C H A P T E R - II

MEIJER BESSEL TRANSFORM AND ITS PROPERTIES

2.1 Introduction :

In this chapter we construct a testing function space $K_{\nu,\mu,a,b}$, its dual space $K'_{\nu,\mu,a,b}$ and extend the Meijer Bessel transform defined by the equation (1.1-3) to a certain class of generalized functions and study some properties of it.

For real numbers ν, μ and positive numbers a, b; we construct a testing function space $K_{\nu,\mu,a,b}$ which contains the kernel $\frac{2}{\pi} pq \sqrt{pqxy}\ K_{\nu}(px)K_{\mu}(qy)$ as a function on $0 < x < \infty$, $0 < y < \infty$ for each fixed p and q.

The Meijer Bessel transform F(p, q) of a distribution f in the dual space $K'_{\nu,\mu,a,b}$ is defined by

$$F(p,q) = K'_{\nu,\mu}(f) = \langle f(x,y), \frac{2}{\pi} pq \sqrt{pqxy}\ K_{\nu}(px)K_{\mu}(qy)\rangle$$

for suitably restricted p and q.

Let I denotes the open set ($0 < x < \infty$, $0 < y < \infty$). D(I) is the space of all smooth functions on I having compact support on I and D'(I) is the dual space of Schwartz distributions on I.

2.2 The Testing Function Spaces $K_{\nu,\mu,a,b}$ and Their Duals.

Let a and b be any real numbers and ν, μ be either

Let a and b be any real numbers and ν, μ be either zero or complex numbers such that $R_e\,\nu > 0$, $R_e\,\mu > 0$.

Let h(t) be a continuous function on $0 < t < \infty$ defined by

$$h(t) = \begin{cases} \log t & \text{if } 0 < t < e^{-1} \\ -1 & \text{if } e^{-1} < t < \infty \end{cases}$$

and we set

$$j_{\nu,\mu}(x,y) = \begin{cases} x^{\nu-1/2}\, y^{\mu-1/2} & \text{if } R_e\,\nu > 0,\ R_e\,\mu > 0 \\ [\sqrt{x}\, h(x)]^{-1}\, [\sqrt{y}\, h(y)]^{-1} & \text{if } \nu = \mu = 0. \end{cases}$$

Let $K_{\nu,\mu,a,b}$ be the space of all infinitely differentiable functions $\phi(x,y)$ on $0 < x < \infty$, $0 < y < \infty$ such that for each $k_1, k_2 = 0, 1, 2, \ldots$

$$\rho_{k_1,k_2}(\phi) \triangleq \rho^{\nu,\mu,a,b}_{k_1,k_2}(\phi) =$$

$$= \sup_{\substack{0<x<\infty \\ 0<y<\infty}} \left| e^{ax+by}\, j_{\nu,\mu}(x,y)\, S^{k_1,k_2}_{\nu,\mu,x,y}\, \phi(x,y) \right| < \infty \qquad \ldots (2.2.-1)$$

where $S^{k_1,k_2}_{\nu,\mu,x,y} = S^{k_1}_{\nu,x}\, S^{k_2}_{\mu,y}$ and

$$S_{\nu,x} = x^{-\nu-1/2} D_x x^{2\nu+1} D_x x^{-\nu-1/2}$$

$$S_{\mu,y} = y^{-\mu-1/2} D_y y^{2\mu+1} D_y y^{-\mu-1/2}.$$

where $D_x = \frac{\partial}{\partial x}$; $D_y = \frac{\partial}{\partial y}$

Clearly $K_{\nu,\mu,a,b}$ is a linear space over the field of complex numbers.

Moreover $\left\{ \rho_{k_1,k_2}^{\nu,\mu,a,b} \right\}_{k_1,k_2=0}^{\infty}$ is a multinorm on $K_{\nu,\mu,a,b}$.

Indeed for any complex number β and $\phi \in K_{\nu,\mu,a,b}$;

$$\rho_{k_1,k_2}^{\nu,\mu,a,b} (\beta \phi) = |\beta| \rho_{k_1,k_2}^{\nu,\mu,a,b} (\phi).$$

Also for each $\phi_1, \phi_2 \in K_{\nu,\mu,a,b}$

$$\rho_{k_1,k_2}^{\nu,\mu,a,b} (\phi_1+\phi_2) \leq \rho_{k_1,k_2}^{\nu,\mu,a,b} (\phi_1) + \rho_{k_1,k_2}^{\nu,\mu,a,b} (\phi_2).$$

Hence each $\rho_{k_1,k_2}^{\nu,\mu,a,b} (\phi)$ is a seminorm and in addition $\rho_{0,0}^{\nu,\mu,a,b} (\phi)$ is a norm on $K_{\nu,\mu,a,b}$.

We assign to $K_{\nu,\mu,a,b}$ the topology generated by the multinorm $\left\{ \rho^{\nu,\mu,a,b}_{k_1,k_2} \right\}^{\infty}_{k_1 k_2 = 0}$ and this makes $K_{\nu,\mu,a,b}$; a countably multinormed space. The dual space $K'_{\nu,\mu,a,b}$ consists of all continuous linear functions on $K_{\nu,\mu,a,b}$.

The dual is a linear space to which we assign the weak topology generated by multinorm $\left\{ \xi_{\emptyset} (f) \right\}$ where

$\left\{ \xi_{\emptyset} \right\} = | < f, \emptyset > |$ and $\emptyset$ varies through $K_{\nu,\mu,a,b}$.

A sequence $\left\{ \emptyset_m \right\}^{\infty}_{m=1}$ converges in $K_{\nu,\mu,a,b}$ to $\emptyset$ if and only if for each pair of non-negative integers k_1, k_2

$$\rho^{\nu,\mu,a,b}_{k_1,k_2} (\emptyset_m - \emptyset) \to 0 \text{ as } m \to \infty.$$

A sequence $\left\{ \emptyset_m \right\}^{\infty}_{m=1}$ is a Cauchy sequence in $K_{\nu,\mu,a,b}$ if and only if

$$\rho^{\nu,\mu,a,b}_{k_1,k_2} (\emptyset_m - \emptyset_n) \to 0 \text{ for every } k_1, k_2 \text{ as m and}$$

n tend to infinity independently.

<u>Lemma 2.2-1</u>

$K_{\nu,\mu,a,b}$ is complete and therefore a Fre'chet space.

<u>Proof</u> :

Let $\left\{ \phi_m \right\}_{m=1}^{\infty}$ be a Cauchy sequence in $K_{\nu,\mu,a,b}$.

Then by equation (2.2-1), we have a uniform Cauchy sequence $\left\{ \psi_m \right\}_{m=1}^{\infty}$ on I for each k_1, k_2 where

$$\psi_m(x,y) = e^{ax+by} \, j_{\nu,\mu}(x,y) \; S^{k_1,k_2}_{\nu,\mu,x,y} \left[\phi_m(x,y)\right] \qquad \ldots (2.2\text{-}2)$$

By Cauchy criterion $\left\{ S^{k_1,k_2}_{\nu,\mu,x,y} \psi_m \right\}$ converges uniformly to $\left\{ S^{k_1,k_2}_{\nu,\mu,x,y} \psi \right\}$ on I for all k_1, k_2. Hence by standard theorem [1, P.402], there is a smooth function $\phi(x,y)$ on I such that $\psi_m(x,y) \rightarrow \psi(x,y)$ uniformly on I and

$$S^{k_1,k_2}_{\nu,\mu,x,y} \psi_m(x,y) \rightarrow S^{k_1,k_2}_{\nu,\mu,x,y} \psi(x,y)$$

where $\psi(x,y) = e^{ax+by} \, j_{\nu,\mu}(x,y) \; S^{k_1,k_2}_{\nu,\mu,x,y} \left[\phi(x,y)\right]$

$$\ldots (2.2\text{-}3)$$

Since $\psi_m(x,y)$ is a uniform Cauchy sequence hence for each $\epsilon > 0$, there is an integer N_{k_1,k_2} such that

$$\sup_{\substack{0<x<\infty \\ 0<y<\infty}} \left| \psi_m(x,y) - \psi_n(x,y) \right| < \epsilon$$

for every m, n $< N_{k_1,k_2}$.

Taking limit as $n \to \infty$, we have

$$\sup_{\substack{0<x<\infty \\ 0<y<\infty}} \left| \psi_m (x,y) - \psi (x,y) \right| \leq \epsilon , \quad m > N_{k_1 k_2}. \qquad \ldots (2.2\text{-}4)$$

Thus for each k_1, k_2

$$\rho_{k_1,k_2}^{\nu,\mu,a,b} (\phi_m - \phi) \to 0 \text{ as } m \to \infty.$$

Finally according to the uniformity of the convergence and the fact that each $\psi_m(x,y)$ is bounded on I, there exists a constant C_{k_1,k_2} not depending on m such that

$$\left| \psi_m (x,y) \right| < C_{k_1,k_2} \text{ for all } (x,y)$$

Hence by (2.2-4) we get

$$\sup_{\substack{0<x<\infty \\ 0\leq y<\infty}} \left| \psi(x,y) \right| \leq \epsilon + C_{k_1,k_2} .$$

which shows that $\psi(x,y)$ is bounded on I.
Hence a function $\phi(x,y)$ which is the limit of a given sequence $\{\phi_m\}$ is a member of $K_{\nu,\mu,a,b}$.

Thus the sequence $\{\phi_m\}$ converges in $K_{\nu,\mu,a,b}$ to the unique limit ϕ. Hence $K_{\nu,\mu,a,b}$ is complete.

Since $K_{\nu,\mu,a,b}$ is countably multinormed space which is complete hence $K_{\nu,\mu,a,b}$ is a Fre'chet space.

$K_{\nu,\mu,a,b}$ is complete and hence $K'_{\nu,\mu,a,b}$ is complete.

Theorem 2.2-1 : $K_{\nu,\mu,a,b}$ is a testing function space.

Proof : Clearly $K_{\nu,\mu,a,b}$ satisfies first two conditions of testing function space. Now we shall prove the third.

Let $\left\{ \phi_m \right\}_{m=1}^{\infty}$ converges to zero in $K_{\nu,\mu,a,b}$.

In view of (2.2-2) and the seminorms defined in (2.2-1), it follows by induction on k_1, k_2 that, for each pair of k_1, k_2, $\left\{ D_x^{k_1} \; D_y^{k_2} \; (\phi_m) \right\}_{m=1}^{\infty}$ converges uniformly to zero function on every compact subset of I.
which completes the proof of the Theorem 2.2-1.

Now we list some properties of these spaces.

(i) Let $\nu, \mu \geq -\frac{1}{2}$. For a fixed complex number (p, q) belonging to the strip

$$\Omega = \left\{ (p,q) \in C^2 \; / \; p,q \neq 0 \text{ or a negative number} \right\},$$

$$pq\sqrt{pqxy} \; K_\nu(px) \; K_\mu(qy) \in K_{\nu,\mu,a,b} \, .$$

Indeed by the analyticity of $\sqrt{zw} \; K_\nu(z) \; K_\mu(w)$, $z, w \neq 0$, it follows that $pq\sqrt{pqxy} \; K_\nu \; (px) \; K_\mu(qy)$ is smooth on I.

Also in the view of the property

$$S^{k_1,k_2}_{\nu,\mu,x,y}\left[pq\sqrt{pqxy}\;K_\nu(px)K_\mu(qy)\right] = p^{2k_1}\, q^{2k_2}\cdot (pq\sqrt{pqxy}\;K_\nu(px)\,K_\mu(qy) \qquad \ldots (2.2\text{-}5)$$

and by the equations (1.2-2), (1.2-3) and asymptotic estimate (1.2-4) and the fact that $\left| e^{ax+by}\,(px)^{\nu}\,(qy)^{\mu}\cdot K_\nu(px)\,K_\mu(qy)\right|$ is bounded for $0 < x < \infty$, $0 < y < \infty$, $(p,q) \in \Omega$, the quantities $\rho^{\nu,\mu,a,b}_{k_1,k_2}\left[pq\sqrt{pqxy}\;K_\nu(px)\,K_\mu(qy)\right]$ are finite for all $k_1, k_2 = 0, 1, 2\ldots.$

Thus our assertion is verified.

(ii) Let $\nu > -\frac{1}{2}$, $\mu \geq -\frac{1}{2}$. For a fixed complex number (p,q) belonging to the strip

$$\Omega = \left\{ (p,q) \in C^2 \;/\; p,q \neq 0 \text{ or a negative number} \right\}$$

$$D\left[\frac{2}{\pi}\,pq\sqrt{pqxy}\;K_\nu(px)\,K_\mu(qy)\right] \in K_{\nu,\mu,a,b}$$

where $D = \dfrac{\partial}{\partial p}$ or $\dfrac{\partial}{\partial q}$

Hence $\rho^{\nu,\mu,a,b}_{k_1,k_2}\left[D\left(\frac{2}{\pi}\,pq\,\sqrt{pqxy}\;K_\nu(px)\,K_\mu(qy)\right)\right] < \infty$

for any fixed $(p,q) \in \Omega$.

(iii) Let $0 < c < a$, $0 < d < b$. Then

$$K_{\nu,\mu,c,d} \subset K_{\nu,\mu,a,b} \text{ and the topology of}$$

$K_{\nu,\mu,c,d}$ is stronger than the topology induced on it by $K_{\nu,\mu,a,b}$.

This follows from the inequality

$$\rho^{\nu,\mu,a,b}_{k_1,k_2}(\phi) \leq \rho^{\nu,\mu,c,d}_{k_1,k_2}(\phi) \quad \text{for } \phi \in K_{\nu,\mu,a,b}.$$

Let $0 < e^{ax+by} < e^{cx+dy}$ on I,

Then

$$\left| e^{ax+by} j_{\nu,\mu}(x,y) \; S^{k_1,k_2}_{\nu,\mu,x,y} \phi(x,y) \right| \leq$$

$$\left| e^{cx+dy} j_{\nu,\mu}(x,y) \; S^{k_1,k_2}_{\nu,\mu,x,y} \phi(x,y) \right|$$

Therefore $\rho^{\nu,\mu,a,b}_{k_1,k_2} \phi(x,y) \leq \rho^{\nu,\mu,c,d}_{k_1,k_2} \phi(x,y)$.

Thus our assertion is implied by the last inequality. Hence the restriction of $f \in K'_{\nu,\mu,a,b}$ to $K_{\nu,\mu,c,d}$ is in $K'_{\nu,\mu,c,d}$.

(iv) D(I) is a subspace of $K_{\nu,\mu,a,b}$ and the convergence in D(I) implies convergence in $K_{\nu,\mu,a,b}$ i.e. the topology of D(I) is stronger than that induced on it by $K_{\nu,\mu,a,b}$. Hence the restriction of $f \in K'_{\nu,\mu,a,b}$ to D(I) is in D'(I). Thus, members of $K'_{\nu,\mu,a,b}$ are distributions in Zemanian sense [14, P.39].

Here I denotes the first quadrent $\{(x, y) / 0 < x < \infty, 0 < y < \infty\}$.

(v) Each of the differential operators $S_{\nu,x}$ and $S_{\mu,y}$ is a continuous linear mapping of $K_{\nu,\mu,a,b}$ into itself since

$$\rho_{k_1,k_2}(S_{\nu,x}\,\phi) = \rho_{k_1+1,\,k_2}(\phi) \quad \text{and}$$

$$\rho_{k_1,k_2}(S_{\mu,y}\,\phi) = \rho_{k_1,\,k_2+1}(\phi)$$

We define the weak differential operators $S'_{\nu,x}$ and $S'_{\mu,y}$ on $K'_{\nu,\mu,a,b}$ as

$$< S'_{\nu,x}\, f, \phi > \; = \; < f, S_{\nu,x}\, \phi >$$

$$< S'_{\mu,y}\, f, \phi > \; = \; < f, S_{\mu,y}\, \phi >$$

for $\phi \in K_{\nu,\mu,a,b}$.

Since $\phi \to S_{\nu,x}\,\phi$ and $\phi \to S_{\mu,y}\,\phi$ are continuous linear mappings of $K_{\nu,\mu,a,b}$ into itself, the mappings $f \to S'_{\nu,x}$ and $f \to S'_{\mu,y}$ are continuous linear mappings from $K'_{\nu,\mu,a,b}$ into itself.

(vi) Let f(x,y) be locally integrable function on $I\,\{0 < x < \infty,\ 0 < y < \infty\}$ such that

$$\int_0^\infty \int_0^\infty \frac{e^{-ax-by}}{j_{\nu,\mu}(x,y)} \left| f(x,y) \right| dxdy < \infty$$

Then f generates a regular generalized function in $K'_{\nu,\mu,a,b}$; defined by

$$< f, \phi > = \int_0^\infty \int_0^\infty f(x,y)\, \phi(x,y)\, dx\, dy. \qquad \ldots (2.2\text{-}6)$$

Indeed

$$< f,\phi > = \left| \int_0^\infty \int_0^\infty \frac{f(x,y)}{e^{ax+by}\, j_{\nu,\mu}(x,y)}\, e^{ax+by}\, j_{\nu,\mu}(x,y) \times \phi(x,y)\, dxdy \right|$$

$$\leq \rho^{\nu,\mu,a,b}_{0,0}\, \phi(x,y) \int_0^\infty \int_0^\infty \left| \frac{f(x,y)}{e^{ax+by}\, j_{\nu,\mu}(x,y)} \right| dxdy$$

which shows that (2.2-6) truely defines a functional f on $K_{\nu,\mu,a,b}$. This functional is clearly a linear one. Moreover, if $\{\phi_m\}_{m=1}^{\infty}$ converges in $K_{\nu,\mu,a,b}$ to zero then $\rho_{0,0}^{\nu,\mu,a,b}(\phi_m) \to 0$ so that $|\langle f, \phi m \rangle| \to 0$. Thus f is also continuous on $K_{\nu,\mu,a,b}$. Hence f generates a regular generalized function in $K'_{\nu,\mu,a,b}$.

(vii) For each $f \in K'_{\nu,\mu,a,b}$; there exists a non-negative integer r and a positive constant C such that for all $\phi \in K_{\nu,\mu,a,b}$

$$|\langle f, \phi \rangle| \leq C \max_{\substack{0<k_1<r \\ 0\leq k_2<r}} \rho_{k_1,k_2}^{\nu,\mu,a,b}(\phi)$$

The proof of this statement is similar to that of [13, Theorem 3.3-1].

2.3 The Distributional Meijer Bessel Transforms.

We call f, a $K_{\nu,\mu}$ - transformable generalized function if it is a member of $K'_{\nu,\mu,a,b}$ for some pair (a,b) of real numbers. In view of note (iii) sec.2.2; to every $f \in K'_{\nu,\mu,a,b}$ there exists the unique reals $\sigma_f, \rho_f > 0$ (Possibly $\sigma_f = \infty$, $\rho_f = \infty$) such that $f \in K'_{\nu,\mu,c,d}$.

if $c < a < \sigma_f$, $d < b < \rho_f$ and $f \notin K'_{\nu,\mu,c,d}$ if $c > \sigma_f$, $d > \rho_f$. We define the $(\nu, \mu)^{th}$ order Meijer Bessel transform $k'_{\nu,\mu}(f)$ of f as the application of f to the kernel $\frac{2}{\pi} pq \sqrt{pqxy}\, K_\nu(px)\, K_\mu(qy)$; i.e.

$$F(p,q) = \langle f(x,y), \frac{2}{\pi} pq \sqrt{pqxy}\; K_\nu(px)\, K_\mu(qy) \rangle \qquad \ldots (2.3\text{-}1)$$

where $(p,q) \in \Omega_f = \Big\{ (p,q) \in C^2 / \text{Re}.P > \rho_f,\ R_e\, q > \sigma_f$

$(p,q) \neq (0,0)$; $-\pi < \arg P < \pi$

$-\pi < \arg q < \pi \Big\}$

The right hand side of (2.3-1) has a sense because the application of $f \in K'_{\nu,\mu,a,b}$ to $\frac{2}{\pi} pq \sqrt{pqxy}\, K_\nu(px) \cdot K_\mu(qy) \in K_{\nu,\mu,a,b}$.

where a and b are any real numbers such that $\rho_f < a < R_e\, p$; $\sigma_f < b < R_e\, q$.

The subset Ω_f of C^2 is the region of definition of the transform of f and the numbers ρ_f and σ_f as the abscissae of the definition.

We refer to the operator $K'_{\nu,\mu}: f \longrightarrow F$ as the generalized Meijer Bessel transform of $(\nu, \mu)^{th}$ order.

whenever $F(p,q) = k'_{\nu,\mu}(f)$ for $(p,q) \in \Omega_f$ then f is a $K_{\nu,\mu}$ - transformable generalized function when ν and μ are either zero or complex numbers with $R_e \nu > 0$, $R_e \mu > 0$ and $F(p,q)$ as in (2.3-1) and Ω_f in usual sense.

Theorem 2.3-1 :

If $F(p,q) = k'_{\nu,\mu}(f)$ for $(p,q) \in \Omega_f$, then for any positive integer m

$$k'_{\nu,\mu}\, S^m_{\nu,x}\, f = p^{2m}\, F(p,q) \qquad \ldots (2.3\text{-}2)$$

$$k'_{\nu,\mu}\, S^m_{\mu,y}\, f = p^{2m}\, F(p,q) \qquad \ldots (2.3\text{-}3)$$

Proof : By property (iii) and the equation (1.2-8)

$$k'_{\nu,\mu} S^m_{\nu,x}\, f = \langle f(x,y),\ S^m_{\nu,x} [\tfrac{2}{\pi}\, pq \sqrt{pqxy}\, K_{\nu}(px)\, K_{\mu}(qy)]$$

$$= \langle f(x,y),\ S^m_{\nu,x} [\ \tfrac{2}{\pi}\, pq \sqrt{px}\, K_{\nu}(px) \times \sqrt{qy}\, K_{\mu}(qy)\ ]$$

$$= \langle f(x,y),\ p^{2m}\ \tfrac{2}{\pi}\, pq \sqrt{pqxy}\, K_{\nu}(px) \times K_{\mu}(qy) \rangle$$

$$= p^{2m}\, F(p,q).$$

Similarly we can prove (2.3-3).

<u>Lemma 2.3-1:</u>

Let $\nu_R \triangleq R_e \nu > 0$, $\mu_R \triangleq R_e \mu > 0$ and let a, b, σ_f, ρ_f be the fixed real numbers such that $0 < a < \sigma_f$, $0 < b < \rho_f$ for all (p, q) in the strip Ω where $\Omega = \{ (p, q) \in C^2 / p, q \neq 0$ or negative number $\}$

For $0 < x < \infty$, $0 < y < \infty$ and $\nu \geq -\frac{1}{2}$, $\mu \geq -\frac{1}{2}$

$$\left| e^{ax+by} (px)^{\nu} (qy)^{\mu} K_{\nu}(Px) K_{\mu}(qy) \right| <$$

$$< A_{\nu,\mu} [1 + |p|^{\nu}][1 + |q|^{\mu}]$$

where $A_{\nu,\mu}$ is the constant with respect to p, q, x and y.

<u>Proof</u> : $(px)^{\nu} (qy)^{\mu} K_{\nu} (px) K_{\mu}(qy)$ is entire and hence bounded on any bounded domain. Moreover by the series expansions of $K_{\nu}(px) K_{\mu}(qy)$ as $|px| \to \infty$, $|qy| \to \infty$; we have that there exists a constant $B_{\nu,\mu}$ such that

$$(px)^{\nu} (qy)^{\mu} K_{\nu}(px) K_{\mu}(qy) < B_{\nu,\mu}, \quad \begin{matrix} |px| \leq 1 \\ |qy| \leq 1. \end{matrix}$$

and by the asymptotic expansion in (1.2-5) there exists a constant $C_{\nu,\mu}$ such that

$$\left| (px)^{\nu} (qy)^{\mu} K_{\nu}(px) K_{\mu}(qy) \right| = \left| (px)^{\nu-1/2} (qy)^{\mu.1/2} \right| .$$

$$. \left| (px)^{1/2} (qy)^{1/2} K_{\nu}(px) K_{\mu}(qy) \right|$$

$$< C_{\nu,\mu} \; |px|^{\nu} \; |qy|^{\mu} \; e^{-(px+qy)},$$

$$|px| > 1, \quad |qy| > 1.$$

Hence for all values of p, q, x, y

$$\left| (px)^{\nu} \; (qy)^{\mu} \; K_{\nu}(px) \; K_{\mu} \; (qy) \right| < E_{\nu,\mu}[1+ \; |px|^{\nu}][1+ |qy|^{\mu}] .$$
$$. \; e^{-(px \; +qy)}$$

where $E_{\nu,\mu}$ is another constant.

It follows that for x, y, p, q and ν, μ restricted as stated

$$\left| \; e^{ax+by} \; (px)^{\nu} (qy)^{\mu} \; K_{\nu}(px) K_{\mu}(qy) \right| \; <$$

$$< E_{\nu,\mu}[1+ \; |px|^{\nu}][1+ \; |qy|^{\mu}] \; e^{(ax+by)-(px+qy)}$$

$$= E_{\nu,\mu} \; [1+ \; |px|^{\nu}][1 \; + \; |qy|^{\mu}] . e^{(a-p)x+(b-q)y}$$

$$< E_{\nu,\mu}[1+ \; |p|^{\nu}][1+ \; |q|^{\mu}](1+x)^{\nu} \; .$$

$$. \; (1+y)^{\mu} \; e^{(a-p)x+(b-q)y}.$$

$$< A_{\nu,\mu} \; [1 \; + |p|^{\nu} \;][1 \; + \; |q|^{\mu} \;].$$

Hence the proof where $A_{\nu,\mu}$ is constant with respect to p, q, x and y.

We shall now prove the analyticity theorem for the generalized Meijer-Bessel transform.

Theorem 2.3.2

If $k'_{\nu,\mu}(f)(p,q) = F(p,q)$ for $(p,q) \in \Omega_f$

then $F(p,q)$ is analytic in p and for fixed q, $(p,q) \in \Omega_f$

$$DF(p,q) = \langle f(x,y), D\left[\frac{2}{\pi} pq\sqrt{pqxy}\, K_\nu(px)\, K_\mu(qy)\right]\rangle \qquad \ldots (2.3\text{-}4)$$

where $D \equiv \frac{\partial}{\partial p}$ or $\frac{\partial}{\partial q}$

Proof : Let (p,q) be an arbitrary but the fixed point of Ω_f.

Fix q Construct two concentric circles of radii r and r_1 with centre p such that both the circles are in Ω_f.

Let $r < r_1$ and let $|\Delta p|$ be a non-zero complex increment in p-plane such that $|\Delta p| < r$.

Consider,

$$\frac{F(p+\Delta p,q)-F(p,q)}{\Delta p} - \langle f(x,y), \frac{\partial}{\partial p}\left[\frac{2}{\pi} pq\sqrt{pqxy}\; K_\nu(px)\, K_\mu(qy)\right]\rangle = \langle f(x,y), \psi_{\Delta p}(x,y)\rangle \qquad \ldots (2.3\text{-}5)$$

where

$$\psi_{\Delta p}(x,y) = \frac{2}{\pi\Delta_p}\left\{ (p+\Delta_p)\sqrt{p+\Delta_p}\, q\sqrt{q}\sqrt{xy}\ \cdot \right.$$

$$\cdot\ K_\nu[(p+\Delta_p)\,x]\,.\,K_\mu(qy)\ -$$

$$\left. -\ [pq\sqrt{pqxy}\ \ K_\nu(px)\ K_\mu(qy)\ ]\right\} -$$

$$-\ \frac{\partial}{\partial p}\,[pq\sqrt{pqxy}\ \ K_\nu(px)\ K_\mu(qy)\ ].$$

our theorem will be proven when we show that (2.3-5) converges to zero as $|\Delta_p| \longrightarrow 0$.

This can be done by showing that $\psi_{\Delta p}(x)$ converges in $K_{\nu,\mu,a,b}$ to zero as $|\Delta_p| \longrightarrow 0$.

Using the fact that

$$S^k_{\nu,x}\,[p\sqrt{px}\,K_\nu(px)\ \ q\sqrt{qy}\,K_\mu(qy)\ ] = p^{2k}(p\sqrt{px}\ q\sqrt{qy}\,K_\nu\ (\ \cdot\ (px)\ K_\mu(qy)\ )$$

and by interchanging differentiation on p with differentiation on x, we may write $S^k_{\nu,x}\,\psi\,\Delta_p\ (x,y)$ using Cauchy's integral formulas [5] as follows :

$$S^k_{\nu,x}\,\psi\,\Delta_p(x,y) = \frac{1}{\pi^2 i}\int_C \frac{\eta^{2k}\,\eta\sqrt{\eta x}\ q\sqrt{qy}\ K_\nu(\eta x)K_\mu(qy)}{(\eta - p - \Delta_p)}\,d\eta\ -$$

$$- \frac{1}{\pi^2 i} \int_C \frac{\eta^{2k}\, \eta \sqrt{\eta x}\, q \sqrt{qy}\, K_{\nu}(\eta x)\, K_{\mu}(qy)}{(\eta - p)\Delta_p}\, d\eta -$$

$$- \frac{1}{\pi^2 i} \int_C \frac{\eta^{2k}\, \eta \sqrt{\eta x}\, q \sqrt{qy}\, K_{\nu}(\eta x) K_{\mu}(qy)}{(\eta-p)^2}\, d\eta$$

$$= \frac{1}{\pi^2 i} \int_C \frac{[(\eta-p)^2 - (\eta-p-\Delta_p)(\eta-p) - (\eta-p-\Delta_p)\Delta_p]}{\Delta_p(\eta-p-\Delta_p)(\eta-p)^2} \cdot$$

$$\cdot\, \eta^{2k}\, \eta \sqrt{\eta x}\, q \sqrt{qy}\, K_{\nu}(\eta x)\, K_{\mu}(qy)\, d\eta .$$

$$= \frac{1}{\pi^2 i} \int_C \frac{\Delta_p}{(\eta-p)^2\, (\eta - p - \Delta_p)}\, \eta^{2k} \cdot$$

$$\cdot\, \eta \sqrt{\eta x}\, q \sqrt{qy}\, K_{\nu}(\eta x)\, K_{\mu}(qy)\, d\eta .$$

Hence

$$S^k_{\nu, x}\, \psi_{\Delta p}(x, y) = \frac{\Delta_p}{\pi^2 i} \int_C \frac{\eta^{2k}\, \eta \sqrt{\eta x}\, K_{\nu}(\eta x) q \sqrt{qy}\, K_{\mu}(qy)}{(\eta - p)^2\, (\eta - p - \Delta_p)}\, d\eta$$

Now for all $\eta \in C$ and $0 < x < \infty$, $0 < y < \infty$

$$\left| e^{ax + by}\, x^{\nu - 1/2}\, y^{\mu - 1/2}\, S^k_{\nu, x}\, \psi_{\Delta p}(x, y) \right| \leq$$

$$\frac{|\Delta_p|}{\pi^2} \int_C \frac{\eta^{2k + \frac{3}{2} - \nu}\, q^{\frac{3}{2} - \mu}}{(\eta-p)^2\, (\eta-p-\Delta_p)} \cdot$$

..

$$\cdot \left| e^{ax+by} (\eta x)^{\nu} (qy)^{\mu} K_{\nu}(\eta x) K_{\mu}(qy) \right| d\eta.$$

$$\leq \frac{K |\Delta_p| A_{\nu,\mu} [1 + |\eta|^{\nu}][1 + |q|^{\mu}]}{r_1^2 (r_1 - r)}$$

... By lemma (2.2-1)

where $A_{\nu,\mu}$ is a bound on

$\left| e^{ax+by} (\eta x)^{\nu} (qy)^{\mu} K_{\nu}(\eta x) K_{\mu}(qy) \right|$ and K is constant independent of p and x.

Moreover $\left| \eta - p - \Delta_p \right| > r_1 - r > 0$ and $\left| \eta - p \right| = r_1$.

Thus as $|\Delta_p| \to 0$, $\psi_{\Delta p}(x,y)$ converges to zero in $K_{\nu,\mu,a,b}$.

Consequently,

$$< f(x,y), \quad \psi_{\Delta p}(x,y) > \to 0 \text{ as } |\Delta_p| \to 0.$$

<u>Theorem 2.3-3</u> :

If $k'_{\nu,\mu}(f)(p,q) = F(p,q)$ for $(p,q) \in \Omega_f$

then (p, q) is analytic in q and for fixed P, $(p,q) \in \Omega_f$,

$$\frac{\partial}{\partial q} F(p,q) = < f(x,y), \frac{\partial}{\partial q} \left[\frac{2}{\pi} pq \sqrt{pqxy} \; K_{\nu}(px) K_{\mu}(qy) \right] >$$

The proof of this theorem is very similar to that of theorem 2.3-2.

Theorem 2.3-4

If $k'_{\nu,\mu}(f)\ (p,q) = F(p,q)$ for $(p,q) \in \Omega_f$ then $F(p,q)$ is analytic on Ω_f and

$$DF(p,q) = \langle f(x,y),\ D\left[\frac{2}{\pi}\, pq\sqrt{pqxy}\ K_{\nu}(px)\, K_{\mu}(qy)\right]\rangle$$

where $D = \dfrac{\partial}{\partial p}$ or $\dfrac{\partial}{\partial q}$.

Proof : By the theorems 2.3-2 and 2.3-3, at every point $(p',q') \in \Omega_f$, each of the functions $F(p,q')$ and $F(p',q)$ is analytic in the single variable p and q respectively. Therefore invoking the Hartog's theorem [2, p.140], we see that $F(p,q)$ is analytic on Ω_f.

Theorem 2.3-5 [Boundedness of $F(p,q)$].

Let $k'_{\nu,\mu}(f)\ (p,q) = F(p,q)$ for $(p,q) \in \Omega_f$ and a, b be any real numbers such that $a > \rho_f$, $b > \sigma_f$. Also let $\Lambda_{c,d}$ be the subset of Ω_f defined by

$$\Lambda_{c,d} = \Big\{ (p,q) \in \Omega_f \,/\, R_e\, p \geq c,\ R_e\, q \geq d \text{ and } |p| \geq c - \rho_f,\ |q| \geq d - \sigma_f .$$

then $F(p,q)$ is bounded according to

$$|F(p,q)| \leq P_{c,d}(|p|\ \ |q|) \text{ where } P_{c,d}(|p|\cdot|q|)$$

is a polynomial in $|p|$ $|q|$ depending on c and d.

__Proof__ : For we choose two real numbers a and b such that $\rho_f < a < c$ and $\sigma_f < b < d$. Then for each fixed (p, q) with $(p,q) \neq (0,0)$ and for $R_e\, P \geq c$, $R_e\, q \geq d$,

$$\frac{2}{\pi}\, pq\sqrt{pqxy}\, K_{\nu}(px)\, K_{\mu}(qy) \in K_{\nu,\mu,a,b}.$$

Moreover $f \in K'_{\nu,\mu,a,b}$ and in view of general result [14, Theorem 1.8.11] there exists a constant $c > 0$ and a non-negative integer r such that for $(p,q) \neq (0,0)$, $R_e\, P > c$, $R_e\, q \geq d$

$$|F(p,q)| \leq C. \max_{0 \leq k_1,k_2 < r} \rho_{k_1,k_2}\left[P\sqrt{px}\; q\sqrt{qy}\, K_{\nu}(px) K_{\mu}(qy)\right]$$

$$= C \max_{0 \leq k_1,k_2 \leq r}\; \sup_{\substack{0<x<\infty \\ 0<y<\infty}} \Big| e^{ax+by}\, j_{\nu,\mu}(x,y)\, S^{k_1,k_2}_{\nu,\mu} \ldots$$

$$P\sqrt{px}\;\; q\sqrt{qy}\, K_{\nu}(px)\;\; K_{\mu}(qy)\Big|$$

$$= C \max_{0 \leq k_1,k_2 \leq r}\; \sup_{\substack{0<x<\infty \\ 0<y<\infty}} \Big| e^{ax+by}\, j_{\nu,\mu}(x,y)\, .$$

$$.\; p^{2k_1}\, q^{2k_2}\, p\sqrt{px}\; q\sqrt{qy}\;\; K_{\nu}(px)\; K_{\mu}(qy)\Big|$$

If $\nu = 0$, $\mu = 0$ then

$$\left| F(p,q) \right| \leq C \max_{0 \leq k_1, k_2 \leq r} |p|^{2k_1+\frac{3}{2}} \; |q|^{2k_2+\frac{3}{2}} \; .$$

$$\cdot \sup_{\substack{0<x<\infty \\ 0<y<\infty}} \left| \frac{e^{ax+by} K_o(px) \, K_o(qy)}{h(x) \; h(y)} \right|$$

Since for all x, y > 0, $\left| \frac{e^{ax+by} K_o(px) \, K_o(qy)}{h(x) \; h(y)} \right| < A_o$

where A_o is constant with respect to p, q, x and y [14, lemma 6.5-2]

Hence

$$\left| F(p,q) \right| < A_o \max_{0 \leq k_1, k_2 \leq r} |p|^{2k_1+\frac{3}{2}} \; |q|^{2k_2+\frac{3}{2}}$$

$$\left| F(p,q) \right| \leq C \max_{0 < k_1, k_2 \leq r} |p|^{2k_1+\frac{3}{2}-\nu} \; q^{2k_2+\frac{3}{2}-\mu} .$$

$$\cdot \sup_{\substack{0<x<\infty \\ 0<y<\infty}} \left| e^{ax+by} (px)^{\nu} (qy)^{\mu} K_{\nu}(px) \, K_{\mu}(qy) \right|$$

and since

$$\left| e^{ax+by} (px)^{\nu} (qy)^{\mu} K_{\nu}(px) \, K_{\mu}(qy) \right| < A_{\nu,\mu} [1+ |p|^{\nu}] . [1 + |q|^{\mu}]$$

Hence

$$\left| F(p,q) \right| < A_{\nu,\mu} \max_{0 \le k_1, k_2 \le r} \Big[|p|^{2k_1 + \frac{3}{2} - \nu} \cdot |q|^{2k_2 + \frac{3}{2} - \mu} \cdot (1 + |p|^{\nu})(1 + |q|^{\mu'}) \Big]$$

where $A_{\nu,\mu}$ is constant with respect to p, q, x and y and the theorem follows.

2.4 Inversion Theorem

The inversion formula for distributional Meijer Bessel transform determines the restriction to D(I) of any $K_{\nu,\mu}$ - transformable generalized function from its Meijer Bessel transform. From this we will obtain an incomplete version of uniqueness theorem, which states that two $K_{\nu,\mu}$ - transformable generalized functions having the same transform must have the same restriction to D(I).

Theorem 2.4-1

Let $k_{\nu,\mu}(f)\ (p,q) = F(p,q),\ (p,q) \in \Omega_f$ where (p,q) is restricted to the real positive axis and let $\nu \ge -\frac{1}{2},\ \mu \ge -\frac{1}{2}$.

Then for each $\phi \in D(I)$

$$< \frac{1}{(2\pi i)^2} \int_{\sigma - iR}^{\sigma + iR} \int_{\sigma - iR'}^{\sigma + iR'} F(p,q) \left(\frac{x}{p} \right)^{\frac{1}{2}} I_{\nu}(px) I_{\mu}(qy)\, dp\, dq,$$

$, \phi(x,y) \;\; > \longrightarrow \; < f, \phi > \text{ as } R, R' \longrightarrow \infty$ ... (2.4-1)

__Proof__ : Let $\phi \in D(I)$. Choose real numbers a and b such that $0 < a < \sigma_f$ and $0 < b < \rho_f$. Since the integral in (2.4-1) is a continuous function of (x,y), it generates a regular distribution in $D(I)$. Hence we have

$$< \frac{1}{2\pi i^2} \int_{\sigma-iR}^{\sigma+iR} \int_{\sigma-iR'}^{\sigma+iR'} F(p,q) \left(\frac{x}{p}\right)^{\frac{1}{2}} \left(\frac{y}{q}\right)^{\frac{1}{2}} I_\nu(px) I_\mu(qy)\, dp\,dq,$$

$$\phi(x,y) >$$

$$= \frac{1}{2\pi i^2} \int_0^\infty \int_0^\infty \phi(x,y)\,dx\,dy \int_{\sigma-iR}^{\sigma+iR} \int_{\sigma-iR'}^{\sigma+iR'} F(p,q) \,.$$

$$. \left(\frac{x}{p}\right)^{\frac{1}{2}} \left(\frac{y}{q}\right)^{\frac{1}{2}} I_\nu(px)\; I_\mu(qy)\; dp\,dq.$$

Since ϕ is of bounded support and the integrand on right hand side is a continuous function of (x,y,p,q); we can change the order of integration and obtain

$$< \frac{1}{2\pi i^2} \int_{\sigma-iR}^{\sigma+iR} \int_{\sigma-iR'}^{\sigma+iR'} F(p,q) \left(\frac{x}{p}\right)^{\frac{1}{2}} \left(\frac{y}{q}\right)^{\frac{1}{2}} I_\nu(px) \,.$$

$$. I_\mu(qy)\; dp\,dq,\; \phi(x,y) \;\; >$$

...

$$= \frac{1}{2\pi} \int_{-R}^{R} \int_{-R'}^{R'} < f(t_1, t_2), \frac{2}{\pi} pq \sqrt{pqt_1t_2} \; K_{\nu}(t_1 p) .$$

$$. K\mu(t_2 q) \; dpdq. \int_0^{\infty} \int_0^{\infty} \emptyset(x,y) \; \left(\frac{x}{p}\right)^{\frac{1}{2}} \; I_{\nu}(px) \; .$$

$$. I_{\mu}(qy) \; dxdy. \qquad \ldots (2.4\text{-}2)$$

Now let

$$\Phi (p,q) = \int_0^{\infty} \int_0^{\infty} \emptyset(x,y) \; \left(\frac{x}{p}\right)^{\frac{1}{2}} \left(\frac{y}{q}\right)^{\frac{1}{2}} I_{\nu}(px) I_{\mu}(qy) dxdy.$$

and

$$M_{R_1 R'}(t_1, t_2) = \frac{1}{\pi^2} \int_{-R}^{R} \int_{-R'}^{R'} \Phi (p,q) \; (pq \sqrt{pqt_1t_2} \; K_{\nu}(pt_1).$$

$$. K\mu(q \; t_2) dpdq.$$

Since

$$\left| e^{at_1+bt_2} \; t_1^{\nu-1/2} \; S^{k_1,k_2}_{\nu,\mu,t,t_2} \; M_{R,R'}(t_1,t_2) \right|$$

$$= \frac{1}{\pi^2} \int_{-R}^{R} \int_{-R'}^{R'} p^{2k_1+\frac{1}{2}-\nu} \; q^{2k_2+\frac{1}{2}-\mu} \; (pt_1)^{\nu} (qt_2)^{\mu} .$$

$$. e^{at_1+bt_2} \; K_{\nu}(pt_1) \; K\mu(qt_2) \; \Phi (p,q) dpdq$$

$$\leq B \int_{-R}^{R} \int_{-R'}^{R'} \Phi (p,q) \; p^{2k_1+\frac{1}{2}-\nu} \; q^{2k_2+\frac{1}{2}-\mu} dpdq$$

where B is a suitable constant bound on $e^{at_1+bt_2} \; (pt_1)^{\nu}$.

. $(qt_2)^{\mu}$ (pt_1) $K\mu(qt_2)$.

Hence $M_{R,R'}(t_1,t_2) \in K_{\nu,\mu,a,b}$ for each $R,R' > 0$

Using the Riemann sum technique we can write equation (2.4-2) as

$$< \frac{1}{2\pi i^2} \int_{-R}^{R} \int_{-R'}^{R'} F(p,q) \left(\frac{x}{p}\right)^{\frac{1}{2}} \left(\frac{y}{q}\right)^{\frac{1}{2}} I_\nu(px)\, I_\mu \,.$$

$$.\ (qy)\, dpdq,\ \phi(x,y) >$$

$$= < f(t_1,t_2),\ M_{R,R'}(t_1,t_2) >,\ R,R' > 0.$$

This has a sense because $M_{R,R'}(t_1,t_2) \in K_{\nu,\mu,a,b}$.

Hence the theorem will be proven when we show that

$$M_{R,R'}(t_1,t_2) \longrightarrow \phi(t_1,t_2) \text{ in } K_{\nu,\mu,a,b} \text{ as } R,R' \longrightarrow \infty.$$

Since

$\Phi(p,q)\ \frac{2}{\pi}\ pq\sqrt{pqt_1t_2}\ \ K_\nu(pt_1)\, K_\mu(qt_2)$ is smooth and $\phi \in D(I)$, we may repeatedly differentiate under integral sign and use equations (1.2-8), (1.2-9) to write

$$S^{k_1,k_2}_{\nu,\mu,t_1,t_2}\ M_{R,R'}(t_1,t_2) = \frac{1}{\pi^2} \int_{-R}^{R} \int_{-R'}^{R'} S^{k_1,k_2}_{\nu,\mu,t_1,t_2} \Phi(p,q).$$

$$.\ pq\sqrt{pqt_1t_2}\ \ K_\nu(pt_1) K_\mu.$$

$$.\ (qt_2)\, dpdq.$$

...

$$= \frac{1}{\pi^2} \int_{-R}^{R} \int_{-R'}^{R'} p^{2k_1} q^{2k_2} pq \sqrt{pqt_1t_2} \; K_{\nu}(pt_1) \cdot$$

$$\cdot K_{\mu}(qt_2)dpdq.$$

$$\cdot \left\{ \int_0^{\infty} \int_0^{\infty} \emptyset(x,y) \left(\frac{x}{p}\right)^{\frac{1}{2}} \left(\frac{y}{q}\right)^{\frac{1}{2}} I_{\nu}(px).\right.$$

$$\left. \cdot \; I_{\mu}(qy) \; dxdy \right\}.$$

$$= \frac{1}{\pi^2} \int_{-R}^{R} \int_{-R'}^{R'} \sqrt{t_1t_2} \; p^{-1/2} \; q^{-1/2} \; K_{\nu}(pt_1) \cdot$$

$$\cdot K_{\mu}(qt_2) \; dpdq.$$

$$\cdot \int_0^{\infty} \int_0^{\infty} \emptyset(x,y) \; p^{2k_1} \; q^{2k_2} [pq \sqrt{pqxy} \; I_{\nu}(px).$$

$$\cdot I_{\mu}(qy) \;]dxdy.$$

$$= \frac{1}{\pi^2} \int_{-R}^{R} \int_{-R'}^{R'} \sqrt{t_1t_2} \; p^{-1/2} \; q^{-1/2} \; K_{\nu}(pt_1).$$

$$\cdot K_{\mu}(qt_2) \; dpdq.$$

$$\cdot \int_0^{\infty} \int_0^{\infty} \emptyset(x,y) \; S^{k_1,k_2}_{\nu,\mu,x,y} [pq \sqrt{pqxy} \; I_{\nu}($$

$$\cdot \; (px) \; I_{\mu}(qy) \;] \; dxdy$$

$$= \frac{1}{\pi^2} \int_{-R}^{R} \int_{-R'}^{R'} pq \sqrt{t_1t_2} \; K_{\nu}(pt_1) \; K_{\mu}(qt_2) dpdq.$$

$$\cdot \int_0^{\infty} \int_0^{\infty} \sqrt{xy} \; I_{\nu}(px) I_{\mu}(qy) \quad S^{k_1,k_2}_{\nu,\mu,x,y} \; \emptyset(x,y) dxdy.$$

The last equality is obtained by integrating by parts the inner integral $2k_1$ times with respect to x first and then $2k_2$ times with respect to y and noting that ϕ is of compact support. Let us reverse the order of integration and use equation (1.2-10) to obtain

$$L_{R,R'}^{(x,y,t_1,t_2)} = \frac{\sqrt{xyt_1t_2}}{\pi^2} \int_{-R}^{R} \int_{-R'}^{R'} pq\, K_\nu(pt_1) \,.$$

$$\cdot\, K_\mu(qt_2)\, I_\nu(px)\, I_\mu(qy)\, dpdq$$

$$= \frac{\sqrt{xyt_1t_2}\, R\, R'}{\pi^2(x^2-t_1^2)(y^2-t_2^2)} \Big\{ [xI_{\nu+1}(Rx)\, K_\nu(Rt_1)+t_1\, K_{\nu+1}\, \cdot(Rt_1)\, I_\nu(Rx)\,]. \cdot[yI_{\mu+1}(R'y)\, K_\mu(R't_2) + t_2\, K_{\mu+1}(R't_2)I_\mu(R'y)] \Big\}$$

... (2.4-3)

Hence we obtain

$$S_{\nu,\mu t_1,t_2}^{k_1,k_2}\, M_{R,R'}^{(t_1,t_2)} = \int_0^\infty \int_0^\infty L_{R,R'}^{(x,y,t_1,t_2)}\, S_{\nu,\mu,x,y}^{k_1,k_2}\, \phi(x,y)\, dxdy \,.$$

... (2.4-4)

Denote $S^{k_1,k_2}_{\nu,\mu,x,y}\ \phi(x,y)$ by $\phi_k(x,y)$

Now suppose that the support of $\phi(x,y)$ is contained in $[A,B] \times [C,D]$ where $0 < A < B < \infty$, $0 < C < D < \infty$.

Let us break the integral in (2.4-4) into

$$S^{k_1,k_2}_{\nu,\mu,t_1,t_2}\ M^{(t_1,t_2)}_{R,R'} = \int_0^{t_2-\delta}\int_0^{\infty} + \int_{t_2-\delta}^{\infty}\int_0^{t_1-\delta} +$$

$$+ \int_{t_2-\delta}^{t_2+\delta}\int_{t_1-\delta}^{t_1+\delta} + \int_{t_2-\delta}^{t_2+\delta}\int_{t_1+\delta}^{\infty} +$$

$$+ \int_{t_2+\delta}^{\infty}\int_{t_1-\delta}^{\infty}$$

$$= V_1(t_1,t_2) + V_2(t_1,t_2) + V_3(t_1,t_2) + V_4(t_1,\ t_2) +$$

$$+ V_5(t_1,t_2).$$

We shall first show that

$$N^{(t_1,t_2)}_{R,R'} \triangleq e^{at_1+bt_2}\ j_{\nu,\mu}(t_1,t_2)\ [V_3(t_1,t_2) - \phi_k(t_1,t_2)\ ].$$

converges uniformly to zero on $0 < t_1 < \infty$, $0 < t_2 < \infty$ as $R,\ R' \longrightarrow \infty$.

If either $0 < t_1+\delta \leq A$, $0 < t_2 < \infty$ or

$t_1-\delta \geq B$, $0 < t_2 < \infty$ and either $0 < t_1 < \infty$,

$0 < t_2 + \delta \leq C$ or $0 < t_1 < \infty$, $t_2 - \delta \geq D$ then

$V_3(t_1,t_2) \equiv 0$ and $\phi_k(t_1,t_2) \equiv 0$

Therefore we have to consider the rectangle $A-\delta < t_1 < B+\delta$,

$C-\delta < t_2 < D+\delta$.

Moreover since the support of $\phi C[A,B] \times [C,D]$, we take the integral in (2.4-4) on $[A,B] \times [C,D]$.

For $R,R' > 0$, using the asymptotic expansions (1.2-4), (1.2-5), (1.2-6) to estimate $N_{R,R'}^{(t_1,t_2)}$ we have for large R,R'

$$N_{R,R'}^{(t_1,t_2)} = \frac{e^{at_1+bt_2} \, j_{\nu,\mu}(t_1,t_2)}{\pi^2} \int_{t_2-\delta}^{t_2+\delta} \int_{t_1-\delta}^{t_1+\delta}$$

$$\frac{\sin(Rx - Rt_1)\sin(R'y - R't_2)}{(x-t_1)(y-t_2)} \cdot$$

$$\cdot \, e^{\sigma x - \sigma t_1} \, e^{\sigma y - \sigma t_2} \, \phi_k(x,y)\,dy\,dx + \frac{e^{at_1+bt_2} \, j_{\nu,\mu}(t_1,t_2)}{\pi^2}$$

$$\int_{t_2-\delta}^{t_2+\delta} \int_{t_1-\delta}^{t_1+\delta} \frac{\sin(Rx-Rt_1)\sin(R'y-R't_2)}{(x-t_1)(y-t_2)} \, e^{\sigma x-\sigma t_1} \cdot e^{\sigma y-\sigma t_2} \cdot$$

...

$$. \phi(x,y) \left[\left\{ O\left(\frac{1}{|px|}\right) + O\left(\frac{1}{|pt_1|}\right) + O\left(\frac{1}{|px|}\right) O\left(\frac{1}{|pt_1|}\right) \right\} .\right.$$

$$. \left\{ O\left(\frac{1}{|qy|}\right) + O\left(\frac{1}{|qt_2|}\right) + O\left(\frac{1}{|qy|}\right) O \,.\right.$$

$$\left. . \left(\frac{1}{|qt_2|}\right) \right\} \Big] \, dy dx - \frac{+e^{at_1+bt_2} \, j_{\nu,\mu}(t_1,t_2)}{\pi^2} \Big[1 +$$

$$+ O\left(\frac{1}{|pt_1|}\right) \Big] \Big[1 + O\left(\frac{1}{|qt_2|}\right) \Big] .$$

$$\int_{t_2-\delta}^{t_2+\delta} \int_{t_1-\delta}^{t_1+\delta} \frac{\operatorname{Cos}(Rx+Rt_1-\nu\pi) \quad \operatorname{Cos}(R'y + R't_2-\mu\pi)}{(x+t_1)(y+t_2)} .$$

$$e^{-6x-6t_1} \, . \, e^{-6y-6t_2} \, \phi_k(x,y) \Big[1 + O\left(\frac{1}{|px|}\right) \Big] \Big[1 +$$

$$+ O\left(\frac{1}{|qy|}\right) \Big] dy dx \; - e^{at_1+bt_2} \, j_{\nu,\mu}(t_1,t_2) \; \phi_k(t_1,t_2) .$$

... (2.4-5)

First, consider the fourth term of (2.4-5).

Since supp $\phi(x,y) \subset [A,B] \times [C,D]$, the function is bounded on

$$\left\{ (x,y,t_1,t_2) / \; A < x < B, \quad C < y < D \atop A-\delta < t_1 < B+\delta, \; C-\delta < t_2 < D+\delta \right\} .$$

Hence for any given $\epsilon > 0$, the magnitude of this term can be made less than $\epsilon/2$ for all $R, R' > 1$ by choosing δ small enough say $\delta = \delta_1$.

Now consider the sum of the first and last term in (2.4-5).

We can write this sum as

$$\frac{1}{\pi^2} \int_{t_2-\delta}^{t_2+\delta} \int_{t_1-\delta}^{t_1+\delta} H(T_1, T_2, t_1, t_2) \operatorname{Sin}(RT_1) \operatorname{Sin}(R'T_2) dT_2 dT_1 +$$

$$e^{at_1+bt_2} \, j_{\gamma,\mu}(t_1, t_2) \, \phi_k(t_1, t_2) \, .$$

$$. \left[\frac{1}{\pi^2} \int_{(t_2-\delta)R'}^{(t_2+\delta)R'} \frac{\operatorname{Sin} v}{v} dv \, . \int_{(t_1-\delta)R}^{(t_1+\delta)R} \frac{\operatorname{Sin} u}{u} du - 1 \right].$$

... (2.4-6)

where

$$H(T_1, T_2, t_1, t_2) = e^{at_1+bt_2} \, j_{\gamma,\mu}(t_1, t_2) \, .$$

$$. \left[\frac{\phi_k\left\{ (T_1+t_1)(T_2+t_2) e^{6T_1} e^{6T_2} \right\} - \phi_k(t_1, t_2)}{T_1 T_2} \right].$$

Since $H(T_1, T_2, t_1, t_2)$ is a continuous function of (T_1, T_2, t_1, t_2) and supp. $\phi(t_1, t_2) \subseteq [A,B] \times [C,D]$, $H(T_1 T_2, t_1, t_2)$ is bounded function of (T_1, T_2) on $t_1-\delta < T_1 < t_1 + \delta$, $t_2-\delta < T_2 < t_2+\delta$ for all $0 \quad t_1 < \infty$, $0 < t < \infty$.

Hence choosing δ small enough say $\delta = \delta_2$, the first term in (2.4-6) can be made less than $\epsilon/2$ for all $R, R' > 1$.

Now fix $\delta = \min. (\delta_1, \delta_2)$. The second term in (2.4-6) converges uniformly to zero on $0 < t_1 < \infty$, $0 < t_2 < \infty$.

Let us consider the second term in (2.4-5) which is

$$\frac{1}{\pi^2} e^{at_1+bt_2} j_{\nu,\mu}(t_1, t_2) \int_{t_2-\delta}^{t_2+\delta} \int_{t_1-\delta}^{t_1+\delta} \frac{\sin (Rx-Rt_1) \sin (R'y - R't_2)}{(x-t_1)(y-t_2)} \cdot$$

$$\cdot e^{\sigma x-\sigma t_1} e^{\sigma y-\sigma t_2} \phi_k(x, y) \left[\left\{ O\left(\frac{1}{|px|}\right) + O\left(\frac{1}{|pt_1|}\right) + O\left(\frac{1}{|px|}\right). O\left(\frac{1}{|pt_1|}\right) \right\} . \left\{ O\left(\frac{1}{|qy|}\right) + O\left(\frac{1}{|qt_2|}\right) + O\left(\frac{1}{|qy|}\right) O\left(\frac{1}{|qt_2|}\right) \right\} \right] dydx.$$

Since

$$\left| \frac{\sin(Rx-Rt_1)\sin (R'y-R't_2)}{(x-t_1)(y-t_2)} \cdot O(|px|^{-1})\, O(|qy|^{-1}) \right.$$

...

$$\leq \left| \frac{\sin(Rx-Rt_1)\ \sin\ (R'y-R't_2)}{(Rx-Rt_1)(R'y-R't_2)} \right| \cdot \frac{C_{R,R'}}{|6 \pm iR|\ |6 \pm iR'|}$$

$$\leq \ C$$

and similarly

$$\left| \frac{\sin(Rx-Rt_1)\sin\ (R'y-R't_2)}{(x-t_1)(y-t_2)} \cdot 0\ (|pt_1|^{-1})\ 0(\ |qt_2|^{-1}) \right| \leq C$$

Hence for given $\epsilon > 0$, the magnitude of this term can be made less than $\epsilon/2$ for all $R, R' > 1$ by choosing δ small enough say $\delta = \delta_3$. Similarly we can show the convergence of the third term in (2.4-5).

Thus $\left| N_{R,R'}^{(t_1, t_2)} \right| < \epsilon$ on $0 < t_1 < \infty$, $0 < t_2 < \infty$.

Since $\epsilon > 0$ is arbitrary, we conclude that $N_{R,R'}^{(t_1, t_2)}$ converges uniformly to zero on $0 < t_1 < \infty$, $0 < t_2 < \infty$ as $R, R' \longrightarrow \infty$.

Now consider

$$P_{R,R'}^{(t_1, t_2)} = e^{at_1+bt_2}\ j_{\gamma,\mu}{}^{(t_1, t_2)}\ \ V_1(t_1, t_2)$$

$$= e^{at_1+bt_2}\ j_{\gamma,\mu}(t_1, t_2) \int_0^{t_2-\delta} \int_0^{\infty} L_{R,R'}^{(x, y, t_1, t_2)}$$

$$\cdot\ \phi_k(x, y)\ dydx$$

For $0 < t_1 < \infty$, $t_2 - \delta < C$, $P_{R,R'}^{(t_1, t_2)} \equiv 0$

Therefore, we have to consider the range

$0 < t_1 < \infty, \quad C < t_2-\delta < \infty$

Let $D' = \min(D,\ t_2-\delta)$

For large R,R'; using the asymptotic expansions as before, we get

$$P_{R,R'}^{(t_1,t_2)} = \frac{e^{(at_1+bt_2)-\sigma(t_1+t_2)}\ J_{\nu,\mu}^{(t_1,t_2)}}{\pi^2} \cdot$$

$$\cdot \left[\int_0^{t_2-\delta} \int_0^{\infty} e^{\sigma x}\ e^{\sigma y}\ \phi_k(x,y) \frac{\sin(Rx-Rt_1)\ \sin(R'y-R't_2)}{(x-t_1)(y-t_2)}\, dydx \,- \right.$$

$$- \int_0^{t_2-\delta} \int_0^{\infty} \frac{e^{-\sigma x}\ e^{-\sigma y}\ \phi_k(x,y)\ \cos(Rx+Rt_1 - \nu\pi)\cos(R'y + R't_2 - \mu\pi)}{(x+t_1)(y+t_2)}\, dydx \,+$$

$$+ \int_0^{t_2-\delta} \int_0^{\infty} \left\{ \frac{e^{\sigma x}\ e^{\sigma y}\ \phi_k(x,y)\sin(Rx - Rt_1)\sin(R'y - R't_2)}{(x-t_1)(y-t_2)} \right. +$$

...

$$+ \frac{e^{-\sigma x}\, e^{-\sigma y}\, \phi_k(x,y) \mathrm{Cos}(Rx+Rt_1 - \nu\,\pi)\ \mathrm{Cos}\ (R'y+R't_2-\mu\pi)}{(x+t_1)\ (y+t_2)} \Bigg\} .$$

$$.\left\{ 0\left(\frac{1}{|px|}\right) + 0\left(\frac{1}{|pt_1|}\right) + 0\left(\frac{1}{|px|}\right) 0\left(\frac{1}{|pt_1|}\right) \right\} .$$

$$.\left\{ 0\left(\frac{1}{|qy|}\right) + 0\left(\frac{1}{|qt_2|}\right) + 0\left(\frac{1}{|qy|}\right) 0\left(\frac{1}{|qt_2|}\right) \right\} . \Big] dydx$$

$$\ldots (2.4\text{-}7)$$

We note that $e^{(at_1+bt_2)-\sigma(t_1+t_2)}\ j_{\nu,\mu}(t_1, t_2)$

is bounded function for $t_1 - \delta > A$, $t_2 - \delta > B$.

Similarly the quantity

$$\frac{e^{\sigma x}\, e^{\sigma y}\, \phi_k(x,y)\ \mathrm{Sin}\ (Rx-Rt_1)\mathrm{Sin}\ (R'y - R't_2)}{(x-t_1)(y-t_2)} +$$

$$+ \frac{e^{-\sigma x}\, e^{-\sigma y} \phi_k(x,y)\ \mathrm{Cos}(Rx+Rt_1 - \nu\pi)\ \mathrm{Cos}(R'y + R't_2 - \mu\pi)}{(x+t_1)(y+t_2)} \text{ is}$$

bounded on the domain

$$\text{Ⓗ} = \left\{ (x, y, t_1, t_2) \Big/ \begin{array}{l} A < x < B, \quad C < y < D, \\ 0 < t_1 < \infty, \quad C < t_2 - \delta < \infty \end{array} \right\}$$

which implies that the last term in (2.4-7) converges uniformly to zero for $0 < t_1 < \infty$, $C < t_2 - \delta < \infty$ as $R, R' \longrightarrow \infty$.

Now integrating by parts the inner integral in the first term within the braces in (2.4-7) and noting that for any $y > 0$, the limits at A and at B are zero, the integral in the first term equals

$$\int_{C}^{D'} \frac{e^{6y}}{(y-t_2)} \frac{1}{R} \left\{ \int_{A}^{B} \left[D_x \left(\frac{e^{6x} \phi_k(x,y)}{(x-t_1)} \right] \cos (Rx-Rt_1)\, dx \right\} . \right.$$

$$\sin (R'y - R't_2)\, dy.$$

Again integrating by parts with respect to y, we get

$$\frac{1}{RR'} \left\{ \frac{e^{6y}}{(y-t_2)} \int_{A}^{B} \left[D_x \left(\frac{e^{6x} \phi_k(x,y)}{(x-t_1)} \right) \right] \cos(Rx-Rt_1)\, dx \right. .$$

$$\left\{ -\cos (R'y - R't_2) \right\}_{C}^{D'} + \frac{1}{RR'} \int_{C}^{D'} dy \int_{A}^{B} \left[\frac{\partial^2}{\partial x\, \partial y} \right. .$$

$$. \left(\frac{e^{6x}\, e^{6y}\, \phi_k (x,y)}{(x-t_1)(y-t_2)} \right) . [\cos(Rx-Rt_1)][\cos(R'y-R't_2)]\, dx$$

... (2.4-8)

The lower limit term is zero and if $D \leq t_2-\delta$, the upper limit term is also zero.

Since $D_x \left(\frac{e^{6x} \phi_k(x,y)}{(x-t_1)} \right)$ is bounded by a constant

say M on $\left\{ (x,y,t_1)/A < x < B,\ C < y < D,\ 0 < t_1 < \infty \right\}$

for $t_2-\delta < D$, the upper term is bounded by $\frac{1}{RR'} M(B-A)$,

consequently the upper limit term converges to zero for

$0 < t_1 < \infty$, $C < t_2-\delta < \infty$ as $R, R' \to \infty$.

Moreover, $\frac{\partial^2}{\partial x\, \partial y} \left(\frac{e^{6x}\, e^{6y}\, \phi_k(x,y)}{(x-t_1)(y-t_2)} \right)$ is bounded on the

domain Ⓗ which implies that the second term in (2.4-8)

also converges uniformly to zero for $0 < t_1 < \infty$,

$C < t_2 - \delta < \infty$ as $R, R' \to \infty$.

Now consider the second term in (2.4-7).

Upon integrating by parts the inner integral, we have

$$\left\{ \frac{\phi_k(x,y) e^{-6x} \operatorname{Sin}(Rx+Rt_1 - \nu\pi)}{R(x+t_1)} \right\}_A^B -$$

$$- \int_A^B D_x \left(\frac{\phi_k(x,y)\ e^{-6x}}{(x+t_1)} \right) \frac{\operatorname{Sin}(Rx+Rt_1 - \nu\pi)}{R}\, dx$$

For any $y > 0$, both the upper and lower limits are zero

and hence the second term in (2.4-7) equals to

$$\frac{e^{(at_1+bt_2) - 6(t_1+t_2)}\ j_{\nu,\mu}(t_1, t_2)}{\pi^2} \int_C^{D'} dy \int_A^B$$

$$\cdot \left[D_x \left(\frac{\phi_k(x,y) e^{-6x}}{(x+t_1)} \right) \cdot \frac{\operatorname{Sin}(Rx+Rt_1 - \nu\pi)}{R} \cdot \right.$$

$$\cdot \frac{\operatorname{Cos}\,(R'y + R't_2 - \mu\pi)}{(y + t_2)} \cdot e^{-6y}\, dx.$$

Since

$$\frac{e^{(at_1+bt_2) - \sigma(t_1+t_2)}\, e^{-\sigma y}}{(y + t_2)} \quad Dx \left(\frac{\phi_k(x,y)e^{-\sigma x}}{(x+t_1)} \right)$$

is bounded on the domain (H) , the second term in (2.4-7) converges uniformly to zero on $0 < t_1 < \infty$, $C < t_2-\delta < \infty$ as $R \to \infty$ for all $R' > 1$. Similarly we can show the convergence of the third term. Let us consider the emergence of the fourth term in (2.4-7).

For all $R, R' > 1$, the integrand is bounded on

$$\left\{ (x,y,t_1,t_2) \;/\; A < x < B,\; C < y < D,\; 0 < t_1 < \infty,\; C < t_2-\delta < \infty \right\}$$

by a constant independent of R and R'. Therefore for given $\epsilon > 0$, we can choose δ so small, say $\delta = \delta_4$ such that the magnitude of this term can be made less than $\epsilon/2$ for all $R, R' > 1$.

Thus $P_{R,R'}(t_1,t_2)$ converges uniformly to zero on $0 < t_1 < \infty$, $0 < t_2 < \infty$.

Let $P'_{R,R'}(t_1,t_2) = e^{at_1+bt_2} J_{\nu,\mu}(t_1,t_2) \; V_4(t_1,t_2)$

$$= \frac{e^{at_1+bt_2} J_{\nu,\mu}(t_1,t_2)}{\pi^2} \int_{t_2-\delta}^{t_2+\delta} \int_{t_1+\delta}^{\infty} L_{R,R'}(x,y,t_1,t_2) \cdot$$

...

$\phi_k(x, y)$ dydx.

For the ranges $B \leq t_1+\delta < \infty$, $0 < t_2 < \infty$, $0 < t_1 < \infty$, $0 < t_2+\delta < C$ and $0 < t_1 < \infty$, $D \leq t_2-\delta < \infty$

$$P'_{R,R'}(t_1, t_2) \equiv 0.$$

So we consider the range $0 < t_1+\delta < B$, $C - \delta < t_2 < D+\delta$

Let $A' = \max(A, t_1+\delta)$

Using the asymptotic expansions for $R > 0$ and large R', we have

$$P'_{R,R'}(t_1+t_2) = \frac{e^{(at_1+bt_2)-\sigma(t_1+t_2)} \, j_{\nu,\mu}(t_1, t_2)}{\pi^2} \times$$

$$\times \int_{t_2-\delta}^{t_2+\delta} \int_{A'}^{B} \frac{e^{\sigma x} e^{\sigma y} \phi_k(x, y) \text{Sin}(Rx-Rt_1)\text{Sin}(R'y-R't_2)}{(x-t_1)(y-t_2)} \, dy \, dx$$

$$- \int_{t_2-\delta}^{t_2+\delta} \int_{A'}^{B} \frac{e^{-\sigma x} e^{-\sigma y} \phi_k(x, y)\text{Cos}(Rx+Rt_1-\nu\pi)\text{Cos}\,(}{(x+t_1)(y+t_2)}$$

$$\frac{.(R'y + R't_2 - \mu\pi)}{} \, . \, dy \, dx \quad +$$

$$+ \int_{t_2-\delta}^{t_2+\delta} \int_{A'}^{B} \Big[\frac{e^{\sigma x} e^{\sigma y} \phi_k(x, y)\text{Sin}(Rx-Rt_1)\text{Sin}\,(R'y-R''t_2)}{(x-t_1)(y-t_2)} +$$

...

$$+ \frac{e^{-\delta x}\, e^{-\delta y}\, \phi_k(x, y)\, \text{Cos}(Rx + Rt_1 - \nu\pi)\, \text{Cos}(R'y + \dot{R}t_2 - \mu\pi)}{(x + t_1)(y + t_2)} \Big] .$$

$$.\Big[\, 0\big(\frac{1}{|px|}\big) + 0\big(\frac{1}{|pt_1|}\big) + \big(\frac{1}{|px|}\big)\, 0\big(\frac{1}{|pt_1|}\big) \Big] .$$

$$.\Big[\, 0\big(\frac{1}{|qy|}\big) + 0\big(\frac{1}{|qt_2|}\big) + 0\big(\frac{1}{|qy|}\big).\, 0\big(\frac{1}{|qt_2|}\big) \Big] .$$

$$.\ dy\, dx \qquad \ldots (2.4\text{-}9)$$

Consider the inner integral in the first term in the braces in (2.4-9) which is

$$\int_{A'}^{B} \frac{e^{\delta x}\, \phi_k(x, y)}{(x-t_1)}\, \text{Sin}\,(Rx-Rt_1)\, dx$$

Upon integrating by parts the above integral we get

$$\left\{ \frac{-\, e^{\delta x}\, \phi_k(x, y)}{(x-t_1)} \quad \frac{\text{Cos}\,(Rx - Rt_1)}{R} \right\}_{A'}^{B} +$$

$$+ \frac{1}{R} \int_{A'}^{B} \Big[D_x \Big(\frac{e^{\delta x}\, \phi_k(x, y)}{(x-t_1)}\, \text{Cos}\,(Rx - Rt_1)\, dx. \quad \ldots (2.4\text{-}10)$$

The upper limit term is zero so is the lower limit term if $A > t_1 + \delta$ and on the other hand, if $t_1 + \delta > A$, the lower limit term is bounded by

$$(R\delta)^{-1} \sup_{\substack{A<x<B \\ C<y<D}} \left| \frac{e^{\sigma x} \phi_k(x,y)}{(x-t_1)} \right|$$

Consequently, the lower limit term converges to zero for $0 < t_1+\delta < A$, $C < y < D$ as $R \to \infty$. Moreover

$$D_x \left(\frac{e^{\sigma x} \phi_k(x,y)}{(x-t_1)} \right)$$ is bounded on the domain,

$$\left\{ (x,y,\ t_1)/A < x < B, \quad C < y < D, \quad 0 < t_1+\delta < B \right\}$$

which implies that the second term in (2.4-10) also converges uniformly to zero for $0 < t_1+\delta < B$, $C < y < D$ as $R \to \infty$.

Hence for large R', first, second and third terms in (2.4-9) converge to zero uniformly for $0 < t_1+\delta < B$, $C - \delta < t_2 < D+\delta$ as $R \to \infty$.

Now for all $R, R' > 1$, the integrand in the fourth term in braces is bounded on $\{ (x,y,t_1,t_2)/ A < x < B,$

$C < y < D, \quad 0 < t_1 +\delta < B, \quad C - \delta < t_2 < D+\delta. \}$

by a constant independent of R and R'. Thus $P'_{R,R'}(t_1,t_2)$ converges uniformly to zero on $0 < t_1 < \infty$, $0 < t_2 < \infty$.

Again using the similar arguments as in the previous cases, we can show that $N'_{R,R'}(t_1,t_2)$ and $Q_{R,R'}(t_1,t_2)$ converge uniformly to zero for $0 < t_1 < \infty$, $0 < t_2 < \infty$

as R, R' —> ∞.

where $N'_{R,R'}(t_1,t_2)$ and $Q_{R,R'}(t_1,t_2)$ are given by

$$N'_{R,R'}(t_1,t_2) = \frac{e^{at_1+bt_2} \; j_{\nu,\mu}(t_1,t_2)}{\pi^2} \int_{t_2-\delta}^{\infty} \int_{0}^{t_1-\delta}$$

$$L_{R,R'}(x,y,t_1,t_2) \; \phi_k(x,y) \; dy \; dx$$

and

$$Q_{R,R'}(t_1,t_2) = \frac{e^{at_1+bt_2} \; j_{\nu,\mu}(t_1,t_2)}{\pi^2} \int_{t_2+\delta}^{\infty} \int_{t_1+\delta}^{\infty}$$

$$\cdot \; L_{R,R'}(x,y,t_1,t_2) \quad \phi_k \; (x,y) \; dy \; dx.$$

Thus

$$e^{at_1+bt_2} \; t_1^{\nu-1/2} \; t_2^{\mu-1/2} \; S^{k_1,k_2}_{\nu,\mu,t_1,t_2} \; [\; M_{R,R'}(t_1,t_2)$$

$$- \; \phi \; (t_1,t_2) \;] \longrightarrow 0$$ uniformly on $0 < t_1 < \infty$,

$0 < t_2 < \infty$ as R, R' —> ∞.

which implies that $M_{R,R'}(t_1,t_2) \rightarrow \phi(t_1,t_2)$ in $K_{\nu,\mu,a,b}$

as R, R' —> ∞, and the theorem is proved.

As a result of the inversion theorem we have the following uniqueness theorem.

Theorem 2.4-2

Let $F(p,q) = k_{\nu,\mu}(f)\ (p,q)$ for $(p,q) \in \Omega_f$ and $G(p,q) = k_{\nu,\mu}(f)\ (p,q)$ for $(p,q) \in \Omega_g$.
If $\Omega_f \cap \Omega_g \neq \emptyset$ and $F(p,q) = G(p,q)$ for $(p,q) \in \Omega_f \cap \Omega_g$ then $f = g$ in the sense of equality in $D'(I)$.

Proof : By the inversion theorem, in the sense of convergence in $D'(I)$, we have

$$f(x,y) - g(x,y) = \lim_{R,R' \to \infty} \int_{o-iR}^{o+ir} \int_{o-iR'}^{o+iR'} [F(p,q) - G(p,q)] .$$

$$. \left(\frac{x}{p}\right)^{1/2} \left(\frac{y}{q}\right)^{1/2} I_{\nu}(px)\, I_{\mu}(qy)\, dp dq = 0.$$

Thus $f = g$ in the sense of equality in $D'(I)$.