C H A P T E R - IV

SOME FEATURES OF WEYL TENSOR

IV. SOME FEATURES OF WEYL TENSOR :

1. Weyl Conformal Curvature Tensor :

We know that the Riemann curvature tensor which is invariant characterisation of the gravitational field due to Einstein, plays a vital role in the general theory of relativity. The famous Einstein's equations are based on Ricci tensor which is the trace of Riemann curvature tensor and the Ricci scalar which is the trace of Ricci tensor. The properties of the Riemann curvature tensor have reduced to the total 256 components considerably. Many other tensors of rank four constructed with the help of Riemann curvature tensor and its dual are available in literature. One of the important tensors constructed with Riemann curvature tensor and itstrace is known as Weyl conformal tensor. The defining expression for this Weyl conformal tensor (Carmelli 1982) is of the form

$$C_{abcd} = R_{abcd} + \tfrac{1}{2}(R_{ac}g_{bc} - R_{ad}g_{bc} + R_{bd}g_{ac}) - - \frac{R}{6}(g_{ac}R_{bd} - g_{ad}g_{bc}) \; . \qquad (4.1)$$

The significance of this Weyl tensor is that it satisfies all the properties of the Riemann curvature tensor and in addition is tracefree. That is

$$C_{abcd} = -C_{abdc} = -C_{bacd} \text{ ,} \tag{4.2}$$

$$C_{abcd} = C_{cdab} \text{ ,} \tag{4.3}$$

$$C_{abcd} + C_{acdb} + C_{adbc} = 0 \text{ ,} \tag{4.4}$$

$$C^{a}{}_{bca} = 0. \tag{4.5}$$

These properties imply that the Weyl conformal tensor has only ten independent components. Note that the form of the Weyl tensor is left invariant under conformal mapping and hence the name Weyl conformal tensor. It is clear from the definition that the Riemann curvature tensor consists of two parts, one zero trace part and the other nonzero trace part.

Note 1:

We have the necessary and sufficient conditions for spacetime to be flat is zero Riemann curvature tensor. Hence if all the components of the Riemann curvature tensor vanisn then the spacetime becomes flat. One can prove from Einstein's field equations that the flat spacetime implies empty spacetime (T_{ab}=0) but not the converse. The Weyl tensor gives the non Ricci part of the curvature that no longer vanishes identically. The vanishing of the Weyl conformal tensor is characterised as the conformally flat spacetime . Hence the spacetime is said to be conformally flat if

$$C_{abcd} = 0 \text{ .} \tag{4.6}$$

This concept is analogous to the criterion for flatness (Lord 1976).

Claim 1:

If the spacetime is flat then it is conformally flat but the converse is not true.

[Proof is evident from the defining expression (4.1)].

Remark 1:

We can rewrite (4.1) as

$$R_{abcd} = C_{abcd} - \tfrac{1}{2}\left[(R_{ac}g_{bd} - R_{ad}g_{bc} + R_{bd}g_{ac} - R_{bc}g_{ad}) - \frac{R}{3}(g_{ac}R_{bd} - g_{ad}g_{bc})\right]. \tag{4.7}$$

We observe from this that the total gravitational field characterised by R_{abcd} is the sum of gravitational field from matter part (a gravitational field from second term on right hand side) and the gravitational field from matter free part (given by first term on right hand side). Therefore the Weyl tensor C_{abcd} is said to signify the free gravitational field. The spacetime of constant curvature is characterised by specific value of Riemann curvature tensor.

$$R_{abcd} = \psi(g_{db}g_{ac} - g_{cb}g_{ad}) \; . \tag{4.8}$$

This result immediately produces the value of Ricci tensor.

$$R_{ab} = 3\psi g_{ab} \; . \tag{4.9}$$

This implies that the value of ψ to be

$$\psi = \frac{1}{12} R \; . \tag{4.10}$$

Hence the spacetime of constant curvature is now characterised by the expression

$$R_{abcd} = \frac{R}{12}\,(g_{db}g_{ac} - g_{cb}g_{ad}). \tag{4.11}$$

Remark 2:

The spacetime of constant curvature is essentially unique and maximally symmetric.

Claim 2 :

The spacetime of magnetofluid with constant curvature satisfying the equation of state ($\rho=3p$) is conformally invariant.

Proof :

By making use of (4.11) and the expression of Weyl conformal tensor (4.1) we obtain

$$C_{abcd} = \frac{R}{6}(g_{db}g_{ac} - g_{cb}g_{ad}) \tag{4.12}$$

This under the equation of state $\rho=3p$ gives the result,

$$C^{a}{}_{bcd;a} = 0 . \tag{4.13}$$

This implies that the spacetime is conformally invariant.

Claim 3 :

If the spacetime of magnetofluid is of constant curvature then

$$\rho = 3p + \frac{12\psi}{k} .$$

Proof :

By using the expression (3.12) of R_{ab} for magnetofluid, in equation (4.9) we get

$$12\psi = k(\rho - 3p),$$

i.e., $$\rho = 3p + 12\left(\frac{\psi}{k}\right).$$

Hence the result.

2. Decomposition Of Weyl Tensor :

In electromagnetic field theory the Maxwell field tensor consists of two parts (1) electric and (2) magnetic. These two parts are obtainable from suitable contraction of Maxwell field tensor. The analogous process ~~is~~ to decompose the free gravitational field in electric type part and magnetic type part is suggested by (Glass 1975). In this method these two parts are described through suitable contractions of Weyl tensor. We have defined the expression for electric type tensor E_{ab} and magnetic type tensor H_{ab} as follows (Glass 1974).

$$E_{ab} = C_{abcd}U^bU^d, \tag{4.14}$$

$$H_{ab} = \tfrac{1}{2}\,\eta_{ac}{}^{gh}\,C_{ghbd}U^cU^d, \tag{4.15}$$

where η is the Levi-Civita permutation tensor. These tensors satisfy the following properties.

i) These tensors are $\bar{U}$-normal

i.e., $H_{ab}U^a = 0 = E_{ab}U^a$,

ii) These parts are tracefree

i.e.,
$$E^a{}_a = 0 = H^a{}_a \, . \tag{4.16}$$

iii) These patrs are symmetric in both the indices

i.e.,
$$E_{ab} = E_{ba} \, , \tag{4.17}$$

$$H_{ab} = H_{ba} \, . \tag{4.18}$$

The interlinking between electric type tensor E_{ab} and magnetic type tensor H_{ab} can be obtained with the help of (4.13) and (4.14) in the form

$$C_{abcd} = (g_{abqr}g_{cdst} - \eta_{abqr}\eta_{cdst})U^qU^sE^{rt} - (g_{abqr}\eta_{cdst} + \eta_{abqr}g_{cdst})U^qU^sH^{rt} \, , \tag{4.19}$$

where
$$g_{abcd} = g_{ac}g_{bd} - g_{ad}g_{bc} \, . \tag{4.20}$$

We have the Ricci identities for timelike vector field $\bar{U}$ as

$$2\, U_{b;[cd]} = U^a R_{abcd} \, . \tag{4.21}$$

These identities with the Weyl tensor expression (4.1) and kinematical parameters (   ) yield the results (Glass, 1975)

$$E_{ab} = \tfrac{1}{2}(R_{ac}\, p^c{}_d\, p^d{}_b - \tfrac{1}{3}\, p_{ab}R_{cd}p^{cd}) + \dot{U}_{(c;d)}p^c{}_a p^d{}_b - \dot{U}_a\dot{U}_b - \dot{\sigma}_{cd}p^c{}_a p^d{}_b - \tfrac{2}{3}\,\theta\,\sigma_{ab} - \sigma_{ac}\,\sigma^c{}_b + \omega_{ac}\omega^c_b + \tfrac{1}{3}p_{ab}(2\sigma^2 - 2\omega^2 - \dot{U}^c{}_{;c}) . \tag{4.22}$$

$$H_{ab} = -\eta^{cd}{}_{(a}\,p_{b)}{}^e[\omega_{ec;d} + \sigma_{ec;d}] - 2\dot{U}_{(a}\omega_{b)} \, . \tag{4.23}$$

where

$$\bar{\eta}^{abc} = \eta^{abcd} u_d .$$

3. Shearfree Magnetofluid And Electrictype Tensor E_{ab} :

For shearfree fluid ($\sigma_{ab}=0$) we have from (4.22) the expressions

$$E_{ab} = \tfrac{1}{2}(R_{cd}p^c{}_a p^d{}_b - \tfrac{1}{3}p_{ab}R_{cd}p^{cd}) + \dot{u}_{(c;d)}p^c{}_a p^d{}_b - \dot{u}_a\dot{u}_b + \omega_{ac}\omega^c{}_b + \tfrac{1}{3}p_{ab}(-2\omega^2 - \dot{u}^c{}_{;c}) . \quad (4.24)$$

$$H_{ab} = -\bar{\eta}^{cd}{}_{(a}p_{b)}{}^e\,\omega_{e;cd} - 2\dot{u}_{(a}\omega_{b)} . \quad (4.25)$$

If we use the expression (3.12) of Ricci tensor for shearfree magnetofluid in the equation (4.24) then we obtain the value of electrictype tensor E_{ab} as

$$E_{ab} = \tfrac{1}{2}\{-k[(\rho+p+\mu h^2)u_c u_d - \tfrac{1}{2}(\rho-p+\mu h^2)g_{cd} - \mu h_c h_d] \times (g^c{}_a - u^c u_a)(g^d{}_b - u^d u_b) + \tfrac{k}{3}(g_{ab} - u_a u_b)[(\rho+p+\mu h^2)u_c u_d - \tfrac{1}{2}(\rho-p+\mu h^2)g_{cd} - \mu h_c h_d](g^{cd} - u^c u^d)\} + \dot{u}_{(c;d)}(g^c{}_a - u^c u_a)(g^d{}_b - u^d u_b) - \dot{u}_a\dot{u}_b + \omega_{ac}\omega^c{}_b + \tfrac{1}{3}(g_{ab} - u_a u_b)(-2\omega^2 - \dot{u}^c{}_{;c}) . \quad (4.26)$$

This after simplification provides the following result

$$E_{ab} = k(\mu h_a h_b + \tfrac{1}{3}\mu h^2 p_{ab}) + \dot{u}_{(c;d)}(g^c{}_a - u^c u_a)(g^d{}_b - u^d u_b) -$$

$$- \dot{U}_a \dot{U}_b + \omega_{ac}\omega^c{}_b + \frac{1}{3}(g_{ab} - U_a U_b)(-2\omega^2 - \dot{J}^c{}_{;c}) \,. \quad (4.27)$$

Claim 1 :

For geodesic flow of shearfree magnetofluid with magnetic field normal to plane of rotation,

$$E_{ab}h^a h^b = 0 \Longleftrightarrow \omega^2 = -(k\mu h^2).$$

Proof :

The transvection of the equation (4.27) with $h^a h^b$ and using the given conditions

$$\dot{U}_a = o \quad \text{(geodesic flow)}, \quad (4.28)$$

and $$\omega_{ab}h^b = 0 \,, \quad (4.29)$$

we obtain

$$E_{ab}h^a h^b = \frac{2}{3} h^2 (k\mu h^2 + \omega^2) \quad (4.30)$$

From this we can immediately conclude

$$E_{ab}h^a h^b = 0 \Longleftrightarrow \omega^2 = -(k\mu h^2) \,.$$

This completes the proof.

Remark 1:

For varishing of electric type tensor we have,

$\omega^2 = -(k\mu h^2)$, [vide(4.30)]

Remark 2:

The necessary and sufficient condition for a stationary vaccum spacetime to be static is that the Weyl tensor be of electrictype (Glass 1974).

Corollary 1 :

If $E_{ab} = 0$ and $\dot{U}_a = 0$ then,

$$\omega_{ab}h^b = 0 \Longleftrightarrow \omega^2 = -(k\mu h^2).$$

Proof :

For geodesic flow of shearfree magnetofluid if the electric tensor vanishes then,

$$\omega_{ab}h^b = 0 \Longleftrightarrow \omega^2 = -(k\mu h^2).$$

It follows from (4.27) after inner multiplication with $h^a h^b$ and using the given conditions $E_{ab} = 0$ and $\dot{U}_a = 0$ we get

$$\omega_{ab}\,\omega^b{}_c h^a h^c + \frac{2}{3}\, h^2(k\mu h^2 + \omega^2) = 0 \,. \tag{4.31}$$

If we write $\omega_{ab}h^b = S_a$ then above equation yields,

$$S_a S^a - \frac{2}{3}\, h^2(k\mu h^2 + \omega^2) = 0 \,, \tag{4.32}$$

i.e., $$S^2 = -\frac{2}{3}\, h^2(k\mu h^2 + \omega^2) \,, \tag{4.33}$$

where we have chosen $S_a S^a = -S^2$. (4.34)

Now S^2 vanishes when $(k\mu h^2 + \omega^2)$ is zero. But S^2 is zero implies $\omega_{ab}h^b$ is zero. Hence we have the result.

$$\omega_{ab}h^b = 0 \Longleftrightarrow \omega^2 = -k\mu h^2 \,.$$

Here the proof is complete.

4. Shearfree Magnetofluid And Maxwelllike Equations

The divergence of Weyl tensor is independent of itself and is designated as the matter current $J^{*}{}_{abc}$ (Szekeres 1964), viz.,

$$C^{a}{}_{bcd;a} = J^{*}{}_{bcd} \; . \tag{4.35}$$

So that the divergence equation

$$J^{*abc}{}_{;c} = 0 \; , \tag{4.36}$$

is the conservation equation for the source of the free gravitational field. The well known Bianchi identities imply (Kundt and Trumper 1962)

$$J^{*abc} = R^{c[a;b]} - \frac{1}{6} g^{c[a}R^{;b]} \; . \tag{4.37}$$

This after the use of Einstein field equations gives

$$J^{*}{}_{abc} = T_{c[a;b]} + \frac{1}{3} g_{c[a}T_{;b]} \; . \tag{4.38}$$

For shearfree magnetofluid ,this equaiton then gives

$$\begin{aligned} J^{*}{}_{abc} = {} & U_c(\rho+p+\mu h^2)_{;a}U_b + (\rho+p+\mu h^2)U_{c;[a}U_{b]} + \\ & + (\rho+p+\mu h^2)U_c U_{[b;a]} - \mu h_{c;[a}h_{b]} - \mu h_{c;[a}h_{b]} \\ & - \mu h_c h_{[b;a]} + (\tfrac{1}{3}\rho + \tfrac{1}{2}\mu h^2)_{;[b}g_{a]c} \; . \end{aligned} \tag{4.39}$$

The result (4.37) in combination with (4.39) and the expression for Weyl conformal tensor given by (4.19) yields the following Maxwelllike equations under typical inner multiplication (Asgekar, 1979)

i) $$3H_{ab}\omega^{b} + E_{bc;d}p^{b}{}_{a}p^{cd} = -\frac{1}{3}\rho_{,b}p^{b}{}_{a} + \tfrac{1}{2}h_{a;b}h^{b} - \tfrac{1}{2}\mu h_{b;c}h^{c}U^{b}U_{a}\,. \tag{4.40}$$

ii) $$H_{bc;d}p^{cd}p^{ab} - 3E^{a}{}_{b}\omega^{b} = -(\rho+p+\mu h^{2})\omega^{a} + \tfrac{1}{2}\mu\eta^{abcd}U_{b}U^{e}h_{c}h_{e;d}. \tag{4.41}$$

iii) $$H_{ac}p^{ma}p^{tc} - p_{a}{}^{(m}\eta^{t)rsd}U_{r}E^{a}{}_{s;d} + 2E_{a}{}^{(t}\eta^{m)bpa}U_{b}\dot{U}_{p} + \frac{1}{3}\theta H^{mt} - H_{s}{}^{(m}\omega^{t)s} = -\frac{\mu}{2}h_{a;c}p^{a(t}\eta^{m)ebc}U_{e}h_{b} - \frac{\mu}{2}h_{b;c}h^{(t}\eta^{m)ebc}U_{e}. \tag{4.42}$$

iv) $$\dot{E}^{ab}p^{t}{}_{a}p^{m}{}_{b} + H^{a}{}_{s;d}U_{r}p_{a}{}^{(m}\eta^{t)rsd} - 2H_{a}{}^{(t}\eta^{m)bpa}U_{b}\dot{U}_{p} + \frac{1}{3}\theta E^{mt} - E_{s}{}^{(m}\omega^{t)s} = -\tfrac{1}{4}\mu h^{2}{}_{;b}U^{b}p^{tm} - \frac{1}{6}\mu h^{2}\theta p^{tm} - \frac{\mu}{2}h_{a;b}U^{b}h^{m}p^{at} - \mu h_{[b;c]}U^{c}h^{t}p^{bm}\,. \tag{4.43}$$

The conformal flatness is the sufficient condition to ensure that the velocity vector $\bar{U}$ to be shearfree and hypersurface orthogonal. Hence the conformally flat solutions of Stephani, 1982 are the most general conformally flat fluid solutions. Therefore, under the restriction of conformal flatness $E_{ab} = H_{ab} = 0$ which then with the help of equations (4.40) and (4.41) yield

$$2\rho_{,b}p^{b}{}_{a} - 3\mu h_{a;b}h^{b} + 3\mu h_{b;c}h^{c}u^{b}u_{a} = 0. \tag{4.44}$$

$$2(\rho+p+\mu h^{2})\omega^{a} - \mu\eta^{abcd}u_{b}u^{e}h_{c}h_{e;d} = 0. \tag{4.45}$$

Claim 1 :

If the spacetime of the shearfree magnetofluid is conformally flat then the matter density conserves along the magnetic lines if and only if the magnitude of the magnetic field also conserves along these lines.

Proof :

We consider the term

$$h_{a;b}h^{b}u^{a} = -u_{a;b}h^{a}h^{b} \;, \qquad (\because\; u_{a}h^{a}=0)$$

i.e., $$h_{a;b}h^{b}u^{a} = -(\omega_{ab}+\frac{1}{3}\,\theta\, p_{ab}+\dot{u}_{a}u_{b})h^{a}h^{b} \;,$$

i.e., $$h_{a;b}h^{b}u^{a} = \frac{1}{3}\,\theta h^{2} \;. \tag{4.46}$$

This value when inserted in the equation (4.44) gives

$$2\rho_{,b}p^{b}{}_{a} - 3\mu h_{a;b}h^{b} + \theta h^{2}u_{a} = 0. \tag{4.47}$$

This after contraction with h^{a} gives

$$2\rho_{,b}h^{b} + \frac{3}{2}(h^{2}{}_{;b})h^{b} = 0 \;, \tag{4.48}$$

since $\quad h_{a;b}h^{a} = -\frac{1}{2}h^{2}{}_{;b}\;.$

We write from this

$$\rho_{,b}h^{b}=0 \iff h^{2}{}_{,b}h^{b}=0 \;.$$

This is the required result.

Claim 2:

For shearfree magnetofluid with conformally flat space-time the verticity vector is orthonormal to vectors $\bar{U}$, $\bar{h}$ and $\bar{S}$ $(S_a = \omega_{ab}h^b)$

Proof :

On introducing the kinematical parameters, from the equation (4.45) we write

$$2(\rho+p+\mu h^2)\omega^a + \mu\eta^{abcd}\omega_{ed}U_b h_c h^e = 0 ,$$

i.e., $$(\rho+p+\mu h^2)\omega^a + \tfrac{1}{2}\mu\eta^{abcd}U_b h_c S_d = 0 , \qquad (4.49)$$

where we have taken

$$S_d = \omega_{ed}h^e . \qquad (4.50)$$

This then further simplified into

$$(\rho+p+\mu h^2)\omega^a = -\mu D^a , \qquad (4.51)$$

by defining D^a as

$$D^a = \tfrac{1}{2}\eta^{abcd}\omega_{ed}U_b h_c h^e . \qquad (4.52)$$

Here by the definition of D^a, it is orhtogonal to $\bar{U}$, $\bar{h}$ and $\bar{S}$. Hence the claim follows from the result (4.51).

Remark :

Under the condition of uniform magnetofluid $(h_{a;b}=0)$

the results (4.44) and (4.45) tally exactly with the results derived by Barnes (1984). Hence we claim that for shearfree magnetofluid with uniform magnetic field if $\sigma_{ab}=H_{ab}=0$, then either the vorticity vector is zero or it is an eigenvector of E_{ab} with its eigenvalue $\frac{1}{3}(\rho+p+\mu h^2)$, (vide 4.41).

5. Divergence Expressions For The Electric Type And The Magnetic Type Tensors :

We have the divergence equations of E_{ab} and H_{ab} (Glass, 1975)

$$E^a{}_{b;a} = R_{d[b;c]}U^dU^c + \frac{1}{12}p^c{}_bR_{;c} - \dot{U}^cE_{bc} + 3\omega^cH_{bc} - U_b\sigma^{ad}E_{ad}\,. \qquad (4.53)$$

$$H^a{}_{b;a} = \frac{1}{2}\bar{\eta}^{ad}_b\,U^cR_{ca;d} - \dot{U}^cH_{bc} - 3\omega^cE_{bc} - U_b\sigma^{ad}H_{ad}\,. \qquad (4.54)$$

These relations under the condition $\sigma_{ab}=0$ becomes

$$E^a{}_{b;a} = R_{d[b;c]}U^dU^c + \frac{1}{12}p^c{}_bR_{,c} - \dot{U}^cE_{bc} + 3\omega^cH_{bc}\,. \qquad (4.55)$$

$$H^a{}_{b;a} = \frac{1}{2}\bar{\eta}^{ad}_bU^cR_{ca;d} - \dot{U}^cH_{bc} - 3\omega^cE_{bc}\,. \qquad (4.56)$$

We have the following simplification of the result (4.55) for the shearfree magnetofluid.

$$E^a{}_{b;a} = -\frac{k}{2}\Big\{\big[(\rho+p+\mu h^2)U_dU_b - \tfrac{1}{2}g_{db}(\rho-p+\mu h^2) - h_dh_b\big]_{;c} - \big[(\rho+p+\mu h^2)U_dU_c - \tfrac{1}{2}(\rho-p+\mu h^2)g_{dc} - h_dh_c\big]_{;b}\Big\}U^dU^c + \frac{k}{12}(g^c{}_b-U_bU^c)\,\big[(\rho-3p)_{;c}\big] -$$

$$- \dot{U}^c E_{bc} + 3\omega^c H_{bc} .$$

This after simplification yields the following result

$$E^a{}_{b;a} = -\frac{k}{2}\left[(\rho+p+\mu h^2)\dot{U}_b - \mu \dot{h}_d U^d h_b\right] + \frac{k}{12}(\rho-3p)_{,c}P^c{}_b - \dot{U}^c E_{bc} + 3\omega^c H_{bc} . \qquad (4.57)$$

Also in the similar way we can find the simplification of the result (4.56) for shearfree magnetofluid as

$$H^a{}_{b;a} = \frac{1}{2}\bar{\eta}_b{}^{ad}U^c\left\{-k\left[(\rho+p+\mu h^2)U_c U_a - \frac{1}{2}(\rho-p+\mu h^2)g_{ca} - \mu h_c h_a\right]_{;d}\right\} - \dot{U}^c H_{bc} - 3\omega^c E_{bc} .$$

This after simplification supplies

$$H^a{}_{b;a} = -k\,\omega_b(\rho+p+\mu h^2) - \frac{k}{2}\mu h^c h_a \omega_{cd}\eta_b{}^{ade}U_e - \dot{U}^c H_{bc} - 3\omega^c E_{bc} . \qquad (4.58)$$

Theorem 1 :

For shearfree magnetofluid if the electrictype tensor is divergencefree and the magnetic field vector is the eigenvector of both electrictype tensor and magretic type tensor with zero eigenvalue then the density is preserved along the magnetic lines if and only if the pressure is preserved along these lines.

Proof :

On trarsvecting the equation (4.57) with h^b we get

$$- \frac{k}{2}\left[(\rho + p + \mu h^2)\dot{U}_b h^b + \mu h^2 \dot{h}_b U^b \right] + \frac{k}{12}(\rho - 3p)_{;b} h^b -$$

$$- \dot{U}^c E_{bc} h^b + 3\omega^c H_{bc} h^b = 0$$

i.e., $$- \frac{k}{2}(\rho + p)\dot{U}_b h^b + \frac{k}{12}(\rho - 3p)_{;b} h^b - \dot{U}^c E_{bc} h^b + + 3\omega^c H_{bc} h^b = 0. \tag{4.59}$$

since $\dot{U}_a h^a = -\dot{h}_a U^a$.

By using (3.19) we reduce this equation as

$$- \frac{k}{2} p_{;b} h^b + \frac{k}{12} (\rho - 3p)_{;b} h^b - \dot{U}^c E_{bc} h^b + 3\omega^c H_{bc} h^b = 0 ,$$

i.e., $$\frac{k}{12}(\rho_{;b} - 9 p_{;b}) h^b - \dot{U}^c E_{bc} h^b + 3\omega^c H_{bc} h^b = 0 \tag{4.60}$$

Now as $\bar{h}$ is the eigenvector of E_{ab} and H_{ab} with eigenvalue zero we have

$$E_{ab} h^b = H_{ab} h^b = 0. \tag{4.61}$$

This result when substituted in (4.60), we make the assertion

$$\rho_{;b} h^b = 0 \iff p_{;b} h^b = 0 . \tag{4.62}$$

This is the required result.

Remark :

The theorem is also valid if the spacetime of shearfree magnetofluid is conformally flat.

Claim 3 :

For the divergencefree electric type tensor of shearfree magnetofluid the density is invariant along the flowlines if and only if the pressure remains invariant along these lines.

Proof :

The inner multiplication of (4.57) with flowvector $\bar{U}$ and recalling the results

$$\dot{U}_a U^a = 0 , \quad U_a h^a = 0 \qquad \text{and}$$

$$E_{ab} U^b = 0 = H_{ab} U^b , \qquad \text{we deduce}$$

$$(\rho - 3p)_{;a} U^a = 0 .$$

This implies that

$$\rho_{;a} U^a = 0 \Longleftrightarrow p_{;a} U^a = 0 .$$

Here the proof is complete.

Claim 4 :

For shearfree magnetofluid with the geodesic flow and zero electirctype tensor, ~~then~~ the divergencefree magnetic type tensor implies that the vorticity vector is normal to magnetic field lines.

Proof :

For divergencefree magnetictype tensor we write from (4.59)

$$k\omega_b(\rho + p + \mu h^2) + \frac{k}{2}\mu h^c h_a \omega_{cd} \eta_b{}^{ade} U_e + \dot{U}^c H_{bc} +$$

$$+ 3\omega^c E_{bc} = 0 . \qquad (4.60)$$

When the given conditions $\dot{U}_a=0$ and $E_{ab}=0$ are used in above equation then

$$k\omega_b(\rho+p+\mu h^2) + \frac{k}{2}\mu h^c h_a \omega_{cd} \eta_b{}^{ade} U_e = 0 . \qquad (4.61)$$

This after contraction with h^b yields

$$k\omega_b(\rho+p+\mu h^2)h^b = 0 .$$

This equation gives

$$\omega_b h^b = 0 \quad \text{as} \quad (\rho+p+\mu h^2) \neq 0 .$$

This completes the proof.

Note :

Again the equation (4.61) after inner multiplication with ω^b provides

$$- k\omega^2(\rho+p+\mu h^2) + \frac{k}{2}\mu h^c h_a \omega_{cd} \eta_b{}^{ade} U_e \omega^b = 0 ,$$

i.e., $$\omega^2(\rho+p+\mu h^2) + \frac{\mu}{2} s^2 = 0 , \qquad (4.62)$$

where we have written $S_a = \omega_{ab}h^b$ so that $S_a S^a = -S^2$.

We write from this

$$s^2 = m\omega^2 , \qquad (4.63)$$

where, $m = \frac{2(\rho+p+\mu h^2)}{\mu}$.

This shows that the magnitude of the vector $\bar{S}$ is a multiple of ω^2 .

Remark :

Note that this conclusion is made under the condition that $\dot{U}_a=0$, and $E_{ab}=0$.

Theorem :

The necessary and sufficient condition for a shearfree magnetofluid with magnetic field normal to the plane of rotation be rotationfree is that the Weyl tensor be of purely electrictype.

Proof :

Under the stated condition $\omega_{ab}h^b=0$, the equation (4.58) gives rise to

$$H^a{}_{b;a} = -k\omega_b(\rho+p+\mu h^2) - \dot{U}^c H_{bc} - 3\omega^c E_{bc} \ . \tag{4.64}$$

The Necessary Part :

If we use the conditions that vorticity as well as shear vanish in the expression of H_{ab} (4.19) then we get $H_{ab}=0$. This shows that Weyl tensor is purely electrictype.

The sufficient Part :

As the Weyl tensor is of purely electrictype we have $H_{ab}=0$ in equation (4.59). So that we get

$$\omega^c[k(\rho+p+\mu h^2)p_{bc} + 3E_{bc}] = 0 \ ,$$

i.e.,

$$[E_{bc} + \frac{k}{3}(\rho+p+\mu h^2)p_{bc}] = 0 \ . \tag{4.65}$$

Now when the Weyl tensor is purely electric then it must be of Petrovtype I, D, or O (Jordan et,al, 1960). It immediately follows that $\omega^c=0$ for type O. Further in case of type I and D we know that

$$\det\left[E_{bc} + \frac{k}{3}(\rho + p + \mu h^2)p_{b_c}\right] \neq 0.$$

[Since if the determinant were zero then E_{bc} would have three equal eigenvalues, which is not possible]. Thus $\omega^c=0$. Hence we have proved the sufficient part of the theorem that the rotation is zero for purely electric type Weyl tensor.