Chapter - 0

PRELIMINARIES

In this chapter we give some basic definitions and results which we have used in the dissertation.

§ 0.1 DEFINITIONS

Def.0.1.1 : Partially ordered set or poset [5] : Let P be a nonvoid set. Define a relation $\leq$ on P which has following properties for all $a, b, c, \in P$

i) $a \leq a$ (reflexivity)

ii) $a \leq b$ and $b \leq a \Rightarrow a = b$ (antisymmetry)

iii) $a \leq b$ and $b \leq c \Rightarrow a \leq c$ (transitivity)

The relation satisfying above three conditions is called partial ordering relation. And the set P equipped with such relation $\leq$ is called partially ordered set or poset.

A poset P is called a chain (or totally ordered set or linearly ordered set) if it satisfies the following condition for all $a, b, \in P$

iv) $a \leq b$ or $b \leq a$ (linearity).

Def.0.1.2 : Zero element and Unit element of a Poset [5] : A zero element of a poset P is an element 0 with $0 \leq x$ for all $x \in P$. A unit element 1 of a poset is an element with $x \leq 1$ for all $x \in P$.

Def.0.1.3 : Lattice as Poset [5] : A poset $(L, \leq)$ is a lattice if sup $\{a,b\}$ or $a \vee b$ and inf $\{a,b\}$ or $a \wedge b$ exist for all $a,b, \in L$.

Relation!

Def.0.1.4 : Lattice as an algebra [5] : An algebra $(L,\wedge,\vee)$ is called a lattice if L is nonvoid set, $\wedge$ and $\vee$ are binary operations on L satisfying following properties for all $a,b,c \in L$

i) $a \wedge a = a, \quad a \vee a = a$ (idempotency)

ii) $a \wedge b = b \wedge a, \ a \vee b = b \vee a$ (commutativity)

iii) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$.

$(a \vee b) \vee c = a \vee (b \vee c)$ (associativity)

iv) $a \wedge (a \vee b) = a$,

$a \vee (a \wedge b) = a$ (absorption identities)

Def.0.1.5 : Distributive lattice [1] : A lattice L is said to be distributive if for all $a,b,c \in L$, then the following identity hold

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

or
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

|| ?

Def.0.1.6 : 0-distributive lattice [17] : A lattice L with 0 is said to be 0-distributive if it satisfies the condition : $a \wedge b = 0$ and $a \wedge c = 0$ imply $a \wedge (b \vee c) = 0$. for all $a,b,c \in P$

Def.0.1.7 : Modular lattice [5] : A lattice L is called modular if, $x,y \in L$ and $z \leq x$ imply that

$(x \wedge y) \vee z = x \wedge (y \vee z)$ for all $z \in L$.

Def.0.1.8 : O-modular lattice [17] : A lattice L with least element 0 is said to be 0-modular if it satisfies the condition : $a \leq c$ and $b \wedge c = 0$ imply $(a \vee b) \wedge c = a$. , a, b, c $\in$ L

Def.0.1.9 : Bounded lattice [5] : A lattice L is said to be a Bounded lattice if it has both 0 and 1.

Def.0.1.10 : Complement in a lattice [5] : Let L be a bounded lattice, a,b, $\in$ L. Then a is a complement of b if $a \wedge b = 0$ and $a \vee b = 1$.

Def.0.1.11 : Complemented lattice [5] : A complemented lattice is a bounded lattice in which every element has a complement.

Def.0.1.12 : Pseudocomplement in a lattice [5] : Let L be a lattice with 0. An element a^* in L is pseudocomplement of $a \in L$ if $a \wedge a^* = 0$ and $a \wedge x = 0$ implies that $x \leq a^*$.

Def.0.1.13 : Pseudocomplemented lattice [5] : A lattice with 0 is said to be pseudocomplemented if each element of L has a pseudocomplement.

Def.0.1.14 : Ideal [4] : Let L be lattice and let $I \subseteq L$. I is called an ideal, if $a, b \in I$ implies that $a \vee b \in I$ and

$a \in I$, $x \in L$, $x \leq a$ imply that $x \in I$.

Def.0.1.15 : Maximal ideal [5] : Let L be a lattice, A proper ideal I of L is called maximal if it is not contained in any other $(J \neq L)$ proper ideal of L. Proper ideal

Def.0.1.16 : Prime ideal [5] : A proper ideal I of L is prime, if a, $b \in L$ and $a \wedge b \in I$ imply that $a \in I$ or $b \in I$.

Def.0.1.17 : Principal ideal [5] : Let L be a lattice and $a \in L$. Then the intersection of ideals in L containing a is called principal ideal generated by a. It is denoted by (a]. Equivalently ~~principal ideal as~~

$$(a] = \{x \in L : x \leq a\}.$$

The ideal generated by H in L $(H \subseteq L)$ is the intersection of all ideals in L containing H. It is denoted by (H].

The concepts of filter, maximal filter, prime filter, principal filter are defined dually [5].

Def.0.1.18 : Boolean lattice [5] : A lattice L is called Boolean if it is complemented and distributive.

difference

Def.0.1.19 : Boolean algebra [5] : A Boolean algebra is a Boolean lattice in which 0,1 and ' are also considered as operations.

Thus a Boolean algebra is a system $(B, \wedge, V, ', 0.1)$ where $\wedge$ and v are binary, ' is unary operation and 0,1 are nullary operations.

Def.0.1.20 : Maximal element [5] : Let P be a poset. An element $a \in P$ is called maximal if $a \leq b$ $(b \in P)$ implies that $a=b$.

The minimal element of a poset can be defined dually.

Def.0.1.21. : Semilattice [5] : A poset is a join-semilattice (dually; meet-semilattice) if sup $\{a, b\}$ or $a \vee b$ (dually inf $\{a, b\}$ or $a \wedge b$) exists for any two elements a,b, of a poset. Semilattice as an algebra?

Def.0.1.22 : Distributive Semilattice [6]: A join-(meet)Semilattice S is distributive if for any $x,y,z \in S$, $z \leq x \vee y$ $(x \wedge y \leq z)$ implies $z=a \vee b$ $(z=a \wedge b)$ with $a \leq x$, $b \leq y$ $(x \leq a$, $y \leq b)$. Correct the definition

Def.0.1.23 : 0-distributive Semilattice [18] : A Semilattice S with 0 is 0-distributive if and only if $(a)^* = \{x \in S: x \wedge a = 0\}$ is an ideal in S for every $a \in S$. Ideal in a semilattice

Def.0.1.24 : Modular Semilattice [14] : A join-semilattice (meet-semilattice) is modular if for any $x,y,z, \in S$, $y \leq z \leq x \vee y$ $(x \wedge y \leq z \leq y)$ implies there exists $a \in S$ such that $z = x \vee a$ and $a \leq y$ $(z = x \wedge a$ and $y \leq a)$.

<u>Def.0.1.25</u> : <u>0-modular semilattice [7]</u> : A semilattice S with 0 is called 0-modular if $a\leq c$ and $b\wedge c=0$ $(a,b,c\varepsilon,S)$ imply that there exists d in S such that $b\leq d$ and $a=c\wedge d$.

<u>Def.0.1.26</u> : <u>Congruence relation on Semilattice [5]</u> : An equivalence relation θ on join-semilattice (meet-semilattice) is a congruence relation if, $a\equiv b(\theta)$ and $c\equiv d(\theta)$ implies that $a\vee c\equiv b\vee d(\theta)$ $\{a\wedge c\equiv b\wedge d(\theta)\}$.

<u>Def.0.1.27</u> : <u>Retract of Semilattice [15]</u> : A semilattice T is called a retract of a semilattice S if and only if there are homomorphisms $f:S\rightarrow T$ and $g : T\rightarrow S$ such that $f\circ g$ is identify on T. $f\circ g$.

<u>Def.0.1.28</u> : <u>Cover [5]</u> : In the poset $(p,\leq)$ a covers b in notation $a\succ b$ if $b<a$ and for no x, $b<x<a$.

<u>Def.0.1.29</u> : <u>Atom of a poset [5]</u> : An element a of a poset is an atom if $a\succ 0$.

<u>Def.0.1.30</u> : <u>Dual-atom of a poset [5]</u> : An element a of a poset is dual-atom if $1\succ a$.

<u>Def.0.1.31</u> : <u>Principal ideal in a poset [20]</u> : The set of all elements of a poset P such that $x\leq a$ for some fixed $a\varepsilon P$ is called principal ideal generated by a. It is denoted by$(a]$.

Def.0.1.32 : Prime ideal in a poset [19] : A proper ideal I of a poset P is prime, if a, b ε P such that $(a] \cap (b] \subseteq$ I then $(a] \subseteq$ I or $(b] \subseteq$ I ($a \varepsilon I$ or $b \varepsilon I$).

Def.0.1.33 : Pseudocomplements in poset [20] : An element a of a poset P with 0 is said to have pseudocomplement a* ε P if there exists P an element a* in P such that

i) $(a] \cap (a^*] = (0]$.

ii) for $b \varepsilon P$, $(a] \cap (b] = (0] \Rightarrow (b] \subseteq (a^*]$.

Def.0.1.34 : Pseudocomplemented poset [20] : A poset P with 0 is said to be pseudocomplemented if every one of its elements has a pseudocomplement.

Def.0.1.35 : Ascending chain condition [5] : A poset P is said to satisfy A scending chain condition (ACC) if any increasing chain terminates. That is if $x_i \varepsilon P$ i=0, 1,2 ... and $x_0 \leq x_1 \leq x_2 \leq \ldots$ then for some m we have $x_m = x_{m+1} = ---$.

Def.0.1.36 : 0-distributive poset [12] : A poset P is called as a 0-distributive poset if for a, x_1, --- x_n ε P (n finite), $(a] \cap (x_i] = (0]$ $\forall$ i, $1 \leq i \leq n$ imply $(a] \cap (x_1 \vee --- \vee x_n] = (0]$ whenever $x_1 \vee --- \vee x_n$ exists in P.

§ 0.2 RESULTS

Result.0.2.1 [20] : The set I(P) of all ideals of a poset P with 0 is a complete lattice under set inclusion as ordering relation. def.

Result.0.2.2 [5] : Every distributive lattice is modular.

Result. 0.2.3 [15] : Every distributive semilattice is modular.

Result.0.2.4 [18] : Every distributive semilattice with 0 is 0-distributive.

Result.0.2.5 [7] : Every modular semilattice with 0 is 0-modular.

Result.0.2.6 [17] : Every pseudocomplemented lattice is 0-distributive.

Result.0.2.7 [5] : In a poset satisfying ascending chain condition (ACC) every ideal is principal.

Result.0.2.8 [17] : A lattice L with 0 is 0-distributive if and only if the lattice of all ideals is pseudocomplemented.

Result.0.2.9 [11] : A 0-distributive semilattice S is pseudocomplemented if and only if (a)* is principal ideal for every a ε P.

Result.0.2.10 [11] : A semilattice S with 0 is 0-distributive if and only if I(P) is pseudocomplemented.

Result.0.2.11 [11] : In a semilattice P with 0 and ascending chain condition (ACC) 0-distributively is equivalent to pseudocomplementedness.