CHAPTER - II

O-distributive Posets

C H A P T E R-II

I n t r o d u c t i o n

Pseudocomplemented lattices form an important class of lattices and have been studied by many authors. Frink [4] has obtained the generalization of the theory for semi-lattices. Venkatnarasimhan [11] successfully extended some of the results of Frink [4] and Balchandran [1] to partially ordered sets by defining pseudocomplements in a poset. A poset P bounded below is pseudocomplemented if and only if for any a $\in$ P, the subset of elements disjoint from a is a principal ideal. On one hand it is quite reasonable to replace 'principal ideal' by 'ideal'. This weakened condition motivated us to define O-distributivity in a poset.

On the other hand, O-distributivity in a poset is also an extension of O-distributivity in semilattice. Varlet [10] was the first to investigate O-distributive lattices. O-distributive semilattices were introduced by Varlet [10] and also by Pawar [8] in different ways. We have succeeded in pushing the considerations of Pawar [8] for posets.

Cornish [3] studied in detail annihilator ideals in a distributive lattice. He proved that the set of annihilator ideals in a distributive lattice is a Boolean algebra. Similar result was expected for poset and we have proved that this is true. In this chapter we have obtained several characterizations of O-distributive poset. ~~It is~~

shown that the class of O-distributive posets contains distributive poset with O. Disjunction poset was defined by Venkatnarasimhan [12]. We have also studied disjunctivity in a O-distributive poset and we have shown that these two concepts are completely independent. Properties of dense elements are also studied in O-distributive posets.

...

O-DISTRIBUTIVE POSETS

§ 2.1 Definition and Examples

Throughout this Chapter we shall concern with partially ordered sets or posets. We shall denote the ordering relation in a poset by $\leq$. Let $A = \{a_1, a_2, \ldots, a_n\}$ be a finite subset of poset P, then the least upper bound (join) and the greatest lower bound (meet) of a_i $(1 \leq i \leq n)$ are denoted by $a_1 \vee a_2 \vee \ldots \vee a_n$ and $a_1 \wedge a_2 \wedge \ldots \wedge a_n$ respectively. The least and the greatest elements of a poset, whenever they exist are denoted by '0' and '1' respectively. The set inclusion, set union and set intersection are denoted by $\subseteq$, $\cup$ and $\cap$ respectively. Hereafter by a symbol P we mean a poset P with '0'. We define a 0-distributive poset as follows :

<u>Def. 2.1.1</u> : 0-distributive poset : - The poset P is called 0-distributive if for some finite n, $x_1, x_2, \ldots, x_n$ and $a \in P$ such that

$$(x_1] \cap (a] = (0] ,$$
$$(x_2] \cap (a] = (0] ,$$
$$- - - - - - - - - - -$$
$$- - - - - - - - - - -$$
$$(x_n] \cap (a] = (0] .$$

And if $x_1 V x_2 V \ldots V x_n$ exists, then

$$(x_1 V x_2 V \ldots V x_n] \cap (a] = (0]$$

Dually, we can define 1 - distributive poset.

Example 2.1.2 : An example of 0-distributive poset .

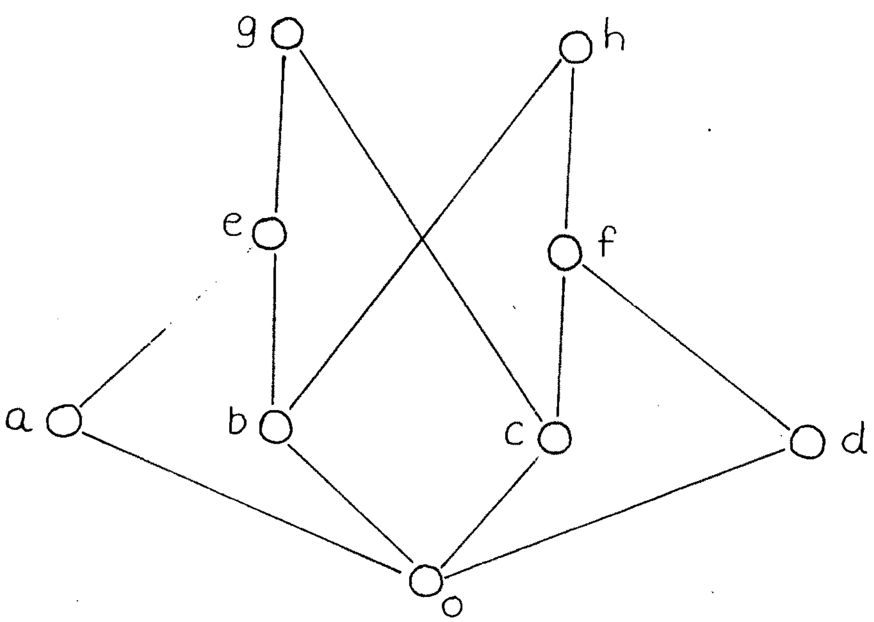


Fig-1

The poset represented by Fig.1 above is 0-distributive.

<u>Example 2.1.3</u> : We furnish here one more example of a 0-distributive poset (This poset contains infinite number of elements)

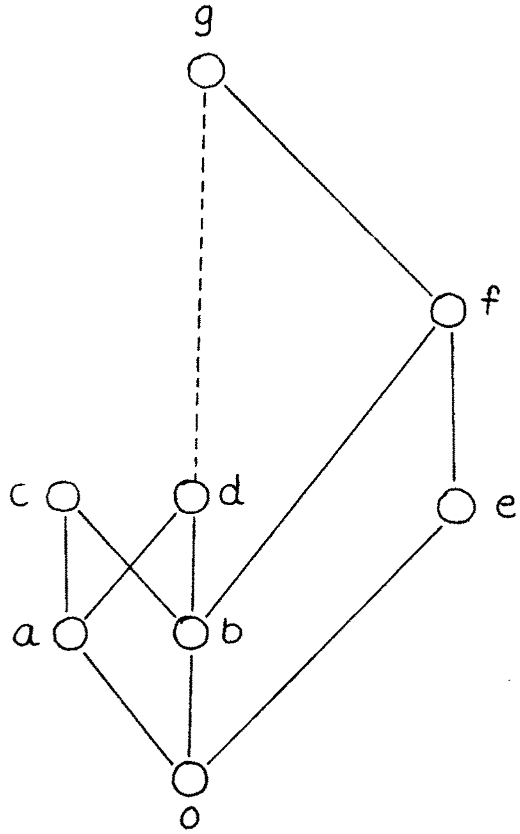


Fig-2

The poset shown in Fig.2 above is 0-distributive.
Note that every poset need not be 0-distributive.

Example 2.1.4 : An example of a poset which is not 0-distributive is given in figure 3.

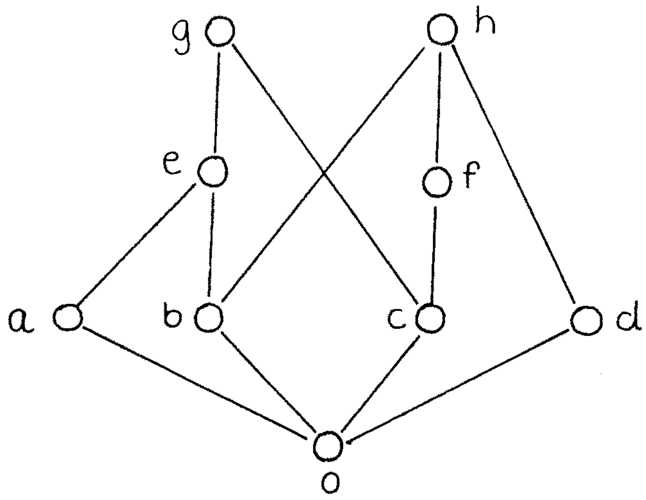


Fig-3

Explanation : In the poset shown in figure 3 above we have

$$(b] \cap (d] = (0]$$
$$(c] \cap (d] = (0]$$
$$(f] \cap (d] = (0]$$

but $(b \vee c \vee f] \cap (d] = (d] \neq (0]$, where $b \vee c \vee f = h$.

Remark : 2.1.5 : A subset of 0-distributive poset, containing '0', need not be 0-distributive (See Fig.1.)

§ 2.2 O-distributivity and pseudocomplementedness

In this article we establish a connection between a O-distributive poset and pseudocomplemented poset.

Let P be a poset a, b $\in$ P. We denote the set of all lower bounds of a,b by $\{a, b\}^{\ell}$.

i.e. $\{a,b\}^{\ell} = \{x \in P \ / \ x \leq a \text{ and } x \leq b\}$

For any poset P the set $\{a\}^* = \{x \in P / \{x,a\}^{\ell} = \{0\}\}$ is a semiideal of P, for a $\in$ P [11]. But $\{a\}^*$ is an ideal for all a $\in$ P if and only if P is O-distributive. Note that Venkatnarasimhan's definition of a poset ideal [11] is different from that given by Frink [5]. Now we have following

Result : 2.2.1 : A poset P is O-distributive if and only if $\{a\}^* = \{x \in P \ / \ \{x,a\}^{\ell} = \{0\}\}$ is an ideal in P for any a $\in$ P.

Proof : Let P be O-distributive poset we have to prove that $\{a\}^*$ is an ideal.

i) Let $y \in \{a\}^*$ and $x \in P$ such that $x \leq y$. If $\{x, a\}^{\ell} \neq \{0\}$ then there exists $z \in P$ such that $z \in \{x,a\}^{\ell}$ and $z \neq 0$. Since $z \leq x$ and $x \leq y$ by

transitivity we get $z \leq y$. Hence $z \in \{y,a\}^{\ell} = \{0\}$
This is a contradiction and hence $\{x,a\}^{\ell} = \{0\}$.
This proves that $x \in \{a\}^*$

ii) Let $x_1, x_2, \ldots, x_n$ be in $\{a\}^*$ and suppose that $x_1 V x_2 V \ldots V x_n$ exists in P. As $x_i \in \{a\}^*$ $(1 \leq i \leq n)$ implies $\{x_i, a\}^{\ell} = \{0\}$. But then

$(x_i] \cap (a] = (0]$ As this is true for each i, $1 \leq i \leq n$ we have

$$
\begin{aligned}
(x_1] \cap (a] &= (0] , \\
(x_2] \cap (a] &= (0] , \\
&\text{-----------------} \\
&\text{-----------------} \\
(x_n] \cap (a] &= (0] .
\end{aligned}
$$

Since P is 0-distributive we must have

$$(x_1 V x_2 V \ldots V x_n] \cap (a] = (0]$$

Hence $x_1 V x_2 V x_3 V \ldots V x_n \in \{a\}^*$.

From i) and ii) it follows that $\{a\}^*$ is an ideal in P for any $a \in P$.

Conversely suppose that $\{a\}^*$ is an ideal for any $a \in P$. Let $x_1, x_2, \ldots, x_n \in P$ such that,

$$(x_1] \cap (a] = (0],$$
$$(x_2] \cap (a] = (0],$$
- - - - - - - - - - - -
- - - - - - - - - - - -
$$(x_n] \cap (a] = (0].$$

And assume that $x_1 V x_2 V \dots V x_n$ exists in P. Now by assumption $\{a\}^*$ being an ideal $x_1 V x_2 V \dots V x_n \in \{a\}^*$. This inturn implies $\{x_1 V x_2 V \dots V x_n, a\}^{\ell} = \{0\}$. That is $(x_1 V x_2 V \dots V x_n] \cap (a] = (0]$ proving that P is 0-distributive. ▮

More generally we have the following

Result 2.2.2 : P is 0-distributive if and only if $A^* = \{x / x \in P, \{x, a\}^{\ell} = \{0\} \; \forall a \in A\}$ is an ideal for $A \subseteq P$.

Proof : - Let A^* be an ideal for any $A \subseteq P$. Then particularly $\{a\}^*$ is an ideal for any $a \in P$. Hence by Result (2.2.1) P is 0-distributive.

Conversely, let P be 0-distributive we have to prove that A^* is an ideal for any $A \subseteq P$. We claim that $A^* = \cap\{\{a\}^* / a \in A\}$. If $x \in A^*$ then for all $a \in A$ we have $\{x, a\}^{\ell} = \{0\}$. This will imply $x \in \{a\}^*$ for all

$a \in A$. Hence $x \in \bigcap \{\{a\}^*/a \in A\}$. Thus $A^* \subseteq \bigcap\{\{a\}^*/a \in A\}$.

For the reverse inclusion let $x \in \bigcap \{\{a\}^*/a \in A\}$ then $\{x, a\}^{\ell} = \{0\}$ for all $a \in A$. But this inturn implies $x \in A^*$. Hence $\bigcap\{\{a\}^*/ a \in A\} \subseteq A^*$. Combining both results we get

$$\bigcap \{\{a\}^* / a \in A\} = A^*$$

Thus A^*, being an arbitrary intersection of ideals, is an ideal in P. ∎

Already there exists a theory of pseudocomplements in lattices [6]. Frink [4] has obtained a generalization of the theory for semilattices. Venkatnarasimhan [11] extended some of the results of Frink [4] and Balchandran [1] the most general systems, posets nicely. Here we prove that 0-distributive poset is a generalization of pseudocomplemented poset defined by Venkatnarasimhan [11].

<u>Result 2.2.3</u> : Every pseudocomplemented poset is 0-distributive.

<u>Proof</u> : Let P be pseudocomplemented poset and let $x_1, x_2, \ldots, x_n, a \in P$ such that,

$$(x_1] \cap (a] = (0] ,$$
$$(x_2] \cap (a] = (0] ,$$
$$- - - - - - - - - - - -$$
$$- - - - - - - - - - - -$$
$$(x_n] \cap (a] = (0] ,$$

Further suppose $x_1 \vee x_2 \vee \ldots \vee x_n$ exists in P. Now since P is pseudocomplemented, a^* exists in P for any $a \in P$ and further $(a]^* = (a^*]$ (see Result 1.2.3). But then we have $(x_1] \subseteq (a^*]$, $(x_2] \subseteq (a^*]$, ... $(x_n] \subseteq (a^*]$. As $x_1 \vee x_2 \vee \ldots \vee x_n$ exists

$$(x_1] \underline{\vee} (x_2] \underline{\vee} \ldots\ldots \underline{\vee} (x_n] = (x_1 \vee x_2 \vee \ldots \vee x_n]$$

(See Result 1.2.2). Therefore,

$$(x_1 \vee x_2 \vee \ldots \vee x_n] \subseteq (a^*] .$$

which inturn implies that

$$(x_1 \vee x_2 \vee \ldots \vee x_n] \cap (a] = (0] .$$

And hence P is 0-distributive. ∎

<u>Remark 2.2.4</u> : The converse of the above result need not be true. Consider the following poset represented by Figure 4.

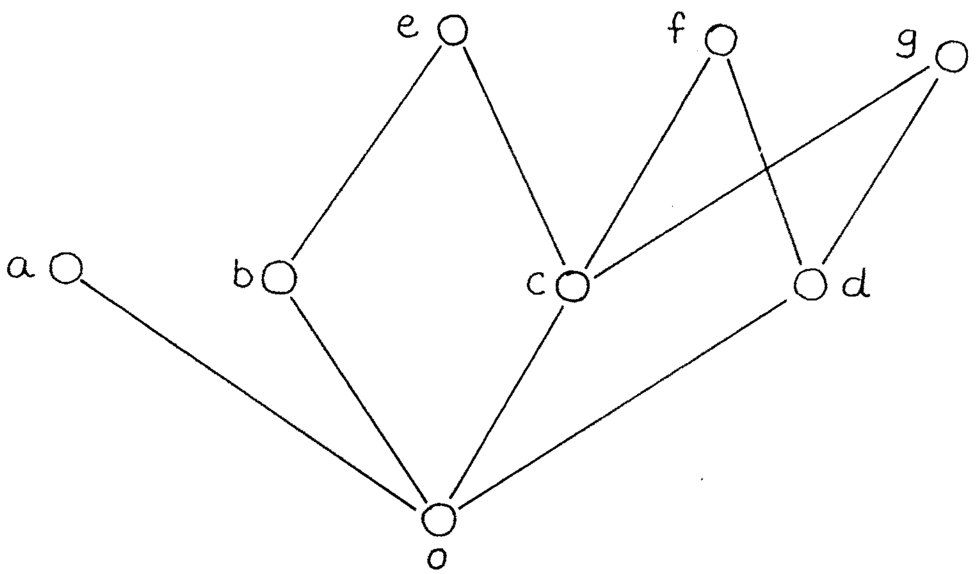


Fig- 4

The poset shown in fig.4 is 0-distributive as,

$$(b] \cap (a] = (o]$$

$$(c] \cap (a] = (o]$$

implies $(b \vee c] \cap (a] = (0]$ where $b \vee c = e$. However it is not pseudocomplemented as a^* does not exists in P.

<u>Remark 2.2.5</u> : 0-distributivity generalizes pseudocomplementedness. As pseudocomplemented poset has greatest element it is always bounded, while a 0-distributive poset need not be bounded above.

Venkatnarasimhan [11] proved that a poset P is pseudocomplemented if and only if $\{a\}^*$ is a principal ideal for every $a \in P$ (Result 1.2.3). Using this together with the Result (1.2.2) we have following

<u>Corollary 2.2.6</u> : 0-distributive poset is pseudocomplemented if and only if $\{a\}^*$ is a principal ideal for all $a \in P$.

<u>Remark 2.2.7</u> : If a poset satisfies ascending chain condition all ideals are principal [6] and hence in poset satisfying the ascending chain condition 0-distributivity is equivalent to pseudocomplementedness.

§ 2.3. O-distributivity and Disjunctivity

Balchandran [1] defined and characterized disjunction lattices. While generalizing the concept of disjunctivity to posets Venkatnarasimhan [12] defined disjunctive posets (Def. 1·1·33). Here we show that disjunctivity and O-distributivity in a poset P are completely independent. For this consider the following examples :

Example 2.3.1 : Example of a poset which is O-distributive but not disjunctive.

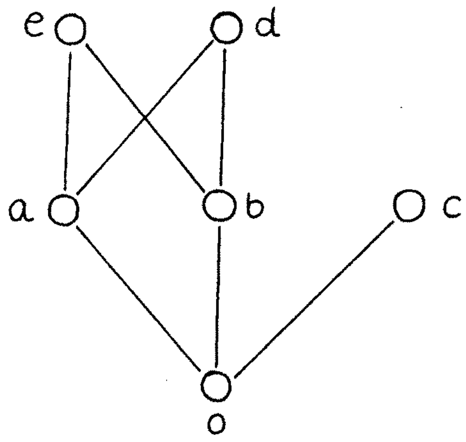


Fig-5

Clearly, poset shown in figure - 5 is O-distributive but it is not disjunctive, since for $c \nleq d$ in P there exists no element in P satisfying the disjunctive property.

<u>Example 2.3.2</u> : Example of a poset which is disjunctive but not 0-distributive.

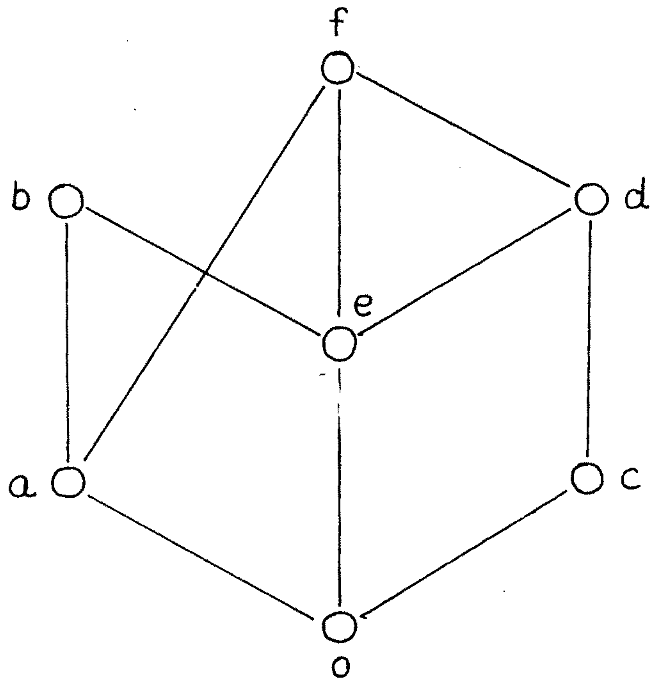


Fig-6

Clearly, the poset shown in Fig.6 is disjunctive But it is not 0-distributive since

$$(a] \cap (e] = (0]$$

$$(c] \cap (e] = (0]$$

and $a \vee c$ exists but

$$(a \vee c] \cap (e] \neq (0] .$$

<u>Example 2.3.3</u> : Example of a poset which is neither 0-distributive nor disjunctive .

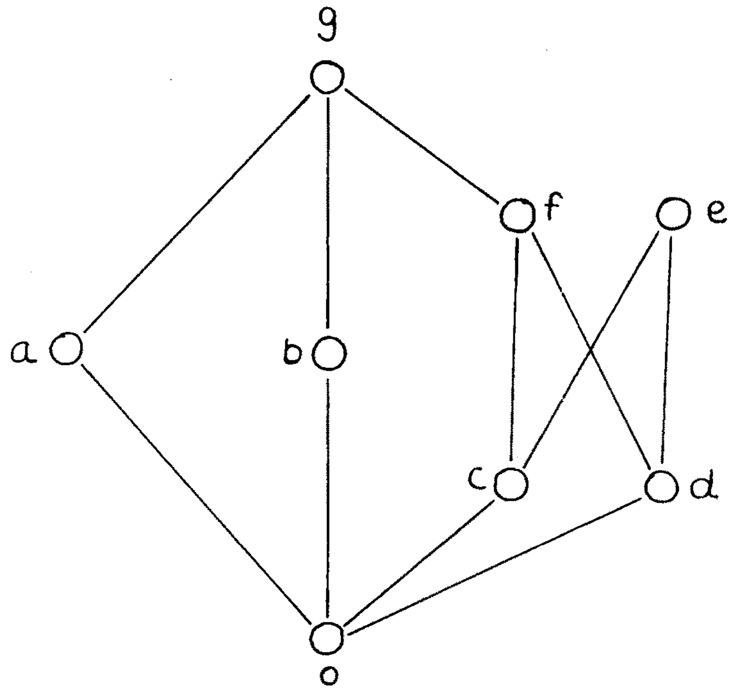


Fig-7

The poset shown in above figure is not 0-distributive since

$$(a] \cap (d] = (0]$$
$$(b] \cap (d] = (0]$$
$$(c] \cap (d] = (0]$$

but $(a \vee b \vee c] \cap (d] = (d] \neq (0]$ where $a \vee b \vee c = g$

Also the poset is not disjunctive since for $e \neq f$ in P there exists no element in P satisfying the disjunctive property.

§ 2.4 Characterization of I_μ

For any poset P we denote the set of all ideals of P by I_μ . . . Here in this section we deal with some important characterization of I_μ.

Venkatnarasimhan [11] proved that I_μ is complete lattice (Result 1.2.1). But under some additional condition on P, I_μ will be pseudocomplemented. This is proved in the following :

Result 2.4.1 : P is 0-distributive if and only if I_μ is pseudocomplemented.

Proof : Let P be 0-distributive poset. Let $A \in I_\mu$. As $A \subseteq P$ by Result (2.2.2) we have A^* is an ideal in P. We claim that A^* is the pseudocomplement of A in I_μ.

Clearly $A \cap A^* = \{0\}$. If there exist $B \in I_\mu$ such that $A \cap B = \{0\}$ then $B \subseteq A^*$, since if $b \in B$ then for any $a \in A$ we have $\{a, b\}^{\ell} = \{0\}$. For if $0 \neq z \in \{a,b\}^{\ell}$ then $z \leq b \in B$ implies $z \in B$ and $z \leq a \in A$ implies $z \in A$ thus $z \in A \cap B = \{0\}$ which is a contradiction. Hence $\{a, b\}^{\ell} = \{0\}$ for all $a \in A$. This inturn implies that $b \in A^*$ for all $b \in B$. Hence $B \subseteq A^*$. Thus A^* is the largest ideal in I_μ satisfying $A \cap A^* = \{0\}$. Hence A^* is the pseudocomplement of A in I_μ. This proves that I_μ is

Pseudocomplemented.

Conversely, let I_μ be pseudocomplemented. We have to prove that P is 0-distributive.

Let $x_1, x_2 \ldots\ldots x_n, a \in P$ such that

$$(x_1] \cap (a] = (0],$$

$$(x_2] \cap (a] = (0],$$

- - - - - - - - - - - -

- - - - - - - - - - - -

$$(x_n] \cap (a) = (0],$$

and suppose that $x_1 V x_2 V \ldots\ldots V x_n$ exists in P. Now $(x_1] \cap (a] = (0] \Longrightarrow (x_1] \subseteq (a]^*$ since $(a] \in I_\mu$ $(a]^*$ exists and is in I_μ, by assumption.

Similarly we have

$$(x_2] \subseteq (a]^*,$$

$$(x_3] \subseteq (a]^*,$$

- - - - - - - -

- - - - - - - -

$$(x_n] \subseteq (a]^*.$$

Therefore $(x_1] \underline{V} (x_2] \underline{V} \ldots\ldots \underline{V} (x_n] \subseteq (a]^*$.

But $(x_1] \underline{V} (x_2] \underline{V} \ldots\ldots \underline{V} (x_n] = (x_1 V x_2 V \ldots V x_n]$

(Result 1·2·2·). Hence $(x_1 V x_2 V \ldots\ldots V x_n \subseteq (a]^*$. But this implies that $(x_1 V x_2 V \ldots\ldots V x_n] \cap (a] = (0]$. Therefore P is O-distributive. ▮

As we know that every pseudocomplemented lattice is O-distributive (Result 1.2.4) we get one more generalization of O-distributivity as

<u>Result 2.4.2</u> : P is O-distributive if and only if I_μ is O-distributive.

<u>Proof</u> : Let I_μ be O-distributive. Let $x_1, x_2, \ldots x_n, a \in P$ such that

$$(x_1] \cap (a] = (0],$$

$$(x_2] \cap (a] = (0],$$

- - - - - - - - - - -

- - - - - - - - - - -

$$(x_n] \cap (a] = (0].$$

And suppose $x_1 V x_2 V \ldots\ldots V x_n$ exists in P.

Now we know that

$$(x_1] \underline{V} (x_2] \underline{V} \ldots\ldots \underline{V} (x_n] = (x_1 V x_2 V \ldots V x_n]$$

(result 1·2·2). Since I_μ is O-distributive lattice we have $(x_1] \underline{V} (x_2] \underline{V} \ldots\ldots \underline{V} (x_n] \cap (a] = (x_1 V x_2 V \ldots\ldots$ $\ldots V x_n] \cap (a] = (0]$, proving that P is O-distributive.

Conversely let P be 0-distributive. By the Result (2·4·1·), I_μ is pseudocomplemented and since every pseudocomplemented lattice is 0-distributive (Result 1.2.4) I_μ is 0-distributive. ∎

Remark 2.4.3 : From the above result Varlet's result (Result 1.2.8) follows as corollary when a poset P becomes lattice.

§ 2.5 O-distributivity and distributivity

Every distributive lattice with 'O' (semilattice with 'O') is O-distributive lattice (semilattice) [8] Hence to keep up such a linking for posets intitutionally we forced to define,

Def. 2.5.1 : Distributive poset : A poset P is called distributive if $(a] \cap (b] \subseteq (c]$ $(a,b,c \in P)$ implies the existance of $x, y \in P$, $x \geq a$, $y \geq b$ such that

$$(x] \cap (y] = (c] .$$

Remark 2.5.2 : It is clear that our definition coincides with the definition of Gratzer [6] in a meet semilattice when a poset becomes meet semilattice.

Example 2.5.3 : Example of a distributive poset

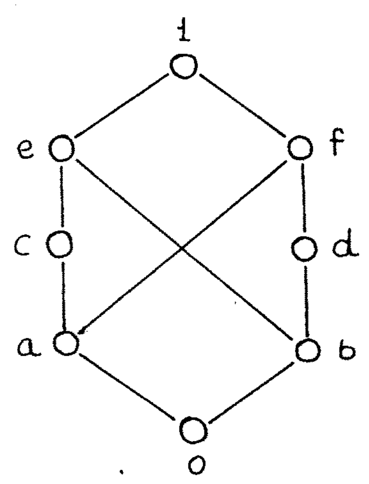


Fig 8

The poset shown in above figure is distributive.

With the definition of distributive poset (Def.2.5.1) we have the following

<u>Result 2.5.4</u> : Every distributive poset with 0 is 0-distributive.

<u>Proof</u> : Let P be distributive poset. Let $x_1, x_2, \ldots, x_n, a \in P$ such that

$$(x_1] \cap (a] = (0],$$
$$(x_2] \cap (a] = (0],$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(x_n] \cap (a] = (0]$$

and suppose $x_1 V x_2 V \ldots\ldots V x_n$ exists in P

Now $(x_1] \supseteq (x_2] \cap (a]$. Hence by distributivity there exist $y_2 \geq x_2$ and $y_1 \geq a$ such that

$$(x_1] = (y_1] \cap (y_2]$$

As $(y_1] \supseteq (y_1] \cap (y_2]$ we get $(y_1] \supseteq (x_1]$

that is $y_1 \geq x_1$. Further we have

$$(x_1] \supseteq (x_r] \cap (a] \;;\; 3 \leq r \leq n.$$

Hence by distributivity there exist $y_r \geq x_r$ and $z_r \geq a$

such that $(x_1] = (y_r] \cap (z_r]$; $3 \leq r \leq n$.

Thus as $y_1 \geq x_1$, $y_2 \geq x_2$, ..., $y_n \geq x_n$ we get

$$
\begin{aligned}
(y_1] &\supseteq (x_1] , \\
(y_2] &\supseteq (x_2] , \\
&\cdots\cdots\cdots \\
&\cdots\cdots\cdots \\
(y_n] &\supseteq (x_n] .
\end{aligned}
$$

Since the set of all ideals I_μ is a lattice (Result 1.2.1) we have

$$
\begin{aligned}
(y_1] \cap (y_2] \cap \cdots\cdots \cap (y_n] &\supseteq (x_1] \ \underline{\vee}\ (x_2] \ \underline{\vee} \cdots \underline{\vee}(x_n] \\
&= (x_1 \vee x_2 \vee \cdots \vee x_n] \text{ (Result 1.2.2)}
\end{aligned}
$$

Thus we get

$$
\begin{aligned}
(a] \cap \{ (y_1] \cap (y_2] \cap \cdots\cdots \cap (y_n] &\supseteq \\
&(a] \cap (x_1 \vee x_2 \vee \cdots\cdots \vee x_n]
\end{aligned}
$$

Now $y_1 \geq a \Longrightarrow (y_1] \supseteq (a] \Longrightarrow (y_1] \supseteq (a] \cap (y_2]$.

This implies

$$
\begin{aligned}
(a] \cap (y_2] &= \{(a] \cap (y_2]\} \cap (y_1] \\
&= (a] \cap \{(y_1] \cap (y_2]\} \\
&= (a] \cap (x_1] \\
&= (0]
\end{aligned}
$$

Thus $(a] \cap (y_i] = (0]$ for any i; $1 \leq i \leq n$

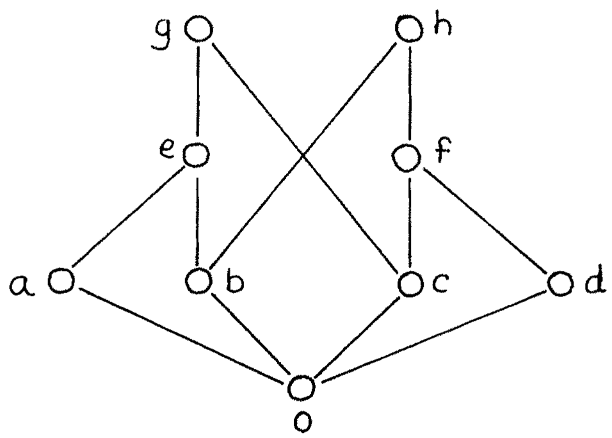
Hence $(a] \cap \{(y_1] \cap (y_2] \cap \ldots\ldots \cap (y_n]\} = (0]$

$\Rightarrow (a] \cap (x_1 \vee x_2 \vee \ldots\ldots \vee x_n] = (0]$

Hence p Is 0-distributive. ∎

<u>Remark 2.5.5</u> : Note that every 0-distributive poset need not be distributive for this consider the following

<u>Example 2.5.6</u> : Example of a poset which is 0-distributive but not distributive.



Fig·9

Clearly, the poset shown in figure-9 is 0-distributive but it is not distributive since

$(b] \cap (c] \subseteq (a]$

but there does not exist $x \geq b$ and $y \geq c$ such that

$(x] \cap (y] = (a]$.

§ 2.6 Annihilator ideals

In this section we deal with annihilator ideals (Def. 1.1.21) in 0-distributive poset P.

Let P be 0-distributive poset. First we find some properties of A^*, $A \subseteq P$. In that direction we have following,

Result 2.6.1 : Let P be 0-distributive poset then

i) For any nonempty subset A of P the disjoint complement A^* is an ideal satisfying $A^* = A^{***}$

ii) An ideal $I \in P$ satisfying $I = A^*$ for some nonempty subset A of P if and only if $I = I^{**}$

Proof : i) By Result (2.2.2) we get that A^* is an ideal. Therefore it only remains to show that $A^* = A^{***}$

Now if B is any nonempty subset of P then we have $B \subseteq B^{**}$. This implies $A^* \subseteq A^{***}$. Now $A \subseteq A^{**}$ implies $A^* \supseteq A^{***}$. Thus we get $A^* = A^{***}$.

ii) If $I = A^*$ for some nonempty subset A of P. Then $I^{**} = A^{***} = A^* = I$.

Conversely if $I = I^{**}$ then $I = A^*$ for $A = I^*$. Hence the proof. ▮

Let P be a O-distributive poset. We denote the set of all annihilator ideals of P by A(P). Here we have the following,

<u>Result 2.6.2</u> : For a O-distributive poset P, the set of all annihilator ideals A(P) forms a Boolean algebra.

<u>Proof</u>. Let I, J $\in$ A(P). We define the greatest lower bound of I, J by $I \cap J$.

$$\text{i.e.} \quad I \wedge J = I \cap J$$

and least upper bound of I, J by $(I^* \cap J^*)^*$

$$\text{i.e.} \quad I \underline{\vee} J = (I^* \cap J^*)^*$$

We claim that $\langle A(P), \wedge, \underline{\vee}, *, (0], P \rangle$ is a Boolean algebra.

We first prove that if I, J $\in$ A(P) then $I \wedge J \in A(P)$

Let I, J $\in$ A(P), $\Rightarrow$ $I = I^{**}$, $J = J^{**}$

Now $I \supseteq I \cap J \Rightarrow I^{**} \supseteq (I \cap J)^{**}$. Since $I \in A(P)$ this imply $I^{**} = I \supseteq (I \cap J)^{**}$. Similarly $J \supseteq (I \cap J)^{**}$ from this we have $I \cap J \supseteq (I \cap J)^{**}$. Thus $I \cap J = (I \cap J)^{**}$ Hence $I \cap J \in A(P)$.

Let K $\in$ A(P) such that $K \subseteq I$, $K \subseteq J$. Thus $K \subseteq I \cap J \Rightarrow I \cap J$ is the greatest lower bound of I and J in $A(P)$. Hence $I \wedge J \in A(P)$ for all $I, J \in A(P)$ ...(1)

Let $I, J, K \in A(P)$. Now $I^* \supseteq I^* \cap J^*$, this imply $I^{**} \subseteq (I^* \cap J^*)^*$. But since $I \in A(P)$ we have $I = I^{**}$. Hence $I \subseteq (I^* \cap J^*)^*$. Similarly $J \subseteq (I^* \cap J^*)^*$. Let $I \subseteq K$ and $J \subseteq K$, imply $I^* \supseteq K^*$ and $J^* \supseteq K^* \Longrightarrow (I^* \cap J^*) \supseteq K^* \Longrightarrow (I^* \cap J^*)^* \subseteq K^{**} = K$. Therefore $(I^* \cap J^*)^*$ is least upper bound of $I, J \in A(P)$. Thus if $I, J \in A(P)$ then

$I \underline{\vee} J \in A(P)$ - ... (2)

From (1) and (2) we get that $A(P)$ is a lattice.

Since $(0] = P^*$ and $P = (0]^*$, $(0]$ and P are the elements of $A(P)$. Further $(0]$ and P are the least and the greatest elements of $A(P)$.

Now we show that $A(P)$ is complemented. Let $I \in A(P)$; Then $I \underline{\vee} I^* = (I^* \cap I^{**})^* = (I^* \cap I)^* = (0]^* = P$ and $I \cap I^* = (0]$ shows that I has complement I^* in $A(P)$ Thus $A(P)$ is complemented. ... (3)

It only remains to show that $A(P)$ is distributive for $I, J, K \in A(P)$ we have to show that

$$I \underline{\vee} (J \wedge K) = (I \underline{\vee} J) \wedge (I \underline{\vee} K) \quad \text{that is}$$

$$[I^* \cap (J \cap K)^*]^* = [(I^* \cap J^*)^* \cap (I^* \cap K^*)^*]$$

We first claim that

$$[I^* \cap (J \cap K)^*]^* \subseteq [(I^* \cap J^*)^* \cap (I^* \cap K^*)^*]$$

Now $J \cap K \subseteq J \Longrightarrow (J \cap K)^* \supseteq J^*$

$$\Longrightarrow I^* \cap (J \cap K)^* \supseteq I^* \cap J^*$$

$$\Longrightarrow [I^* \cap (J \cap K)^*]^* \subseteq (I^* \cap J^*)^*$$

Similarly we can prove that

$$[I^* \cap (J \cap K)^*]^* \subseteq (I^* \cap K^*)^*$$

Thus we have

$$[I^* \cap (J \cap K)^*]^* \subseteq (I^* \cap J^*)^* \cap (I^* \cap K^*)^* \qquad \ldots (4)$$

Next we claim that

$$(I^* \cap J^*)^* \cap (I^* \cap K^*)^* \subseteq [I^* \cap (J \cap K)^*]^*$$

To prove this we need to prove the following set inclusion

$$(I^* \cap J^*)^* \cap K \subseteq [I^* \cap (J \cap K)^*]^*$$

Let $I, J, K \in A(P)$. Now $I \cap K \subseteq I \subseteq [I^* \cap (J \cap K)^*]^*$.

Similarly, $J \cap K \subseteq [I^* \cap (J \cap K)^*]^*$

Now $I \cap K \subseteq [I^* \cap (J \cap K)^*]^* \Longrightarrow I \cap K \cap [I^* \cap (J \cap K)^*]^{**} = (0]$

That is $I \cap K \cap [I^* \cap (J \cap K)^*] = (0]$

Similarly, $J \cap K \cap [I^* \cap (J \cap K)^*] = (0]$.

Thus $J \cap [K \cap I^* \cap (J \cap K)^*] = (o)$.

$\Longrightarrow [K \cap I^* \cap (J \cap K)^*] \subseteq J^*$, similarly

$[K \cap I^* \cap (J \cap K)^*] \subseteq I^*$

$\Longrightarrow [K \cap I^* \cap (J \cap K)^*] \subseteq I^* \cap J^*$

$\Longrightarrow [K \cap I^* \cap (J \cap K)^*] \cap [(I^* \cap J^*)^*] = (o)$

That is $I^* \cap (J \cap K)^* \cap [K \cap (I^* \cap J^*)^*] = (o)$

$\Longrightarrow [K \cap (I^* \cap J^*)^*] \subseteq [I^* \cap (J \cap K)^*]^*$

That is $(I^* \cap J^*) \cap K \subseteq [I^* \cap (J \cap K)^*]^*$

Thus $(I \underline{\vee} J) \wedge K \subseteq I \underline{\vee} (J \wedge K)$ ... (5)

Let $I = N$ and $J = Q$ and $K = N \underline{\vee} R$ then from (5) we have $(N \underline{\vee} Q) \wedge (N \underline{\vee} R) \leq N \underline{\vee} [Q \wedge (N \underline{\vee} R)$

$= N \underline{\vee} [(N \underline{\vee} R) \wedge Q]$

Now $(N \underline{\vee} R) \wedge Q \leq N \underline{\vee} (R \wedge Q)$

Therefore we have ,

$(N \underline{\vee} Q) \wedge (N \underline{\vee} R) \leq N \underline{\vee} (Q \wedge R)$

Replacing N, Q, R by I, J, K we have

$(I \underline{\vee} J) \wedge (I \underline{\vee} K) \leq I \underline{\vee} (J \wedge K)$

That is $(I^* \cap J^*)^* \cap (I^* \cap K^*)^* \subseteq [I^* \cap (J \cap K)^*]^*$ ...(6)

From (4) and (6) we have

$$(I^* \cap J^*)^* \cap (I^* \cap K^*)^* \subseteq [I^* \cap (J \cap K)^*]^* \quad \ldots (7)$$

This implies that A(P) is distributive.

From (3) and (7) we get that A(P) is distributive complemented lattice.

That is A(P) is a Boolean algebra. ▮

§ 2.7 Properties of I**

In this section we study some properties of I**. Here a nice property of I**, where I is an ideal in a 0-distributive poset P is furnished.

Result 2.7.1 : Let P be a 0-distributive poset then for any ideal $I \in P$ the set I** is the largest among all the ideals $A \in P$ with the property that for every $0 \neq x \in A$ there exist $0 \neq y \in I$ such that $y \leq x$.

Proof : We first prove that I** has the stated property Given $0 \neq x \in I^{**}$ assume that no nonzero $y \in P$ satisfies $y \in I$ and $y \leq x$. Then $\{x, z\}^{\ell} = \{0\}$ for every $z \in I$ for if $\{x, z\}^{\ell} \neq \{0\}$, let $0 \neq p \in \{x, z\}^{\ell}$ then $p \leq x$ and $p \leq z$. Now $z \in I$ and I is an ideal together imply that $p \in I$ and $p \leq x$ contradicting our assumption. Thus it follows from $\{x, z\}^{\ell} = \{0\}$, for every $z \in I$, that $x \in I^*$. But x is also the element of I**, hence we get $x \in I^* \cap I^{**} = \{0\}$; contradiction to the fact that x is nonzero. Thus there exists some nonzero $y \in I$ for every $x \in I^{**}$ such that $y \leq x$.

Now assume that A is an ideal with the stated property and suppose $A \subseteq I^{**}$ does not hold. Then there must exists an element $x \in A$ such that x does not belong I^{**}. So $\{x, y\}^{\ell} \neq \{0\}$ for some $y \in I^*$. Let $q \in \{x, y\}^{\ell} \neq \{0\}$. Then $q \in A$ and $q, \in I^*$ imply that $q \in I^* \cap A$. Now by hypothesis there exist $0 \neq s \in I$ such that $s \leq q$. But then $0 \neq s \in I \cap I^*$; which is impossible. Hence $A \subseteq I^{**}$. ∎

Further we have the following,

<u>Result 2.7.2</u> : Let P be 0-distributive poset. If $I_1, I_2, \ldots, I_n$ are ideals in P then

$$\left(\bigcap_{i=1}^{n} I_i\right)^{**} = \bigcap_{i=1}^{n} I_i^{**}$$

<u>Proof</u> : Obviously, as $I_i \subseteq I_i^{**}$ for every $i (1 \leq i \leq n)$ it is sufficient to establish that $I = I_1 \cap I_2$ satisfies $I_1^{**} \cap I_2^{**} = I^{**}$. Suppose $0 \neq x$ is in $I_1^{**} \cap I_2^{**}$ then there exists by previous result on element $0 \neq y \in I_1$ such that $y \leq x$. Since $0 \neq y$ is in $I_1 \cap I_2^{**}$ there exists on element $z \in I_2$ such that $z \leq y$. Hence $0 \neq z$ is in $I_1 \cap I_2 = I$ and $z \leq x$. Thus the ideal $I_1^{**} \cap I_2^{**}$ possesses the property in the Result (2.7.1.) . Hence $I_1^{**} \cap I_2^{**} \subseteq I^{**}$. which proves that $I_1^{**} \cap I_2^{**} = (I_1 \cap I_2)^{**}$. ∎

§ 2.8 Some more results.

A sufficient condition for $(a]^* = (b]^*$ in a 0-distributive poset for $a \neq b$ is stated in the following,

Result 2.8.1 : If a and P are the elements of 0-distributive poset such that $(a] \cap (d] = (b] \cap (d]$ for some dense element $d \in P$. Then $(a]^* = (b]^*$

Proof : $(a]^{**} = (a]^{**} \cap P$

$= (a]^{**} \cap (d]^{**}$ [$\because$ d is dense element in P]

$= [(a] \cap (d]]^{**}$ [by Result (2.7.2)]

$= [(b] \cap (d]]^{**}$ [by assumption]

$= (b]^{**} \cap (d]^{**}$ [$\because$ by Result (2.7.2)]

$= (b]^{**} \cap P$ [$\because$ d is dense in P]

$= (b]^{**}$

Hence we get

$$(a]^* = (a]^{***}$$
$$= \{(a]^{**}\}^*$$
$$= \{(b]^{**}\}^*$$
$$= (b]^{***}$$
$$= (b]^* .$$

A property of the set of dense elements in a 0-distributive poset is investigated in the following

Result 2.8.2 : In a 0-distributive poset P is $\{0\} \neq A$ is the intersection of all nonzero ideals of P then $A^* = P - D$ where D is set of all dense elements of P.

Proof : Now $A \neq \{0\}$, so we get for any $x \in A^*$. $\{x\}^* \neq \{0\}$. That x is nondense element of P. Therefore $x \in A^* \Longrightarrow x \in P-D$ that is $A^* \subseteq P-D$.

On the other hand as P is 0-distributive $\{d\}^*$ is nonzero ideal of P for every $d \notin D$. Now since A is the intersection of all nonzero ideals of P we have $A \subseteq \{d\}^*$. Therefore $A^* \supseteq \{d\}^{**}$. Now $d \in \{d\}^{**}$ and $\{d\}^{**} \subseteq A^*$. Thus $d \in A^*$. Hence $P - D \subseteq A^*$. Therefore we get $A^* = P - D$. ∎

Next we have

Result 2.8.3 : If intersection of all prime ideals is $(0]$ then P is 0-distributive.

Proof : Let P be a poset. Let the intersection of all prime ideals of P is $(0]$.

Let $\mathcal{P} = \{M \,/\, M \subseteq P \text{ and } M \text{ is prime}\}$.

Now we have to prove that P is 0-distributive

Let $x_1, x_2 \ldots\ldots x_n, a \in P$ such that

$(x_1] \cap (a] = (o]$,

$(x_2] \cap (a] = (o]$,

- - - - - - - - -

- - - - - - - - -

$(x_n] \cap (o] = (o]$,

and suppose $x_1 V x_2 V \ldots V x_n$ exists in P

Now $(x_i] \cap (a] = (o]$, $1 \leq i \leq n$

imply either $x_i \in M$ or $a \in M$ since M is prime.

we claim that

$$(x_1 V x_2 V \ldots V x_n] \cap (a] \subseteq M, \forall M \in \wp .$$

Suppose the contradictory; that is $\exists$ $M \in \wp$ such that

$(x_1 V x_2 V \ldots V x_n] \cap (a] \nsubseteq M$

Then $x_1 V x_2 V \ldots V x_n \notin M$ and $a \notin M$ because M is prime.

Now if $(x_i] \cap (a] = (o]$ $1 \leq i \leq n$ then $x_i \in M$ for each i; and this would imply $x_1 V x_2 V \ldots V x_n \in M$, which is a contradiction. Therefore $(x_1 V x_2 V \ldots$ $\ldots V x_n] \cap (a] \subseteq M$ $\forall M \in \wp$ But $\bigcap M (M \in \wp) = (o] \Rightarrow$

$$(x_1 V x_2 V \ldots V x_n] \cap (a] = (o] .$$

Hence P is O-distributive. ▮

oOo