi_{m,n}) \\ &= (\chi_o, R^k \psi_{m,n})\end{aligned}$$

Q.E.D.

Thus as shown in lemma 2,6,2 $\{\alpha_k\}_{k=o}^{\infty}$ is multinorm on S(I) . We assign to S(I) the topology generated by $\{\alpha_k\}_{k=o}^{\infty}$ and this makes S(I) a countably multinormed space. Moreover, S(I) is complete and therefore a Frénchet space. Under this formation S(I) turns out to be testing function space.

**LEMMA 2.6.5**

The differential operator R is linear and continuous mapping of S(I) into itself.

**PROOF**

The proof of this lemma follows directly from the definition of testing function space S(I). Because if $\phi \in$ (S(I) then R $\phi(x,y) \in S(I)$. Since S(I) consists of infinitely smooth functions therefore R is mapping from S(I) into S(I).

Also

$$\alpha_k (R\phi) = \left[ \int_0^a \int_0^b |R^k R\phi(x,y)|^2 dxdy \right]^{\frac{1}{2}}$$

$$= \alpha_{k+1}(\phi) < \infty$$

And it exists for any nonnegative integer k. Again,

$$R(\alpha\phi + \beta\Psi) = \alpha R(\phi) + \beta R(\psi) \quad \phi, \psi \in S(I)$$

and $\alpha\ \beta \in C$.

Hence R is linear.

Furthermore, if $\phi_\nu \to 0$ in S(I) then

$$\alpha_k(\phi_\nu) \to 0 \text{ in } S(I) \Longrightarrow \alpha_{k+1}(\phi_\nu) \to 0 \text{ in } S(I)$$

Thus $\phi_\nu \to 0$ in S(I) $\Longrightarrow R\phi_\nu \to 0$ in S(I)

Hence R is continuous.

Thus the mapping $\phi(x,y) \to R\ \phi(x,y)$ is linear and continuous mapping of S(I) into S(I).

**LEMMA 2.6.6**

If $\phi(x,y) \in S(I)$, then for $(x,y) \in I$

$$\phi(x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\phi(x,y), \psi_{m,n}(x,y))\, \psi_{m,n}(x,y)$$

where the series converges in S(I)

PROOF

In view of condition (ii) equation (2.6.1) of definition of S(I) , truely $R^k \phi$ is a member of $L_2$ (I) for each nonnegative integer k and for any $\phi(x,y) \varepsilon S(I)$. Hence we may expand $R^k \phi$ into a series of the orthonormal functions $\psi_{m,n}(x,y)$ . By using (2.6.2) and the fact that

$R^k \psi_{m,n} = \lambda^k_{m,n} \psi_{m,n}$, we obtain

$$
\begin{aligned}
R^k \phi &= \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} (R^k \phi , \psi_{m,n}) \psi_{m,n} \\
&= \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} (\phi , R^k \psi_{m,n}) \psi_{m,n} \\
&= \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} ( \lambda^k_{m,n} \psi_{m,n}) \psi_{m,n} \\
&= \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} (\phi , \psi_{m,n}) \lambda^k_{m,n} \psi_{m,n} \\
&= \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} (\phi , \psi_{m,n}) R^k \psi_{m,n} \qquad 2.6.7
\end{aligned}
$$

This series definitely converges in $L_2(I)$.

Consequently, for each k

$$\alpha_k [ \phi - \sum_{m=o}^{N} \sum_{n=o}^{N} (\phi , \psi_{m,n}) \psi_{m,n}]$$

$$\alpha_o [ R^k \psi - \sum^{N} \sum^{N} ( \phi , \psi_{m,n}) R^k \psi_{m,n}] \to 0 \quad \text{as } N \to \infty$$

This implies

$$[\phi - \sum_{m=o}^{N} \sum_{n=o}^{N} (\phi , \psi_{m,n}) \psi_{m,n}] \to o \text{ as } N \to \infty$$

Hence

$$\phi = \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} (\phi , \psi_{m,n}) \psi_{m,n} \quad \text{in } S(I)$$

Q.E.D

LEMMA 2.6.7

The operator R is self-adjoint on S(I)

PROOF

By virtue of (2.6.2) and (2.6.7) and the fact that the inner product is continuous with respect to each of its argument implies that R satisfies the relation

$$( R \phi_1 ,\phi_2) = ( \phi_1 , R\phi_2) \qquad 2.6.8$$

where $\phi_1(x,y)$ and $\phi_2(x,y)$ are arbitrary members of S(I). This shows that R is slef adjoint on S(I).

self

Indeed, for any $\phi_1$ , $\phi_2 \varepsilon$ S(I) and by equation (2.6.7) we have

$$(R\phi_1 ,\phi_2) = \int_o^a \int_o^b R \phi_1 \overline{\phi_2} \, dxdy$$

$$= \int_o^a \int_o^b \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} ( \phi_1 \psi_{m,n}) R \psi_{m,n} \overline{\phi_2} \, dxdy$$

$$= \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} ( \phi_1 , \psi_{m,n}) \int_o^a \int_o^b \overline{\phi_2} R \psi_{m,n} \, dxdy$$

$$= \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} ( \phi_1 , \psi_{m,n}) \int_o^a \int_o^b \psi_{m,n} \overline{R \phi_2} \, dxdy$$

$$= \int_o^a \int_0^b \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} ( \phi_1 , \psi_{m,n}) \psi_{m,n} \overline{R \phi_2} \, dxdy$$

$$= \int_0^a \int_0^b \phi_1 \overline{R \phi_2} \, dxdy$$

$$= (\phi_1, R\phi_2)$$

Q.E D

Now the next lemma gives a characterization of orthonormal series that converges in S(I).

**LEMMA 2.6.8**

Let $a_{m,n}$ denote complex numbers. then, for $(x,y) \in I$, the series $\sum_{m=o}^{\infty} \sum_{n=o}^{\infty} a_{m,n} \psi_{m,n}(x,y)$ converges in S(I) if and only if the series $\sum_{m=o}^{\infty} \sum_{n=o}^{\infty} |\lambda_{m,n}|^{2k} |a_{m,n}|^2$ converges for every nonnegative integer k.

**PROOF**

We employ the fact that $\psi_{m,n}$ form an orthonormal set to write

$$\int_0^a \int_0^b |R^k \sum_{m=q}^{N} \sum_{n=q}^{N} a_{m,n} \psi_{m,n}|^2 \, dxdy$$

$$= \int_0^a \int_0^b | \sum_{m=q}^{N} \sum_{n=q}^{N} a_{m,n} R^k \psi_{m,n}|^2 \, dxdy$$

$$= \int_0^b \int_0^b \sum_{m=q}^{N} \sum_{n=q}^{N} \sum_{r=q}^{N} \sum_{s=q}^{N} a_{m,n} \overline{a}_{rs} \lambda^k_{m,n} \lambda^k_{r,s} \psi_{m,n} \overline{\psi}_{rs} \, dxdy$$

$$= \sum_{m=q}^{N} \sum_{n=q}^{N} |\lambda_{m,n}|^{2k} |a_{m,n}|^2 \int_0^a \int_0^b \psi_{m,n} \overline{\psi}_{r,s} \, dxdy$$

$$= \sum_{m=q}^{N} \sum_{n=q}^{N} |\lambda_{m,n}|^{2k} |a_{m,n}|^2$$

Thus our assertion follows directly from this equation

THE DUAL SPACE S'(I) OF S(I)

Let S'(I) denotes the dueal of S(I), which consists of all continuous linear functionals on S(I). The members of S'(I) are called generalized functions. Instead of working with the number $< f, \phi>$ that $f\varepsilon$ S'(I) assigns to $\phi$ $\varepsilon$ S(I) is more convenient to work with the number that f assigns to the complex conjugate number of $\phi$, we write

$$(f,\phi) = < f, \overline{\phi} > \qquad 2.6.10$$

This is consistent with the inner produft c notation in $L_2$ (I). That is the use of notation (·· , ·· ) both as an inner product in $L_2$ (I), and for the number that f $\varepsilon$ S'(I) assigns to $\phi\varepsilon$ S(I) does not lead to any inconsistency.

S'(I) is linear space. Since S(I) is complete, so does S'(I) also [37 Theorem 1.8.3 p.21]. In view of the slef adjoint self nature of the operator R, we define the generalized differential operator R'=R on S'(I) through the definition

$$(Rf,\phi) = (f,R\phi), f \varepsilon S'(I), \phi \varepsilon S(I) \qquad 2.6.11$$

since $(f,R\phi) = < f,R \phi > = < R'f,\phi> = (R'f,\phi) = (Rf, \phi)$

LEMMA 2.6.9

The generalized differential operator R' is linear and continuous mapping of S'(I) into S'(I)

**PROOF**

In view of (2.6.10) and (2.6.11) we shall show that $R'f = Rf$ is a member of $S'(I)$.

For any $\phi$ , $\psi \in S(I)$ and $\alpha$ , $\beta \in C$ we have

$$< R'f, \alpha\beta + \beta\phi> = < f, R(\alpha\phi + \beta\psi) > \qquad \because (R \text{ is linear on } S(I))$$

$\alpha\phi + \beta\psi.$

$$= < f, \alpha R\phi + \beta R\psi >$$

$$= \alpha < f, R\phi > + \beta < f, R\psi >$$

$$= \alpha < R'f, \phi > + \beta < R'f, \psi >$$

which shows that $R'f$ is linear functional on $S(I)$

Furthermore let $\{\phi_\nu\}_{\nu=1}^{\infty}$ converges in $S(I)$ to zero, then as $\nu \to \infty$ , $R\phi_\nu \to 0$ in $S(I)$ and so that,

$$< R'f, \phi_\nu > = < f, R\phi_\nu > \to 0 \text{ in } S(I),$$

Which shows that $R'f$ is continuous functional on $S(I)$. Thus $R'f$ is continuous and linear functional on $S(I)$ therefore $R'f \in S'(I)$.

Thus $f \in S'(I)$ and $R'f \in S'(I)$ , which means that $R'$ is mapping of $S'(I)$ onto $S'(I)$.

Now we shall show that $R'$ is linear and continuous

Let $f, g \in S'(I)$ , $\phi \in S(I)$ and $\alpha$ , $\beta \in C$ then, we have

$$< R'(\alpha f + \beta g), \phi > = < \alpha f + \beta g, R\phi >$$

$$= < \alpha f, R\phi > + < \beta g, R\phi >$$

$$= \alpha < f, R\phi > + \beta < g, R\phi >$$

$= \alpha < R'f, \phi > + \beta < R'g, \phi >$

$= < \alpha R'f, \phi > + < \beta R'g, \phi >$

$= < \alpha R' + \beta R'g, \phi >$

which implies $R'(\alpha f + \beta g) = \alpha R'f + \beta R'g$ IN $S'(I)$

That is $R'$ is linear on $S'(I)$

Now let $f_\nu \to 0$ in $S'(I)$ , then for every $\phi \varepsilon S(I)$

$$< R'f_\nu , \phi > = < f_\nu , R \phi > \to 0$$

therefore $R'f_\nu \to 0$ in $S'(I)$

Thus $f_\nu \to 0$ in $S'(I) \Longrightarrow R'f_\nu \to 0$ in $S'(I)$

which means that $R'$ is continuous on $S'(I)$.

## 2.7 SOME PROPERTIES OF S(I) AND S'(I)

i) S(I) is a subspace of $L_2(I)$ when we identify that each function in S(I) with the corresponding equivalence class in $L_2(I)$.

Moreover convergence in S(I) implies convergence in $L_2(I)$.

ii) S(I) is a locally convex, sequentially complete Hausdorff topological vector space.

iii) D(I) is a subspace of S(I) AND CONVERGENCE in D(I) implies convergence in S(I). The topology of D(I) is stronger than that induced on it by S(I). Consequently, the restriction of any $f \varepsilon S'(I)$ to D(I) is in D'(I). Moreover, convergence in S'(I) implies convergence in D'(I).

HENCE members of $S'(I)$ are called distributions in the sense of Zemanian [37,pp.39].

iv) $S(I)$ is subspace of $E(I)$. Furthermore, if $\{\phi_\nu\}_{\nu=1}^{\infty}$ converges in $S(I)$ to the limit, then $\{\phi_\nu\}_{\nu=1}^{\infty}$ also converges in $E(I)$ to the same limit .

v) Since $D(I) \subset S(I) \subset E(I)$ and since $D(I)$ is dense in $E(I)$, implies that $S(I)$ is also dense in $E(I)$. The topology of $S(I)$ is stronger than that induced on it by $E(I)$. Hence $E'(I)$ can be identified with a subspace of $S'(I)$ which follows from [37,corollary 1.8.2a and lemma 9.3.4].

vi) We imbed $L_2(I)$ (and, therefore, $S(I)$ since $S(I)$ is subspace of $L_2(I)$ into $S'(I)$ by defining the number that $f \in L_2(I)$ assigns to any $\phi \in S(I)$ as

$$(f, \phi) = \int_0^a \int_0^b f(x,y)\ \overline{\phi}(x,y)\ dxdy \qquad (*)$$

f is clearly linear on $S(I)$. Its continuity on $S(I)$ follows from the fact that if sequence $\{\phi_m\}_{m=1}^{\infty}$ converges in $S(I)$ to zero, then by the Schwarz inequality

$$|(f, \phi_m)| \leq \alpha_o(f)\ \alpha_o(\phi_m) \to 0 \text{ , as } m \to \infty$$

The above definition (*) is consistent with the facts that he dual of $L_2(I)$ is $L_2(I)$ and that the inner product$(h, \chi)$ is precisely the number that an arbitrary, continuous linear functional h on $L_2(I)$ assigns to an arbitrary $\chi \in L_2(I)$.

 

This imbedding of $L_2(I)$ into $S'(I)$ one-to-one . Indeed, if two members f and g of $L_2(I)$ become imbedded as the same element of $S'(I)$ , then

$(f, \phi) = (g, \phi)$ for every $\phi \in D(I)$. But this implies that f=g almost everywhere on I. Hence f and g are in the same equivalence class in $L_2(I)$.

vii) If $f(x,y) = R^k g(x,y)$ for some $g(x,y) \in L_2(I)$ and for some nonnegative integer k, then $f(x,y) \in S'(I)$. This follows directly from the facts that $L_2(I) \subset S'(I)$ and that R maps $S'(I)$ into $S'(I)$.

viii) For each $f \in S'(I)$ there exists a nonnegative integer r and a positive constant C such that

$$|(f, \phi)| \leq C \max_{0 \leq k \leq r} \alpha_k(\phi)$$

for every $\phi \in S(I)$.

Here r and C depends upon f but not on $\phi$.

ix) Since the conventional operator R is self adjoint on S(I) implies that the conventional operator R and the generalized operator R=R' coinsides on S(I).

## 2.8 ORTHONORMAL SERIES EXPANSION AND GENERALIZED INTEGRAL TRANSFORMATION

The following fundamental theorem provides an orthonormal series expansion with respect to $\psi_{m,n}$ of generalized functions in $S'(I)$ which in turn yields an inversion formula for a certain generalized integral transformation.

THEOREM 2.8.1

Any generalized function in $S'(I)$ possesses an orthonormal series expansion with respect to $\psi_{m,n}$ used in construction of $S(I)$

Symbolically If $f \in S'(I)$ then

$$f = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (f, \psi_{m,n}) \psi_{m,n} \qquad 2.8.1$$

where the series converges in $S'(I)$.

PROOF

To prove this theorem we need merely invoke the previous lemmas (2.6.1) and (2.6.6). Then for any $\phi(x,y) \in S(I)$ we have

$$(f, \phi) = (f, \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\phi, \psi_{m,n}) \psi_{m,n})$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (f, \psi_{m,n}) \overline{(\phi, \psi_{m,n})}$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (f, \psi_{m,n})(\psi_{m,n}, \phi)$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} ((f, \psi_{m,n}) \psi_{m,n}, \phi) \qquad 2.8.2$$

And the right hand side of (2.8.2) truely converges for any $\phi \in S(I)$. Which means that the series (2.8.1) converges in $S'(I)$. Q.E.D.

DEFINITION : DISTRIBUTIONAL FOURIER COSINE-SINE TRANSFORMATION

The members of $S'(I)$ lead to the distributional (generalized) Fourier cosine sine transformation $f'_{cs}$ defined by

$$f'_{cs}(f) = F(m,n) = \sqrt{ab}/2\ (f, \psi_{m,n}) \qquad 2.8.3$$

for all $f \in S'(I)$ ,$m,n=0,1,2,\ldots$

And this $f'_{cs}$ is a mapping of $f \notin \in S'(I)$ into the space of complex valued functions $F(m,n)$ defined on the set of ordered pairs of nonnegative integers.

The inverse distributional Fourier cosine sine transformation $f'^{-1}_{cs}$ is defined by the series (2.8.1) which we rewrite as

$$f'^{-1}_{cs}\,[F(m,n)] = 2/\sqrt{ab} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} F(m,n)\,\psi_{m,n} = f$$

that is

$$f = 2/\sqrt{ab}\ [\sum_{n=1}^{\infty} F_{cs}(0,n)\ \psi_{0,n} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} F_{cs}(m,n)\ \psi_{m,n}]$$

$$= 2/\sqrt{ab} \sum_{n=1}^{\infty} F_{cs}(0,n) \frac{\operatorname{Sin} \frac{n\pi y}{b}}{\sqrt{ab}}$$

$$+ 2/\sqrt{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} F_{cs}(m,n)\ 2/\sqrt{ab}\ \operatorname{Cos} \frac{m\pi x}{a}\ \operatorname{Sin}\left(\frac{m\pi y}{b}\right)$$

$$= 2/ab \sum_{n=1}^{\infty} F_{cs}(0,n)\ \operatorname{Sin}\left(\frac{n\pi y}{b}\right) + 4/ab \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} F_{cs}(m,n)\ \operatorname{Cos}\left(\frac{m\pi x}{a}\right)\ \operatorname{Sin}\left(\frac{n\pi y}{b}\right)$$

**THEOREM 2.8.2**

**UNIQUENESS THEOREM**

If $f$, $g \in S'(I)$ and if their transformations $F(m.n)$ and $G(m.n)$ defined by (2.8.1) satisfy $F(m.n)=G(m.n)$ for every pair (m,n) of nonnegative integers then f=g in the sense of equality in S'(I).

**PROOF**

By theorem (2.8.1) for any $\phi \in S(I)$ and $f,g \in S'(I)$ we have

$$(f-g, \phi) = \left( \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} (f-g, \psi_{m,n}) \psi_{m,n}, \phi \right)$$

$$= \left( \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} [ (f, \psi_{m,n}) \psi_{m,n} - (g, \psi_{m,n}) \psi_{m,n} ], \phi \right)$$

$$= 2/\sqrt{ab} \left( \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} [F(m,n) - G(m,n)] \psi_{m,n}, \phi \right)$$

$$= 2/\sqrt{ab} \; (0, \phi)$$

$\Rightarrow$ f-g = 0

In the sense of equality in S'(I). This means that f=g in S'(I). Q.E.D