Chapter I

PRECURSORY NOTIONS AND CONCEPTS

1.1. SECTIONWISE RECONNAISSANCE

The evolution of the stress-energy tensor for the relativistic magnetofluid constitute the prime aim of this chapter. This stress-energy tensor forms the substratum for the work in later chapters. The general notations and conventions are presented in Section 1.2. The time-like congruences and associated kinematical parameters are given in Section 1.3, whereas the terminology underlying particular geometrical symmetries is explored in Section 1.4. The further Section 1.5 deals with the physical properties of the stress-energy tensor of the magnetofluid. The strong energy condition which is supposed to be true for every physically transperent stress-energy tensor is verified. The last Section 1.6 provides some deductions of particular geometrical symmetries admitted by the magnetofluid space-time. No original results of the author are reported in this chapter.

1.2. CONVENTIONS

The four dimensional space-time with the metric form $ds^2 = g_{ab}dx^a dx^b$ is considered here. The signature of the metric is taken as $(-,-,-,+)$. All latin indices assume the values 1,2,3,4. The summation convention on diagonally repeated indices is used throughout the dissertation. Commas (,) are used to denote partial differentiation. Semicolons (;) are employed to mean covariant differentiation.

Round and square brackets around suffixes signify symmetrization and antisymmetrization respectively. The units are such that the speed of light (c) is unity. The Riemann curvature tensor R_{abcd} and the Ricci tensor R_{ab} are defined by

$$R^a_{bcd} = \{bd,a\}_{,c} - \{bc,a\}_{,d} + \{bd,e\}\{ec,a\} - \{bc,e\}\{ed,a\}$$

$$R_{ab} = R^e_{abc} ,$$

where $\{bc,a\}$ are the Christoffel symbols of second kind given by

$$\{bc,a\} = \frac{1}{2} g^{ad} (g_{bd,c} + g_{bc,b} - g_{bc,d})$$

Here $R = R^a_a$ is a scalar known as Curvature scalar.

$\pounds_\alpha$ denotes the Lie derivative with respect to the vector α.

1.3. THE STUDY OF CONGRUENCES

Congruence is an elegant word that means space filling family of curves.

The existence of time-like congruences in cosmological models, space-like congruences in self-gravitating magnetofluids and null congruences in gravitational radiation protends the indispensibility of the study of the congruences in the general theory of relativity. For exemplification, in cosmology, we find expanding universe models by Friedmann (1922), Mivittie (1955), Date (1973a & b), rotating models by Ozsvath (1966), Krasinski (1974). These models have been lucidely illustrated by Ellis (1971). Stromer's (1960) perturbed Friedmann model with shear, dust filled universe with inevitable geodesic congruence by Ellis (1967), Vaidya's (1968) model filled with black body radiation admitting geodesic and irrotational flow; boosting universe by Misra and Udit Narayan (1971), shear-free null geodesic congruence model of a null fluid by Vaidya (1973) have created interest in this subject.

The study of local behaviour of congruences in RMHD is necessary because the comparison of kinematical parameters in the general theory of relativity with Newtonian theory is well understood. Besides, the properties of cosmological models on the basis of relative motion of galaxies can be infered from the knowledge of the congruences.

The general theory of relativity mainly deals with three types of congruences viz. time-like congruences, space-like congruences and null-congruences. The parameters associated with each type of the congruence exist in literature. We mainly deal with kinematical parameters associated with time-like congruence with reference to the space-time of the magnetofluid.

According to Greenberg's (1970a) formalism the kinematical parameters associated with the flow vector U^a which defines a time-like congruences of curves are described through the expressions -

(i) Expansion Scalar :

$$\theta = U^{a}_{;a} \quad . \qquad (1.3.1)$$

(ii) Shear tensor :

$$\sigma_{ab} = U_{(a;b)} - \dot{U}_{(a}U_{b)} - \frac{1}{3}\theta h_{ab} \, . \qquad (1.3.2)$$

(iii) Rotation tensor :

$$\omega_{ab} = U_{[a;b]} - \dot{U}_{[a}U_{b]} \, . \qquad (1.3.3)$$

(iv) Acceleration Vector :

$$\dot{U}_{a} = U_{a;b}U^{b} \, . \qquad (1.3.4)$$

where the tensor

$$h_{ab} = g_{ab} - U_{a}U_{b} \;, \qquad (1.3.5)$$

is defined as the three space projection operator.

We note that the unitary flow vector U^{a} satisfies

$$U^{a}U_{a} = 1 \qquad (1.3.6)$$

Consequently,

$$U_{a;k}U^{a} = 0 \, . \qquad (1.3.7)$$

The term 'kinematical', recognizes that the quantities θ, σ_{ab}, w_{ab} have definite dynamical implications for the time development of the gravitational field.

It follows from (1.3.1) to (1.3.5) that

$$\sigma_{ab} = \sigma_{ba} \;, \qquad (1.3.8)$$

$$w_{ab} = - w_{ba} \;, \qquad (1.3.9)$$

$$\sigma^a_a = w^a_a = 0 , \tag{1.3.10}$$

$$h_{ab} = h_{ba} , \tag{1.3.11}$$

$$h^a_a = 3 , \tag{1.3.12}$$

$$h^a_b = h^a_c h^c_b , \tag{1.3.13}$$

$$\sigma_{ab}U^b = w_{ab}U^b = h_{ab}U^b = 0 , \tag{1.3.14}$$

$$\dot{U}_a U^a = 0 . \tag{1.3.15}$$

Therefore the tensor σ_{ab} is symmetric and traceless and hence there exists atmost five independent components for σ_{ab}. The shear tensor σ_{ab} represents a direction-dependent velocity field which produces an ellipsoid out of a sphere of particles. Since the trace σ^a_a vanishes, this ellipsoid has the same volume as the original sphere and thus we here have a change in shape at constant volume. As well, we have the tensor w_{ab} as antisymmetric tensor, thus it has atmost three independent components.

As the decomposition of tensors relative to projection operator is desirable for the convenience of physical interpretation of tensor relations in relativistic continuum mechanics, we by virtue of (1.3.1) to (1.3.4) decompose the gradient of the four-velocity U^a in terms of expansion θ, acceleration $\dot{U}_a$, shear σ_{ab} and rotation w_{ab} in the form

$$U_{a;b} = \sigma_{ab} + w_{ab} + \dot{U}_a U_b + \frac{1}{3}\theta h_{ab} . \tag{1.3.16}$$

Since this splitting is covariant, the individual components characterise the flow field invariantly.

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1.4. GEOMETRICAL SYMMETRIES

An infinitesimal point transformation

$$\bar{x}^a = x^a + \delta\alpha^a(x), \qquad (1.4.1)$$

(where δ is a positive infinitesimal) in a Riemannian space-time V_4 is defined as (Katzin et.al. 1969).

(i) Motion if

$$\underset{\alpha}{\pounds}\, g_{ab} = 0 = \alpha_{a;b} + \alpha_{b;a} = 0 \quad . \qquad (1.4.2)$$

(well known Killing equations).

(ii) Conformal motion if

$$\underset{\alpha}{\pounds}\, g_{ab} = \lambda g_{ab} = \alpha_{a;b} + \alpha_{b;a} = \lambda g_{ab} \qquad (1.4.3)$$

where λ is a positive scalar function.

These symmetries are fully exploited in the next chapter.

1.5. EVOLUTION OF THE STRESS-ENERGY TENSOR FOR THE RELATIVISTIC MAGNETOFLUID

The study of electromagnetic field in self gravitating matter is due to Maugin (1972). His study deals with stress-energy tensor of electromagnetic field involving currents and potentials. All the related terms like conservation of stress-energy momentum, equations of motion, Maxwell equations, interaction of electromagnetic field with matterfield are introduced very elegantly through well known Action principle. The stress-energy tensor for self-gravitating magnetofluid is given by Maugin (1972) as

$$T^{ab} = T^{ab}_{(M)} + T^{ab}_{(E)} \tag{1.5.1}$$

where $T^{ab}_{(M)}$ is the stress-energy tensor for perfect fluid given through

$$T^{ab}_{(M)} = (\rho + p)\, U^a U^b - p g^{ab} , \tag{1.5.2}$$

and $T^{ab}_{(E)}$ is the stress-energy momentum tensor that results from the electromagnetic field in presence of matter given by

$$T^{ab}_{(E)} = \frac{1}{2}(E_c E^c - B_c B^c) g^{ab} - (E^a D^b + H^a B^b) + B^c H_c h^{ab} + E_c D^c U^a U_b + U^a V^b + W^a U^b , \tag{1.5.3}$$

where ρ is the matter energy density,

p is the isotropic pressure of the fluid,

E_a is the four-electric current vector,

D_a is the electric displacement vector,

H_a is space-like magnetic field vector, such that $H^a H_a = - h^2$ and $U^a H_a = 0$. (1.5.4)

B_a is the magnetic induction vector,

$$V^b = \frac{1}{\sqrt{-1}} \frac{\epsilon^{nbeq}}{\sqrt{-g}} E_n H_e U_q , \tag{1.5.5}$$

$$W^a = \frac{1}{\sqrt{-1}} \frac{\epsilon^{aceq}}{\sqrt{-g}} D_c B_e U_q \ , \qquad (1.5.6)$$

ϵ^{abcd} is the permutation symbol.

But for the perfect magnetohydrodynamics with the fluid having infinite electric conductivity (Maugin,1972), we have

$$E_a = D_a = 0 \ . \qquad (1.5.7)$$

Thus, in this case, making use of the equations (1.5.5) and (1.5.6) the equation (1.5.3) reduces to

$$T^{ab}_{(E)} = -\frac{1}{2} B_c B^c g^{ab} - H^a B^b + B^c H_c h^{ab} \qquad (1.5.8)$$

If the magnetic induction depends linearly on the magnetic field then for the magnetically isotropic and homogeneous fluid, we have

$$B^a = \mu H^a \ , \qquad (1.5.9)$$

where μ is the constant magnetic permability.

Using the equations (1.5.9) and (1.5.4) in the equation (1.5.8) we have

$$T^{ab}_{(E)} = \frac{1}{2} \mu^2 h^2 g^{ab} - \mu H^a H^b - \mu h^2 h^{ab} \ ,$$

ie $$T^{ab}_{(E)} = \frac{1}{2} \mu^2 h^2 g^{ab} - \mu H^a H^b - \mu h^2 (g^{ab} - U^a U^b) ,$$

(by virtue of (1.3.5))

ie $$T^{ab}_{(E)} = \frac{1}{2} \mu h^2 U^a U^b - \mu \left(1 - \frac{\mu}{2}\right) h^2 g^{ab} - \mu H^a H^b \ . \qquad (1.5.10)$$

Now substituting the values of $T^{ab}_{(M)}$ and $T^{ab}_{(E)}$ from the equations (1.5.2) and (1.5.10) respectively into the equation

(1.5.1), we get the final form of T^{ab} as

$$T^{ab} = (\rho + p)\ U^aU^b - pg^{ab} + \mu h^2U^aU^b -$$

$$- \mu(1 - \tfrac{\mu}{2})h^2g^{ab} - \mu H^aH^b\ ;$$

i.e. $$T^{ab} = (\rho + p + \mu h^2)U^aU^b - [p + \mu(1 = \tfrac{\mu}{2})h^2]g^{ab} -$$

$$- \mu H^aH^b\ . \qquad (1.5.11)$$

Remark - This stress-energy tensor reduces to the stress-energy tensor given by Lichnerowicz (1967) if

$$(1 - \tfrac{\mu}{2}) = \tfrac{1}{2}\ .$$

Note - Substituting

$$\tfrac{1}{2}\ \mu h^2 = m\ , \qquad (1.5.12)$$

in the expression (1.5.11), we obtain

$$T^{ab} = (\rho + p + 2m)\ U^aU^b - (p + 2m - m\mu)g^{ab} -$$

$$- \mu H^aH^b\ . \qquad (1.5.13)$$

This stress-energy tensor given by equation (1.5.13) is used throughout the dissertation to represent the self-gravitating perfect magnetofluid (Magnetofluid).

1.6. SEVERAL ASPECTS OF THE STRESS-ENERGY TENSOR FOR MAGNETOFLUID

For the stress-energy tensor (1.5.13) we list the following deductions -

$$T^{ab}U_a = (\rho + m\mu)U^b\ , \qquad (1.6.1)$$

$$T^{ab}U_aU_b = \rho + m\mu ,\qquad (1.6.2)$$

$$T^{ab}H_a = -(p - m\mu)H^b ,\qquad (1.6.3)$$

$$T^{ab}H_aH_b = (p - m\mu)h^2 ,\qquad (1.6.4)$$

$$T^{ab}g_{ab} = T = \rho - 3p - 4m + 4m\mu ,\qquad (1.6.5)$$

$$T^{ab}U_aH_b = 0 ,\qquad (1.6.6)$$

$$T^{ab}T_{ab} = (\rho - m\mu)^2 + (p - m\mu)^2 + 2(p + 2m - m\mu)^2 .\qquad (1.6.7)$$

$$T^{ab}h_{ab} = 3m\mu - 3p - 4m .\qquad (1.6.8)$$

Thus from the equation (1.6.1), we conclude that the energy density of the fluid which is the eigen value corresponding to the time-like eigen vector U^a is $\rho + m\mu$, which is the proper energy density of the magnetofluid as given by the equation (1.6.2). Equation (1.6.5) gives the trace of the stress-energy tensor.

(1.6.1). Energy condition

All known forms of matter and equations of state must satisfy the Hawking and Ellis (1968) energy condition given by

$$T^{ab}U_aU_b - \frac{1}{2} T \geqslant 0 .\qquad (1.6.9)$$

In the case of magnetofluid (1.5.13) we have

$$\rho + m\mu - \frac{1}{2} (\rho - 3p - 4m + 4m\mu) \geqslant 0 .\qquad (1.6.10)$$

(By equations (1.6.2) and (1.6.5))

i.e. $$\rho + 3p + 4m - 2m\mu \geqslant 0 .\qquad (1.6.11)$$

which is obviously true since the magnetic permeability μ is always constant and is less than two. Consequently we can conclude that the stress-energy tensor (1.5.13) for the magnetofluid is physically transparent.

APPENDIX - I.A

The parameters with respect to the other types of congruences (space-like, null etc.) are available in literature. The summary of which is given below :

(1.A.1) Parameters associated with space-like unit congruence V^a as given by (Greenberg(1970b) are

$$\theta^* = \frac{1}{2}\left(V^a_{;a} - V_{a;b}U^aU^b\right) \text{ (expansion)} \tag{1.A.1}$$

$$\sigma^*_{ab} = \gamma^c_a\,\gamma^d_b\,V_{(c;d)} - \frac{1}{2}\,\gamma_{ab}\theta^* \text{ (shear)} \tag{1.A.2}$$

$$\omega^*_{ab} = \gamma^c_a\,\gamma^d_b\,V_{[c;d]}\ , \quad \text{(rotation)} \tag{1.A.3}$$

where

$$\gamma_{ab} = g_{ab} - U_aU_b + V_aV_b\ . \tag{1.A.4}$$

Hence we have

$$U_{a;b}V^b = V_{a;b}U^b - U_aV_{b;c}U^bV^c + V_aV_{b;c}U^bV^c\ . \tag{1.A.5}$$

$$N_{a;b}V^b = U_aV_{b;c}N^bU^c - V_aV_{b;c}N^bV^c\ . \tag{1.A.6}$$

(since N_a is arbitrary unitary space congruence)

Equations (1.A.5) and (1.A.6) are known as Transport laws relative to V_a. Accordingly the expression for the gradient of the vector field V_a is

$$V_{a;b} = \sigma^*_{ab} + \omega^*_{ab} + \theta^*\gamma_{ab} + \dot{V}_aU_b - V_a\dot{U}_b -$$

$$- U_aV_c\left(\sigma^c_dV^dV_b + \sigma_b^{\ c} + \omega^c_b\right)\ . \tag{1.A.7}$$

(1.A.2). Parameters associated with Null congruences L^a as given by (Witten, 1962) are

$$\bar{\theta} = \frac{1}{2} L^{a}_{;a} , \qquad \text{(expansion)} \quad (1.A.8)$$

$$\bar{\sigma} = \left[\frac{1}{2} L_{(a;b)} L^{a;b} - \bar{\theta}^2 \right]^{\frac{1}{2}} , \qquad \text{(shear)} \quad (1.A.9)$$

$$\bar{\omega} = \left[\frac{1}{2} L_{[a;b]} L^{a;b} \right]^{\frac{1}{2}} , \qquad \text{(rotation)} \quad (1.A.10)$$

where $L^a L_a = 0$, $L_{a;b} L^b = 0$.

APPENDIX - I.B

A CASE OF COMOVING COORDINATE SYSTEM (Stephani, 1982)

The comoving system

$$U^a = (0, 0, 0, U^4) \qquad (1.B.1)$$

is very widely used to make calculations easy. Let us consider the two cases -

Case 1 : If the rotation $w_{ab} = 0$

Here, clearly the flow given by U^a is hypersurface orthogonal and the metric takes the form

$$ds^2 = g_{\alpha\beta}\,(x^a)dx^\alpha dx^\beta - U_4^2 dt^2, \qquad (1.B.2)$$

(where α & β takes values from 1 to 3)

If we write down the covariant derivative $U_{a;b}$ explicitly in this metric and compare the result with the equation (1.3.16) then we find that

(i) For $\sigma_{ab} = 0$, the metric $g_{\alpha\beta}$ of the three space, contains the time only in a factor common to all elements :

$$g_{\alpha\beta}(x^a) = V^2(x^\nu, t)\bar{g}_{\alpha\beta}\,(x^\mu) \,. \qquad (1.B.3)$$

(where ν and μ takes values from 1 to 3)

(ii) For $\Theta = 0$, the determinant of the three-metric $g_{\alpha\beta}$ does not depend upon the time,

(iii) For $\dot{U}_a = 0$, we can transform U^4 to C.

<u>Case 2</u> : <u>If the rotation $w_{ab} \neq 0$</u>

and also $\theta = \sigma_{ab} = 0$, then for the comoving observer the distances to neighbouring matter elements do not change, and we have a rigid rotation.