CHAPTER - TWO

2.1.1 The space $H_{\mu,\lambda,\alpha}$ -

The space $H_{\mu,\lambda,\alpha}$ is the space of smooth functions ϕ on $0 < x < \infty$ satisfying

$$\gamma_{m,k}^{\mu,\lambda,\alpha}(\phi) = \sup_{x} \left| x^{m\lambda} (x^{1-2\lambda}D)^{k} x^{-\lambda\mu-\lambda+1/2} \phi(x) \right|$$

$$< A_k B^{m\lambda} m^{m\lambda\alpha}$$

where $m,\ k = 0, 1, 2 \ldots$ and $\alpha \geq 0$.

The constants A_k and B depend upon ϕ. For $m = 0$, we shall take $m^{m\lambda\alpha} = 1$.

2.1.2 The space $H_{\mu,\lambda}^{\beta}$ -

The space $H_{\mu,\lambda}^{\beta}$ is the space of smooth functions ϕ on $0 < x < \infty$ satisfying

$$\gamma_{m,k}^{\mu,\lambda}(\phi) = \sup_{x} \left| x^{m\lambda} (x^{1-2\lambda}D)^{k} x^{-\lambda\mu-\lambda+1/2} \phi(x) \right|$$

$$< A_{m\lambda} B^{k} K^{k\beta}$$

where $m,\ k = 0, 1, 2 \ldots.$ and $\beta \geq 0$

The constants $A_{m\lambda}$ and B depend upon ϕ and for $k = 0$, $K^{k\beta} = 1$.

2.1.3. The space $H_{\mu,\lambda,\alpha}^{\beta}$ -

The space $H_{\mu,\lambda,\alpha}^{\beta}$, $\alpha \geq 0$, $\beta \geq 0$ is the space of smooth functions ϕ on $0 < x < \infty$ satisfying ,

$$\gamma_{m,k}^{\mu,\lambda,\alpha}(\phi) = \sup_{x} \left| x^{m\lambda}(x^{1-2\lambda} D)^{k} x^{-\lambda\mu-\lambda+1/2} (\phi)(x) \right|$$

$$< AB^{m\lambda} C^{k} m^{m\lambda\alpha} K^{k\beta}$$

where $m, k = 0,1,2....$

The constants A, B, C depend upon ϕ.

Note : These spaces are the countable union of the spaces $H^{\beta}_{\mu,\lambda,\alpha}$ written as

$$H^{\infty}_{\mu,\lambda,\alpha} = \bigcup_{\beta_i=1}^{\infty} H^{\beta_i}_{\mu,\lambda,\alpha}$$

$$H^{\beta}_{\mu,\lambda,\infty} = \bigcup_{\alpha_i=1}^{\infty} H^{\beta}_{\mu,\lambda,\alpha_i}$$

$$H^{\infty}_{\mu,\lambda,\infty} = \bigcup_{\alpha_i,\beta_i=1}^{\infty} H^{\beta_i}_{\mu,\lambda,\alpha_i}$$

2.2 Another way of defining the spaces of the type $H_{\mu,\lambda,\alpha}$ -

2.2.1 The space $H_{\mu,\lambda,\alpha}$ -

Definition of the space $B_{\mu,A}$ -

The space $B_{\mu,A}$ is defined as follows -

It is a space of smooth functions ϕ s.t.

$\phi(x) = 0$ for $x > A$ and

$$\gamma_q^{\mu}(\phi) = \sup_{0 < x < \infty} \left| (x^{-1}D)^q \, x^{-\mu-1/2} \, \phi(x) \right| < \infty$$

where $q = 0, 1, 2 \ldots.$

If $\alpha = 0$ and $\lambda = 1$ we have

$$B_{\mu,A} \subseteq H_{\mu,1,0} \, .$$

Now let $\alpha > 0$, we shall prove the following theorem :

Theorem 2.2.1 -

Let $\phi \in H_{\mu,\lambda,\alpha}$, $\alpha > 0$. Then

$$\left| (x^{1-2\lambda} \, D)^k \, x^{-\lambda\mu-\lambda+1/2} \, \phi(x) \right| < A_k' \exp\left(-a \, (x)^{1/\alpha}\right)$$

where $a = \dfrac{\lambda \, \alpha}{e \, B^{1/\alpha}}$

Proof -

We have defined the space $H_{\mu,\lambda,\alpha}$ as the space of smooth functions ϕ on $0 < x < \infty$ s.t.

$$\sup_x \left| x^{m\lambda} (x^{1-2\lambda} D)^k \, x^{-\lambda\mu-\lambda+1/2} \, \phi(x) \right| < A_k B^{m\lambda} \, m^{m\lambda\alpha}$$

where $m, k = 0, 1, 2 \ldots$

Divide both sides by $|x|^{m\lambda}$ and take the infimum, we get

$$\left| (x^{1-2\lambda} \, D)^k \, x^{-\lambda\mu-\lambda+1/2} \, \phi \right| < \inf_m \frac{A_k \, B^{m\lambda} \, m^{m\lambda\alpha}}{|x|^{m\lambda}}$$

$$< A_k \, \mu_\alpha \, (x/B)$$

where $\mu_\alpha(\xi) = \inf\limits_m \dfrac{m^{m\lambda\alpha}}{|\xi|^{m\lambda}}$

Now if $\alpha = 0$, $\mu_0(\xi) = \inf \dfrac{1}{|\xi|^{m\lambda}}$

$$= 1 \text{ if } \xi \leq 1$$

$$= 0 \text{ if } \xi > 1$$

$\therefore \quad \mu_\alpha(x/B) = 1$ if $\dfrac{x}{B} \leq 1$ i.e. $x \leq B$

$$= 0 \text{ if } \frac{x}{B} > 1 \text{ i.e. } x > B$$

Thus the function $\phi(x) = 0$ for $x > B$.

Now let $\alpha > 0$

Let $f(m) = \dfrac{m^{m\lambda\alpha}}{\xi^{m\lambda}}$ ... (1)

To find the minimum of the function $f(m)$, we shall take logarithm, differentiate and equate the result to zero.

$$\log f(m) = \log \frac{m^{m\lambda\alpha}}{\xi^{m\lambda}}$$

$$= m\lambda\alpha \log m - m\lambda \log \xi$$

Differentiating both sides, we get

$$\frac{1}{f(m)} \cdot f'(m) = m\lambda\alpha . \frac{1}{m} + \log m . (\lambda\alpha) - \lambda \log \xi .1$$

$$= \lambda\alpha + \lambda\alpha \log m - \lambda \log \xi$$

Equating R.H.S. to zero we get

$$\lambda\alpha + \lambda\alpha \log m - \lambda \log \xi = 0$$

i.e $\alpha + \alpha \log m - \log \xi = 0$

If m_o is the value of m at which f(m) is minimum, we have

$$\alpha + \alpha \log m_o = \log \xi$$

$$\therefore \quad \log m_o = \frac{1}{\alpha} \log \xi - \frac{1}{\alpha} \cdot \alpha$$

$$= \log \xi^{1/\alpha} - 1$$

$$= \log \xi^{1/\alpha} - \log e$$

Thus $\log m_o = \log \xi^{1/\alpha} - \log e$

i.e. $\log m_o = \log (\xi^{1/\alpha}/ e)$

$$\therefore \quad m_o = \frac{1}{e} \xi^{1/\alpha} \qquad \ldots (2)$$

Now $\log f(m_o) = \log \left[\dfrac{m_o^{m_o\lambda\alpha}}{\xi^{m_o\lambda}} \right]$

$$= m_o\lambda\alpha \log m_o - m_o\lambda \log \xi$$

$$= m_o\lambda [\alpha \log m_o - \log \xi]$$

$$= \frac{1}{e} \xi^{1/\alpha} . \lambda [\alpha \log (\frac{1}{e} \xi^{1/\alpha}) - \log \xi] \text{ by } (2)$$

$$= \frac{\lambda}{e} \xi^{1/\alpha} [\alpha . \frac{1}{\alpha} \log \xi - \alpha . \log e - \log \xi]$$

$$= \frac{\lambda}{e} \xi^{1/\alpha} [\log \xi - \alpha - \log \xi]$$

$$= \frac{\lambda}{e} \xi^{1/\alpha} [- \alpha]$$

$$\therefore \quad \log f(m_o) = -\frac{\lambda\alpha}{e}\xi^{1/\alpha}$$

$$\text{i.e. } \log \min f(m) = -\frac{\lambda\alpha}{e}\xi^{1/\alpha}$$

$$\therefore \quad \min f(m) = \exp.\left[-\frac{\lambda\alpha}{e}\xi^{1/\alpha}\right] \qquad \ldots (3)$$

Differentiating the function f(m) again we get

$$(\log f(m))'' = \frac{\lambda\alpha}{m}$$

Now let the integer m_1 be very close to m_o. Then at m_1

$$\log f(m_1) = \log f(m_o) + \frac{\lambda\alpha}{2m_2}(m_1 - m_2)^2$$

$$< \log f(m_o) + \frac{\lambda\alpha}{2m_o}$$

$$\therefore \quad \log f(m_1) < \frac{-\lambda\alpha}{e}\xi^{1/\alpha} + \frac{\lambda\alpha e}{2}\xi^{-1/\alpha}$$

Thus

$$\min \log f(m) < \min \log f(m_1)$$

$$< \log f(m_o) + \frac{\lambda\alpha}{2m_o}$$

$$< \frac{-\lambda\alpha}{e}\xi^{1/\alpha} + \frac{\lambda\alpha e}{2}\xi^{-1/\alpha}$$

$$\text{i.e. } \min f(m) < \exp\left[-\frac{\lambda\alpha}{e}\xi^{1/\alpha} + \frac{\lambda\alpha e}{2}\xi^{-1/\alpha}\right]$$

$$\text{or } \min f(m) < \exp\left(-\frac{\lambda\alpha}{e}\xi^{1/\alpha}\right).\exp\left(\frac{\lambda\alpha e}{2}\xi^{-1/\alpha}\right)$$

If $\xi \geqslant 1$, $\exp\left(\frac{\lambda\alpha e}{2}\ \xi^{-1/\alpha}\right)$ is bounded by the constant

$C = \exp.\left(\frac{\lambda\alpha e}{2}\right)$

If $0 < \xi < 1$,

$$\min f(m) < 1 < \exp\left(\frac{\lambda\alpha}{e}\right) \exp\left(-\frac{\lambda\alpha}{e}\ \xi^{1/\alpha}\right)$$

$\therefore$ For any ξ, $0 < \xi < \infty$

$$\exp\left(-\frac{\lambda\alpha}{e}\ \xi^{1/\alpha}\right) < \mu_\alpha(\xi) < C.\ \exp.\left(-\frac{\lambda\alpha}{e}\ \xi^{1/\alpha}\right).$$

where $C = \exp\left(\frac{\lambda\alpha e}{2}\right)$ and $\min f(m) = \mu_\alpha(\xi)$

Therefore,

$$\left| (x^{1-2\lambda} D)^k\, x^{-\lambda\mu-\lambda+1/2}\, \phi(x) \right| < A_k\, \mu_\alpha\,(x/B)$$

$$< A_k.\ C.\ \exp\left(\frac{-\lambda\alpha}{e}\left(\frac{x}{B}\right)^{1/\alpha}\right)$$

$$< A_k'\ \exp\left(-a.\,|x|^{1/\alpha}\right)$$

where $a = \dfrac{\lambda\alpha}{e\, B^{1/\alpha}}$

Hence the theorem.

Thus we can say that all functions $\phi(x)$ belong to this space, together with all their derivatives decrease exponentially at infinity, with an order $\geq 1/\alpha$ and a type $\geq a$, dependent on the function ϕ. [see definition 1.1]

2.3 Topological property of the space $H_{\mu,\lambda,\alpha}$ -

The space $H_{\mu,\lambda,\alpha}$ as the union of countably normed spaces -

Let $H_{\mu,\lambda,\alpha,B}$ be the space of all testing functions $\phi \in H_{\mu,\lambda,\alpha}$ such that the condition

$(\bar{B})^{m\lambda}$ better

$$\left| x^{m\lambda} \left(x^{1-2\lambda} D\right)^k x^{-\lambda\mu-\lambda+1/2} \phi(x) \right| < A_{k\bar{A}} \, \bar{B}^{m\lambda} \, m^{m\lambda\alpha}$$

is satisfied.

where $\bar{B}$ is the constant greater than B.

OR

$H_{\mu,\lambda,\alpha,B}$ is the space of all testing functions $\phi(x)$ satisfying the inequality

$$\left| x^{m\lambda} \left(x^{1-2\lambda} D\right)^k x^{-\lambda\mu-\lambda+1/2} \phi(x) \right| < A_{k\delta} \, (B+\delta)^{m\lambda} \, m^{m\lambda\alpha}$$

for any $\delta > 0$.

Then according to theorem 1, the another definition of $H_{\mu,\lambda,\alpha,B}$ is that, it is a space of all functions $\phi(x)$ which satisfy the inequality

$$\left|(x^{1-2\lambda}D)^k \, x^{-\lambda\mu-\lambda+1/2}\, \phi(x)\right| < A'_{k\delta} \exp\left[\,(-a+\delta)\,|x|^{1/\alpha}\right]$$

where $a = \dfrac{\lambda\alpha}{e\,B^{1/\alpha}}$ and $\delta > 0$.

Now let us define

$$M_p(x) = \exp\left[\,a\,\left(1-\frac{1}{p}\right)|x|^{1/\alpha}\right]$$

$$p = 2, 3 \ldots$$

If we define

$$\varrho_p^{\mu,\lambda,\alpha}(\phi) = \max_{0<k<p} \sup_x M_p(x)\left|(x^{1-2\lambda}D)^k \, x^{-\lambda\mu-\lambda+1/2}\phi(x)\right|$$

Then $\varrho_p^{\mu,\lambda,\alpha}$ is a norm for the space

space $H_{\mu,\lambda,\alpha,B}$.

Thus $H_{\mu,\lambda,\alpha,B}$ is a space of all functions $\phi(x)$ such that

$$\varrho_p^{\mu,\lambda,\alpha}(\phi) = \max_{0<k<p} \sup_x M_p(x)\left|(x^{1-2\lambda}D)^k x^{-\lambda\mu-\lambda+1/2}\,\phi(x)\right|$$

is finite.

Since the space $H_{\mu,\lambda,\alpha,B}$ belongs to the class of space $K\{M_p\}$ and every $K\{M_p\}$ is complete countably normed space [1] p. 88-89.

Therefore, the space $H_{\mu,\lambda,\alpha,B}$ is complete countably normed space.

Now the condition for the space to be Perfect is that, for each subscript p, there exists a subscript p' > p such that

$$\lim_{x \to \infty} \frac{M_p(x)}{M_{p'}(x)} = 0.$$

Since $M_p(x) = \exp\left[\, a\,(1 - 1/p)\,|x|^{1/\alpha}\,\right]$

We have for any p' > p

$$\frac{M_p(x)}{M_{p'}(x)} = \exp\left[\, a(1 - 1/p)\,|x|^{1/\alpha} - a(1 - 1/p')\,|x|^{1/\alpha}\right]$$

$$= \exp\left[\, a\left(\frac{1}{p'} - \frac{1}{p}\right)|x|^{1/\alpha}\,\right]$$

$$\longrightarrow 0 \text{ as } x \longrightarrow \infty$$

Hence the condition is satisfied.

Therefore the space $H_{\mu,\lambda,\alpha,B}$ is perfect.

Now let us consider the another system of norms in the space $H_{\mu,\lambda,\alpha,B}$ as

$$\rho_{q\delta}^{\mu,\lambda,\alpha}(\phi) = \sup_m \sup_x \frac{\left| x^{m\lambda}(x^{1-2\lambda}D)^k\, x^{-\lambda\mu-\lambda+1/2}\, \phi(x)\right|}{(B+\delta)^{m\lambda}\; m^{m\lambda\alpha}}$$

Now we shall show that the two norms are equal.

For that, we shall find the suprimum with respect to the index m. We get

$$\sup_m \frac{|x|^{m\lambda}}{(B+\delta)^{m\lambda}\, m^{m\lambda\alpha}}$$

$$= \frac{1}{\inf.\ (B+\delta)^{m\lambda}\, m^{m\lambda\alpha}/|x|^{m\lambda}}$$

$$= \frac{1}{\mu_\alpha\left(\frac{|x|}{B+\delta}\right)} \quad \text{where } \mu_\alpha(\xi) = \inf \frac{K^{k\alpha}}{|\xi|^k}$$

Now using the inequality of the previous section (2.2.1)

$\dfrac{1}{\mu_\alpha(\xi)} < \exp\left(\dfrac{\lambda\alpha}{e}\ \xi^{1/\alpha}\right)$ we get

$$\frac{1}{\mu_\alpha\left(\frac{|x|}{B+\delta}\right)} < \exp\left[\frac{\lambda\alpha}{e}\left(\frac{|x|}{B+\delta}\right)^{1/\alpha}\right] < M_p(x)$$

for some p

Therefore,

$$\rho_{q\delta}^{\mu,\lambda,\alpha}(\phi) < \sup_x M_p(x)\ \left|(x^{1-2\lambda}D)^k\, x^{-\lambda\mu-\lambda+1/2}\, \phi(x)\right|$$

$$= \rho_p^{\mu,\lambda,\alpha}$$

Thus $\rho_{q\delta}^{\mu,\lambda,\alpha}(\phi) < \rho_p^{\mu,\lambda,\alpha}$

Similarly we can show that

$$\rho_p^{\mu,\lambda,\alpha}(\phi) < \rho_{q\delta}^{\mu,\lambda,\alpha}(\phi)$$

Thus the two norms are equal.

Therefore they generate the same topology.

Now if $B_1 < B_2$. Then,

$$H_{\mu,\lambda,\alpha,B_1} \subsetneq H_{\mu,\lambda,\alpha,B_2}$$

and topology of $H_{\mu,\lambda,\alpha,B_1}$ is stronger than the topology induced by $H_{\mu,\lambda,\alpha,B_2}$.

Therefore by definition,

We can construct the union of countably normed spaces over all $B_i^{'s}$

i.e. $H_{\mu,\lambda,\alpha} = \bigcup_{i=1}^{\infty} H_{\mu,\lambda,\alpha,B_i}$

Thus the space $H_{\mu,\lambda,\alpha}$ is a union of countably normed spaces.

Therefore, a sequence $\{\phi_\nu\}_{\nu=1}^{\infty}$ converges to zero in $H_{\mu,\lambda,\alpha}$ that is all $\phi_\nu^{'s}$ belong to some $H_{\mu,\lambda,\alpha,B}$ and they converge to zero in that particular space

This is true only ~~i.e.~~ if $\{\phi_\nu\}$ converges to zero exactly and the norms $\varrho_p^{\mu,\lambda,\alpha}(\phi_\nu)$ are bounded for all p and ν .

2.4)

2.4.1 Another way of defining the space of the type $H^{\beta}_{\mu,\alpha}$-

By definition, $H^{\beta}_{\mu,\alpha}$ consist of smooth functions $\phi(x)$ on $0 < x < \infty$ satisfying

$$\left| x^{m\lambda}(x^{1-2\lambda}D)^k \, x^{-\lambda\mu-\lambda+1/2}\phi(x) \right| < A_{m\lambda} B^k k^{k\beta}$$

where m, k = 0,1,2... and $\beta \geqslant 0$.

The constants $A_{m\lambda}$, B depend upon ϕ. In this case the constraints are imposed on the growth of the derivative of the function $\phi(x)$. The restrictions are stronger, the smaller the value of the β.

<u>Theorem</u> 2.4.1 -

Let $\phi \in H^{\beta}_{\mu,\lambda}$ Then

$$\sup_x \left| x^{m\lambda}(x^{1-2\lambda}D)^k \, x^{-\lambda\mu-\lambda+1/2} \, \phi(x) \right| < A_{m\lambda,k} \, B^k \, k^{k\beta}$$

where m, k = 0,1,2...

where the constants $A_{m\lambda,k}$ depend on ϕ, m and k.

2.4.2 Topological property of the space $H^{\beta}_{\mu,\lambda}$:-

The space $H^{\beta}_{\mu,\lambda}$ as a union of countably normed spaces:

The space $H^{\beta}_{\mu,\lambda}$ is defined as, the space of all smooth functions ϕ satisfying,

$$\left| x^{m\lambda}(x^{1-2\lambda}D)^k \; x^{-\lambda\mu-\lambda+1/2} \; \phi(x) \right| < A_{m\lambda} B^k \; K^{k\beta}$$

where $m, k = 0,1,2\ldots$

The constants $A_{m\lambda}, B$ depend upon ϕ and for $k = 0$ let $K^{k\beta} = 1$.

Let the space $H^{\beta,B}_{\mu,\lambda}$ be defined as the space of all smooth functions $\phi \in H^{\beta}_{\mu,\lambda}$ satisfying

$$\left| x^{m\lambda}(x^{1-2\lambda}D)^k \; x^{-\lambda\mu-\lambda+1/2} \; \phi(x) \right| < A_{m\lambda\,\rho}(B+\rho)^k \; K^{k\beta}$$

for any $\rho > 0$ and $m, k = 0, 1, 2\ldots$

The space $H^{\beta,B}_{\mu,\lambda}$ does not belong to the class of spaces $K\{M(P)\}$.

Let us introduce a system of norms in the space $H^{\beta,B}_{\mu,\lambda}$ by

$$\|\phi\|_{m\lambda\,\rho} = \frac{\sup\limits_{x,k} \left| x^{m\lambda}(1^{1-2\lambda}D)^k \; x^{-\lambda\mu-\lambda+1/2} \; \phi(x) \right|}{(B+\rho)^k \; K^{k\beta}}$$

where $\rho = 1, 1/2, 1/3$

and $K = 0,1,2\ldots$

We shall show that with this system of norms, the space $H^{\beta,B}_{\mu,\lambda}$ becomes complete countably normed space.

We shall prove some lemmas.

Lemma 1 -

If the sequence $\phi_\nu(x)$ converges correctly to some function $\phi(x)$ and for some m, ρ the norms $\|\phi_\nu\|_{m\lambda\,\rho}$ are bounded. ie $\|\phi_\nu\|_{m\lambda\,\rho} < C$, then the norm $\|\cdot\|_{m\lambda\,\rho}$ exists for the function $\phi(x)$ and $\|\phi\|_{m\lambda\,\rho} < C$.

Proof -

Consider the bounded interval $-a < x < a$

In this interval

$$\sup_{\substack{x \\ m<p}} \left| \frac{x^{m\lambda}(x^{1-2\lambda}D)^k\, x^{-\lambda\mu-\lambda+1/2}\, \phi(x)}{(B+\rho)^k\, K^{k\beta}} \right|$$

$$= \lim_{\nu\to\infty} \sup_{\substack{x \\ k<p}} \left| \frac{x^{m\lambda}(x^{1-2\lambda}D)^k\, x^{-\lambda\mu-\lambda+1/2}\, \phi_\nu(x)}{(B+\rho)^k\, K^{k\beta}} \right|$$

$$\leq \overline{\lim_{\nu\to\infty}} \|\phi_\nu\|_{m\lambda\,\rho}$$

$$\leq C$$

Taking limit as $a \to \infty$ and $p \to \infty$ we have

$$\|\phi\|_{m\lambda\,\rho} = \sup_{x,k} \frac{\left| x^{m\lambda}(x^{1-2\lambda}D)^k\, x^{-\lambda\mu-\lambda+1/2}\, \phi(x) \right|}{(B+\rho)^k\, K^{k\beta}}$$

$$< C$$

Hence the lemma.

Lemma 2 -

If the sequence $\phi_\gamma(x)$ converges to zero at each point and is fundamental in the norm $\|\cdot\|_{m\lambda\rho}$. Then $\|\phi_\gamma\|_{m\lambda\rho} \to 0$.

Proof -

The sequence $\phi_\gamma(x)$ converges correctly to zero because $\phi_\gamma(x)$ is fundamental. Therefore the sequence of differences $\phi_\gamma(x) - \phi_\mu(x)$ converges correctly to $\phi_\gamma(x)$ as $\mu \to \infty$.

$$\therefore \quad \|\phi_\gamma\|_{m\lambda\rho} \leq \sup_{\mu \geq \gamma} \|\phi_\gamma - \phi_\mu\|_{m\lambda\rho}$$
$$< \epsilon$$

$$\therefore \quad \|\phi_\gamma\|_{m\lambda\rho} \to 0$$

Hence the lemma.

Theorem 2.4.2 -

The space $H^{\beta,B}_{\mu,\lambda}$ is complete.

Proof -

Let $\phi_\gamma(x) \in H^{\beta,B}_{\mu,\lambda}$ be fundamental in each of the norms $\|\cdot\|_{m\lambda\rho}$. Since $\phi_\gamma \in H^{\beta,B}_{\mu,\lambda}$, $\|\phi_\gamma\|_{m\lambda\rho}$ is bounded.

$\therefore$ According to lemma 1.

each norm exists for the function $\phi(x)$ and is bounded.

Hence $\phi \in H^{\beta, B}_{\mu, \lambda}$

Also ϕ_ν is fundamental.

$\therefore$ the sequence of differences $\phi_\gamma - \phi_\mu$ converges correctly to zero and bounded in each of the norms.

$\therefore$ According to lemma 2.

$$\| \phi_\gamma - \phi \|_{m\lambda \rho} \longrightarrow 0 \text{ as } \gamma \longrightarrow \infty.$$

for any m & ρ

$\therefore$ $\phi_\gamma \longrightarrow \phi$ in $H^{\beta, B}_{\mu, \lambda}$.

Hence the space $H^{\beta, B}_{\mu, \lambda}$ is complete.

<u>Lemma 3</u> -

If the sequence $\phi_\nu(x)$ is bounded in each of the norms. $\|\cdot\|_{m\lambda\rho}$, and converges correctly to zero, Then it tends to zero in the topology of the space $H^{\beta, B}_{\mu, \lambda}$ i.e. in each of norms.

<u>Proof</u> -

Case (1)

Let m, ρ and $\eta > 0$. be given,

Let $\rho' < \rho$.

Now the numbers $\|\phi_\nu\|_{m\lambda\rho'}$ are bounded by constant $C_{m\rho'}$ .... (given)

$\therefore$ For sufficiently large K, $K > K_o$ the inequality

$\dfrac{(B+\rho')^k}{(B+\rho)^k} < \dfrac{\eta}{C_{m\,\rho'}}$ is true.

$\therefore$ For $K > K_o$, we have the inequality

$$\left| x^{m\lambda}\,(x^{1-2\lambda}\,D)^k\,x^{-\lambda\mu-\lambda+1/2}\,\phi_\gamma(x)\right| <$$

$$< C_{m\,\rho'}\,(B+\rho')^k\,K^{k\beta}$$

$$< \eta\,(B+\rho)^k\,K^{k\beta} \qquad \ldots (1)$$

Case (2) - Let $K < K_o$

let $|x|^{\lambda} > \dfrac{C_{m+1,\,\rho}}{\eta}$

Then we have

$$\left| x^{m\lambda}\,(x^{1-2\lambda}\,D)^k\,x^{-\lambda\mu-\lambda+1/2}\,\phi_\gamma(x)\right| =$$

$$= \frac{1}{|x|^{\lambda}}\left| x^{(m+1)\lambda}(x^{1-2\lambda}D)^k\,x^{-\lambda\mu-\lambda+1/2}\,\phi_\gamma(x)\right|$$

$$< \frac{1}{|x|^{\lambda}}\cdot\|\phi_\gamma\|_{(m+1)\lambda,\,\rho}\,(B+\rho)^k\,K^{k\beta}$$

$$< \frac{\eta}{C_{m+1,\,\rho}}\,(C_{m+1,\,\rho})\cdot(B+\rho)^k\,K^{k\beta}$$

$$< \eta\,(B+\rho)^k\,K^{k\beta}.$$

Case (3) - Now let $K < K_o$ and

$$|x|^{\lambda} < \frac{C_{m+1,\,\varrho}}{\eta}$$

Now since $\phi_{\gamma}(x)$ converges correctly to zero, therefore by definition, $\phi_{\gamma}^{(q)}(x)$ converges uniformly to zero, in any interval, hence in $|x| < \frac{C_{m+1,\,\varrho}}{\eta}$

Therefore the inequality

$$\left| x^{m\lambda}(x^{1-2\lambda}D)^k \; x^{-\lambda\mu-\lambda+1/2} \; \phi_{\gamma}(x) \right| < \eta \; (B+\varrho)^k \; K^{k\beta}$$

is true and it is also true for sufficiently large γ . Say $\gamma > \gamma_o$

Thus

$$\|\phi_{\gamma}\|_{m\lambda\,\varrho} = \sup_{x,k} \frac{\left| x^{m\lambda}(x^{1-2\lambda}D)^k \; x^{-\lambda\mu-\lambda+1/2} \; \phi_{\gamma}(x) \right|}{(B+\varrho)^k \; K^{k\beta}}$$

$$< \eta$$

Hence the sequence $\phi_{\gamma}(x)$ tends to zero in the norm $\|\cdot\|_{m\lambda\,\varrho}$

Since ϱ & m are arbitrary $\phi_{\gamma} \to 0$ in the topology.

Hence the lemma.

Theorem 2.4.2 -

If the sequence $\phi_{\gamma}(x)$ is bounded in each of the norms and converges correctly to some function $\phi(x)$

then $\phi(x)$ belongs to $H^{\beta,B}_{\mu,\lambda}$ and is the limit of the sequence $\phi_\gamma(x)$ in the topology of $H^{\beta,B}_{\mu,\lambda}$.

Proof :

According to lemma 1

$\phi \in H^{\beta,B}_{\mu,\lambda}$

Now the difference $\phi(x) - \phi_\gamma(x)$ is bounded in all norms and converges correctly to zero. Then according to lemma 2, this difference converges to zero in the topology of the space $H^{\beta,B}_{\mu,\lambda}$. Thus $\phi(x)$ is limit of sequence ϕ (x) in the topology of $H^{\beta,B}_{\mu\lambda}$.

Thus ϕ_γ is a fundamental sequence in each of the norms and ϕ_γ converges to ϕ with respect to each norm. Then the norms $\|\cdot\|_{m\lambda\varrho}$ are compatible. Thus the topology of the space $H^{\beta,B}_{\mu,\lambda}$ is generated by the collection of is generated by the collection of compatible norms. Hence $H^{\beta,B}_{\mu,\lambda}$ is complete countably normed space.

Now we shall prove that the space $H^{\beta}_{\mu,\lambda}$ is perfect-

We have already proved that the space $H^{\beta}_{\mu,\lambda}$ is complete countably normed space.
Now let us prove that each bounded set $A \in H^{\beta}_{\mu,\lambda}$ is compact.
Let $\phi_\gamma \in A$ $\{\gamma = 1,2,3\ldots\}$ be an arbitrary bounded

sequence.

Now we shall show that it contains a convergent sub-sequence.

Now, since each norm is bounded (by lemma)

$\therefore \quad \|\phi\|_1$ is bounded.

Therefore the function $\left| \dfrac{\partial \phi_{\nu}(x)}{\partial x} \right|$ is uniformly bounded.

Hence by Arzela theorem, there exists a sub sequence $\phi_{11}, \phi_{12}, \phi_{13} \ldots$ which converges uniformly for $|x| < 1$.

Now since $\|\phi_{1\nu}\|_2$ is bounded. Hence according to Arzela theorem it contains a sub sequence, $\phi_{21}, \phi_{22}, \phi_{23} \ldots$ for which the value of the first derivative $\dfrac{\partial \phi_{2\nu}(x)}{\partial x}$

converges uniformly in the domain $|x| < 2$.

Thus from the convergence of functions $\phi_{2\nu}$ for $|x| \leq 1$ and the uniform convergence of their derivatives for $|x| < 2$, there results the uniform convergence of these functions for $|x| < 2$

Continuing in this way we get the sub sequences -

$$
\begin{array}{llll}
\phi_{11} & \phi_{12} & \phi_{13} & \ldots \\
\phi_{21} & \phi_{22} & \phi_{23} & \\
- - - & - - - & - & \\
- - - & - - - & - &
\end{array}
$$

Then applying the diagonalization process, we obtain a bounded sub sequence ϕ_{11}, ϕ_{22}, ϕ_{33} .... which converges uniformly together with all its derivatives to some limit function $\phi_o(x)$ in any bounded domain (Since $H^{\beta}_{\mu,\lambda}$ is complete).

Thus A is compact.

Thus each bounded set is compact.

Hence $H^{\beta}_{\mu,\lambda}$ is perfect.

Thus $H^{\beta}_{\mu,\lambda}$ is a complete, countably normed perfect space.

Now if $B_1 < B_2$ Then

$$H^{\beta, B_1}_{\mu,\lambda} \subseteq H^{\beta, B_2}_{\mu,\lambda}.$$

and the topology of $H^{\beta, B_1}_{\mu,\lambda}$ is stronger than that the topology induced by $H^{\beta, B_2}_{\mu,\lambda}$

Therefore by definition, we can construct the union of countably normed space over all $B_i^{'s}$.

i.e. $$H^{\beta}_{\mu,\lambda} = \bigcup_{i=1}^{\infty} H^{\beta, B_i}_{\mu,\lambda} .$$

Thus the space is a union of countably normed spaces.

Therefore a sequence $\{\phi_{\gamma}(x)\}_{\nu=1}^{\infty}$

Converges to zero in $H^{\beta}_{\mu,\lambda}$ i.e. all $\phi_{\nu}^{'s}$ belong to some $H^{\beta, B}_{\mu,\lambda}$ and they converge to zero in that particular space.

This is true only if $\{\phi_y\}$ converges to zero correctly and the inequalities.

$$\left| x^{m\lambda} \left(x^{1-2\lambda} D\right)^k x^{-\lambda\mu-\lambda+1/2} \phi_{\gamma} (x) \right| < A_{m\lambda} B^k K^{k\beta}$$

are satisfied where the constants are independent of γ .

2.5)

2.5.1 Another way of defining the space of the type $H^{\beta}_{\mu,\lambda,\alpha}$ -

The space $H^{\beta}_{\mu,\lambda,\alpha}$ is the space of smooth functions $\phi(x)$ on $0 < x < \infty$ satisfying the inequality

$$\gamma^{\mu,\lambda,\alpha}_{m,k} (\phi) = \sup_x \left| x^{m\lambda} (x^{1-2\lambda} D)^k x^{-\lambda\mu-\lambda+1/2} \phi(x) \right|$$

$$< A B^{m\lambda} C^k m^{m\lambda\alpha} K^{kB}$$

where $m, k = 0,1,2,3...$

In this space the restrictions are imposed on decrease of ϕ as $x \to \infty$ and also on the growth of the derivative of ϕ.

Therefore the space $H^{\beta}_{\mu,\lambda,\alpha}$ lies in the intersection of both the spaces $H_{\mu,\lambda,\alpha}$ and $H^{\beta}_{\mu,\lambda}$.

From the theorem 2.1.1, we can write

$$\left| (x^{1-2\lambda} D)^k x^{-\lambda\mu-\lambda+1/2} \phi(x) \right| < A'_k C^k K^{k\beta} \exp\left[(-a+\delta) \left| x \right|^{1/\alpha} \right]$$

where the constant $a = \dfrac{\lambda \alpha}{e B^{1/\alpha}}$, $\delta > 0$.

2.5.2 Topological property of the space $H^{\beta}_{\mu,\lambda,\alpha}$ -

The space $H^{\beta}_{\mu,\lambda,\alpha}$ as a union of countably normed spaces-

Let $H^{\beta,C}_{\mu,\lambda,\alpha,B}$ be a space of all testing functions $\phi \in H^{\beta}_{\mu,\lambda,\alpha}$ such that

$$\sup_x \left| x^{m\lambda} (x^{1-2\lambda}D)^k \, x^{-\lambda\mu-\lambda+1/2} \, \phi(x) \right| < A \, \bar{B}^{m\lambda} \, \bar{C}^{k} \, m^{m\lambda\alpha} \, K^{k\beta}$$

where m, k = 0, 1, 2...

is satisfied where the constant $\bar{B}$ is greater than B and $\bar{C}$ is greater than C.

OR

$H^{\beta,C}_{\mu,\lambda,\alpha,B}$ is a space of all functions $\phi \in H^{\beta}_{\mu,\lambda,\alpha}$ satisfying the inequality

$$\sup_x \left| x^{m\lambda} \, (x^{1-2\lambda}D)^k \, x^{-\lambda\mu-\lambda+1/2} \, \phi(x) \right|$$
$$< A_{\delta\,\rho} \, (B+\delta)^{m\lambda} \, (C+\rho)^k \, m^{m\lambda\alpha} \, K^{k\beta}$$

for any $\delta > 0$, $\rho > 0$ and m, k = 0, 1, 2...

Now we shall introduce a system of norms as follows :

$$\|\phi\|^{\mu,\lambda}_{\delta\,\rho} = \sup_{m,k} \sup_x \frac{\left| x^{m\lambda} (x^{1-2\lambda}D)^k \, x^{-\lambda\mu-\lambda+1/2} \, \phi(x) \right|}{(B+\delta)^{m\lambda} \, (C+\rho)^k \, m^{m\lambda\alpha} \, K^{k\beta}}$$

where $\delta, \rho = 1, 1/2, 1/3 \ldots.$

We shall show that with this system of norms the space $H^{\beta, c}_{\mu, \lambda, \alpha, B}$ becomes a complete countably normed space. For this we shall prove some lemmas.

Lemma 1 :

If sequence $\{ \phi_\nu(x) \}$ converges correctly to some function $\phi(x)$ and for some δ and ϱ, the norms $\| \phi_\nu \|_{\delta\varrho}$ are bounded i.e. $\| \phi_\nu \|_{\delta\varrho} < C$. Then the norms exist even for the function $\phi(x)$ and $\| \phi(x) \|_{\delta\varrho} < C$.

<u>Proof</u> -

Consider the bounded interval $-a < x < a$

$$\sup_{\substack{x \\ K \leq p \\ m < l}} \frac{\left| x^{m\lambda} (x^{1-2\lambda}D)^k \, x^{-\lambda\mu-\lambda+1/2} \, \phi(x) \right|}{(B+\delta)^{m\lambda} \, (c+\varrho)^k \, m^{m\lambda\alpha} \, K^{k\beta}}$$

$$< \lim_{\nu \to \infty} \frac{\left| x^{m\lambda} (x^{1-2\lambda}D)^k \, x^{-\lambda\mu-\lambda+1/2} \, \phi_\nu(x) \right|}{(B+\delta)^{m\lambda} \, (c+\varrho)^k \, m^{m\lambda\alpha} \, K^{k\beta}}$$

$$< \lim_{\to \infty} \| \phi_\nu \|_{\delta\varrho}$$

$$< C$$

Thus taking limit as $a \to \infty$, $p \to \infty$ and $l \to \infty$ we get

$$\| \phi \|_{\delta\varrho} = \sup_{x, m, k} \frac{\left| x^{m\lambda} (x^{1-\lambda}D)^k \, x^{-\lambda\mu-\lambda+1/2} \, \phi(x) \right|}{(B+\delta)^{m\lambda} \, (c+\varrho)^k \, m^{m\lambda\alpha} \, K^{k\beta}}$$

$$< C.$$

Thus the norms for the limit function exist and are bounded.

Lemma 2 -

If the sequence $\}\phi_{\gamma}(x)\{$ converges to zero at each point and is fundamental in the norm $||\cdot||_{\delta\varrho}$. Then $||\phi_{\gamma}||_{\delta\varrho} \rightarrow 0$

Proof -

Since ϕ is fundamental, it converges correctly to zero at each point. Therefore the sequence of differences $\phi_{\gamma}(x) - \phi_{\mu}(x)$ converges correctly to $\phi_{\gamma}(x)$ as $\mu \rightarrow \infty$.

$\therefore \quad ||\phi_{\gamma}||_{\delta\varrho} \leq \sup\limits_{\mu \geq \gamma} ||\phi_{\gamma} - \phi_{\mu}||_{\delta\varrho} < \epsilon$ for sufficiently large γ.

$\therefore \quad ||\phi_{\gamma}||_{\delta\varrho} \rightarrow 0$ as $\gamma \rightarrow \infty$.

Theorem 2.5.2 -

The space $H^{\beta,C}_{\mu,\lambda,\alpha,B}$ is complete.

Proof -

Let $\phi_{\gamma}(x) \in H^{\beta,C}_{\mu,\lambda,\alpha,B}$ be fundamental in each of the norms $||\cdot||_{\delta\varrho}$. Since $\phi_{\gamma}(x) \in H^{\beta,C}_{\mu,\lambda,\alpha,B}$

$\therefore$ $\|\phi_\gamma\|_{\delta\varrho}$ is bounded.

According to lemma 1,

each of the norms exists for the limit function $\phi(x)$ and

$\|\phi\|_{\delta\varrho}$ is bounded.

Hence $\phi(x) \in H^{\beta, C}_{\mu, \lambda, \alpha, B}$

Also ϕ_γ is fundamental.

Therefore the sequence of differences $\phi_\gamma - \phi$ converges correctly to zero and bounded in each of the norms. Therefore according to lemma 2 -

$$\|\phi_\gamma - \phi\|_{\delta\varrho} \longrightarrow 0 \text{ for any } \delta \text{ and } \varrho.$$

Thus the sequence $\{\phi_\gamma\}$ converges to ϕ in the space $H^{\beta, C}_{\mu, \lambda, \alpha, B}$

Hence by definition

$H^{\beta, C}_{\mu, \lambda, \alpha, B}$ is complete.

Lemma 3 -

If the sequence $\phi_\gamma(x)$ is b ounded in each of the norms $\|\cdot\|_{\delta\varrho}$ and converges correctly to zero. Then it tends to zero in the topology of the space $H^{\beta, C}_{\mu, \lambda, \alpha, B}$ i.e. in each of the norms.

Proof -

Let $\eta > 0$, $\delta > 0$ and $\varrho > 0$ be given.

Let $\delta' < \delta$ and $\varrho' < \varrho$.

Since all ϕ_γ's are bounded.

$\therefore$ $\|\phi_\gamma\|_{\delta'\varrho'}$ are bounded i.e. $\|\phi_\gamma\|_{\delta'\varrho'} < C_1$

OR

for any m, k and x

$$\left| x^{m\lambda}(x^{1-2\lambda}D)^k \, x^{-\lambda\mu-\lambda+1/2} \, \phi_\gamma(x) \right| < C_1 \, (B+\delta')^{m\lambda} \, (C+\varrho')^k \cdot m^{m\lambda\alpha} \, K^{k\beta}$$

Now for sufficiently large m, $m > m_o$ the inequality

$(B+\delta')^{m\lambda} < \frac{\eta}{C_1} \, (B+\delta)^{m\lambda}$ is true.

Therefore, for any k, x and $m > m_o$

$$\left| x^{m\lambda}(x^{1-2\lambda}D)^k \, x^{-\lambda\mu-\lambda+1/2} \, \phi_\gamma(x) \right|$$

$$< C_1 \frac{\eta}{C_1} \, (B+\delta)^{m\lambda} \, (C+\varrho)^k \, m^{m\lambda\alpha} \, K^{k\beta}$$

ie $\left| x^{m\lambda}(x^{1-2\lambda}D)^k \, x^{-\lambda\mu-\lambda+1/2} \, \phi_\gamma(x) \right|$

$$< \eta \, (B+\delta)^{m\lambda} \, (C+\varrho)^k \, m^{m\lambda\alpha} \, K^{k\beta} \qquad \ldots (3A)$$

Thus for $m > m_o$, the inequality holds for any K and C.

Also since $\|\phi_\gamma\|_{\delta\rho'}$ are bounded.

$\therefore$ We have $\|\phi_\gamma\|_{\delta\rho'} < C_2$

OR for any m, k and x

$$|x^{m\lambda}(x^{1-2\lambda}D)^k x^{-\lambda\mu-\lambda+1/2} \phi_\gamma(x)|$$

$$< C_2(B+\delta)^{m\lambda} (C+\rho')^k m^{m\lambda\alpha} K^{k\beta}$$

Now for sufficiently large K, $K > K_o$ the inequality

$(C+\rho')^k < \frac{\eta}{C_2} (C+\rho)^k$ is true.

$\therefore$ for any x, m and $k > K_o$

$$|x^{m\lambda}(x^{1-2\lambda}D)^k x^{-\lambda\mu-\lambda+1/2} \phi_\gamma(x)|$$

$$< C_2 \frac{\eta}{C_2} (B+\delta)^{m\lambda} (C+\rho)^k m^{m\lambda\alpha} K^{k\beta}$$

$$< \eta (B+\delta)^{m\lambda} (C+\rho)^k m^{m\lambda\alpha} K^{k\beta} \quad \dots (3B)$$

Thus for $k > K_o$, the inequality holds for any m and x.

Now we shall consider the cases when $m < m_o$ and $K < K_o$.

When $m < m_o$, $|x| > 1$

By (3A) we have for any ka and x

$$|x^{m\lambda}(x^{1-2\lambda}D)^k x^{-\lambda\mu-\lambda+1/2} \phi_\gamma(x)|$$

$$= \frac{|x^{m_o\lambda}(x^{1-2\lambda}D)^k x^{-\lambda\mu-\lambda+1/2} \phi_\gamma(x)|}{|x|^{m_o-m}}$$

$$< \frac{1}{|x|} \eta \ (B+\delta)^{m_o\lambda} \ (C + \rho)^k \ m_o^{m_o\lambda\alpha} \ K^{k\beta}$$

for sufficiently large x, $|x| > x_o$

$$\frac{(B + \delta)^{m_o\lambda} \ m_o^{m_o\lambda\alpha} \ K^{k\beta}}{|x|} < (B + \delta)^{m\lambda} \ m^{m\lambda\alpha} \ K^{k\beta}$$

for $m = 0, 1, 2 \ldots\ldots m_o-1$

Thus

$$\left| x^{m\lambda} (x^{1-2\lambda} D)^k \ x^{-\lambda\mu-\lambda+1/2} \ \phi_\gamma(x) \right|$$

$$< \eta (B + \delta)(C + \rho)^k \ m^{m\lambda\alpha} \ K^{k\beta} \qquad \ldots (3)$$

Similarly when $K < K_o$,

we can show that the inequality (3) is satisfied.

Thus for $m < m_o$ and $k < k_o$, $|x| > x_o$

the inequality (3) is satisfied.

Now if $m < m_o$ and $K < K_o$ and when ρ and δ are fixed, then the constants $(B+\delta)^{m\lambda}$, $(C+\rho)^k$, $m^{m\lambda\alpha}$, $k^{k\beta}$ are bounded. Since $\phi_\gamma^k(x)$ converges uniformly to zero in the segment $|x| < x_o$, then for any $\eta > 0$ and sufficiently large $\gamma > \nu_o$, the inequality holds.

Thus for $\gamma > \gamma_o$, the inequality holds for any m, k and x.

i.e. $\|\phi_\gamma\|_{\delta\rho} < \eta$ for $\gamma > \gamma_o$

$\therefore$ It follows that $\phi_\gamma \to 0$ in each of the norms $\|\cdot\|_{\delta\rho}$.

Since δ and ρ are arbitrary,

$\phi_\gamma \longrightarrow 0$ in the topology of the space $H^{\beta,C}_{\mu,\lambda,\alpha,B}$ as $\gamma \longrightarrow 0$.

Theorem 2.5.2 -

If the sequence $\phi_\gamma(x)$ is bounded in each of the norms. $\|\cdot\|_{\delta\rho}$ and converges correctly to some function $\phi(x)$. Then $\phi(x) \in H^{\beta,C}_{\mu,\lambda,B}$ and $\phi(x)$ is the limit of the sequence $\phi_\gamma(x)$ in the topology of the space $H^{\beta,C}_{\mu,\lambda,\alpha,B}$

<u>Proof</u> -

By lemma 1,

$$\phi(x) \in H^{\beta,C}_{\mu,\lambda,\alpha,B}$$

Since $\phi_\gamma(x)$ is bounded in each of the norms, the difference $\phi(x) - \phi_\gamma(x)$ is also bounded in each of the norms and converges correctly to zero

$\therefore$ By lemma 3,

$\phi(x) - \phi_\gamma(x)$ converges to zero in the topology of the space.

Thus ϕ_γ is a fundamental sequence in each of the norms and ϕ_γ converges to ϕ with respect to each norm. Then the norms $\|\cdot\|_{\delta\rho}$ are compatible. Thus the topology of the space $H^{\beta,C}_{\mu,\lambda,\alpha,B}$ is generated by the collection of compatible norms.

Hence $H^{\beta,C}_{\mu,\lambda,\alpha,B}$ is a countably normed space.

The space $H^{\beta,C}_{\mu,\lambda,\alpha,B}$ is perfect.

The proof of this is similar to that of the proof of the result '$H^{\beta}_{\mu,\lambda}$ is perfect'.

Thus $H^{\beta,C}_{\mu,\lambda,\alpha,B}$ is complete countably normed perfect space.

Now if $B_1 < B_2$ and $C_1 < C_2$

The space

$$H^{\beta,C_1}_{\mu,\lambda,\alpha,B_1} \subseteq H^{\beta,C_2}_{\mu,\lambda,\alpha,B_2}$$

Hence we can construct the union of countably normed spaces over all B and C's. Thus

$$H^{\beta}_{\mu,\lambda,\alpha,B} = \bigcup_{B,C=1}^{\infty} H^{\beta,C}_{\mu,\lambda,\alpha,B} .$$

Hence sequence ϕ_{ν} converges to zero in $H^{\beta}_{\mu,\lambda,\alpha}$ if ϕ_{ν} belongs to some $H^{\beta,C}_{\mu,\lambda,\alpha,B}$ and converges to zero in that space. This is true if and only if, the sequence ϕ_{ν} converges correctly to zero and the inequality

$$\left| x^{m\lambda}(x^{1-2\lambda}D)^k \, x^{-\lambda\mu-\lambda+1/2} \, \phi_{\nu}(x) \right|$$

$$< A \ (B + \delta)^{m\lambda}(C + \rho)^k \, m^{k\lambda\alpha} \, K^{k\beta}$$

is satisfied. where the constants A, B, C are independent of γ.

2.6

2.6.1 Property -

The space D(I) is subspace of the space $H_{\mu,\lambda,\alpha}$ -

<u>Proof</u> -

Let $\phi \in D(I)$

Then according to the definition of space D(I)

(1) ϕ is smooth function having compact support.

(2) $\sup_k \left| D^k \phi(t) \right|$ is finite.

Now $\phi \in D(I)$ ===> ϕ is infinitely smooth function on some compact set K or we can say on $0 < x < \infty$.

$\therefore$ the first condition of the space $H_{\mu,\lambda,\alpha}$ is satisfied.

Now since $\sup_k \left| D^k \phi(t) \right|$ is finite.

$$\sup_{0 < x < \infty} \left| x^{m\lambda} (x^{1-2\lambda} D)^k \, x^{-\lambda\mu-\lambda+1/2} \, \phi(x) \right|$$

$$= \sup_k \left| x^{m\lambda} (x^{1-2\lambda} D)^k \, \psi(x) \right|$$

where $\psi(x) = x^{-\lambda\mu-\lambda+1/2} \, \phi(x)$.

Since $\phi(x)$ is smooth function.

$\therefore$ $\psi(x)$ is smooth function.

$\therefore$ $(x^{1-2\lambda}D)^k \, \psi(x)$ is continuous

$\therefore$ $x^{m\lambda}\,(x^{1-2\lambda}D)^k \psi(x)$ is continuous and continuous function over some compact set K is always bounded.

$\therefore$ $\sup\limits_{k} \left| x^{m\lambda}(x^{1-2\lambda}D)^k \psi(x) \right|$ is bounded.

Thus

$\sup\limits_{x} \left| x^{m\lambda}(x^{1-2\lambda}D)^k \, x^{-\lambda\mu-\lambda+1/2} \, \emptyset(x) \right|$ is bounded.

Hence $\emptyset \in H_{\mu,\lambda,\alpha}$.

Hence D(I) is subspace of $H_{\mu,\lambda,\alpha}$.

By the same method, we can show that D(I) is subspace of the space $H^{\beta}_{\mu,\lambda}$ and the space $H^{\beta}_{\mu,\lambda,\alpha}$.

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