CHAPTER - III

Strongly 0-distributive Lattices

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Introduction

Varlet [9] defined a 0-distributive lattice as a lattice L with 0 in which $\{a\}^* = \{x \in L \,/\, x \wedge a = 0\}$ is an ideal for every $a \in L$. NieminenJuhani [7] defined 2-ideal in a finite lattice L as a generalization of the concept of ideal in a lattice. A nonempty subset I of finite lattice L is called as a 2-ideal if i) $x \leq i \in I \Rightarrow x \in I$, ii) If $i, j \in I$, $i \neq j$ and x covers i, j then $x \in I$. Hence it is natural to pose following :

Problem : Find a necessary and sufficient condition for a finite lattice in which $\{a\}^*$ is a 2-ideal.

In an attempt to answer this question we are led to define a new concept called "strongly 0-distributive lattice".

It is shown that the class of all strongly 0-distributive lattices contains distributive lattices with 0, pseudocomplemented lattices, 0-distributive lattices. Nieminen Juhani [7] characterized lattices for which the lattice of 2-ideals is distributive. As distributivity and pseudocomplementedness are completely independent here we characterize those lattices for which the lattice of 2-ideals in pseudocomplemented. Mainly we have proved that the lattice of all 2-ideals of L(L - finite lattice with 0) is pseudocomplemented if and only if L is strongly 0-distributive.

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Strongly 0-distributive lattices

In the second chapter our setting was a poset now we shall build up our theory for lattices.

Throughout this chapter $L = \langle L; \wedge, \vee \rangle$ will denote a finite lattice with '0'.

At the outset we give detailed proof of the known result.

Result 3.1 : Every ideal in any finite lattice is 2-ideal. (see dedinition 1.1.17).

Proof : Let L_1 be a finite lattice and $I \subseteq L_1$ is an ideal.

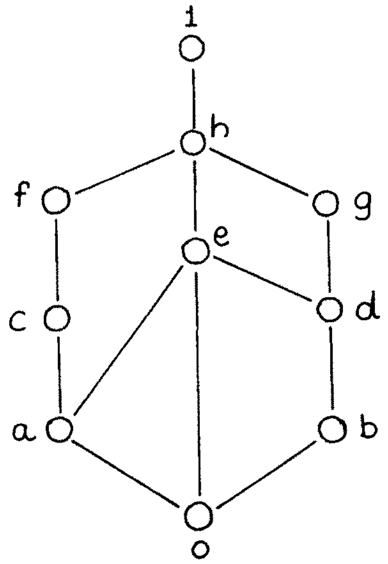
i) Let $x \leq y \in I$. Then by the definition of an ideal $x \in I$.

ii) Let $x, y \in I$ such that $x \neq y$ and $t \in L_1$ such that $t \succ x, y$. Then as t covers x and y, $t = x \vee y$. As I is an ideal, $x, y \in I \Rightarrow x \vee y \in I$. This inturn implies that $t \in I$.

From i) and ii) we get I is a 2-ideal. ∎

<u>Remark 3.2</u> : Every 2-ideal need not be an ideal.

For the consider the following example



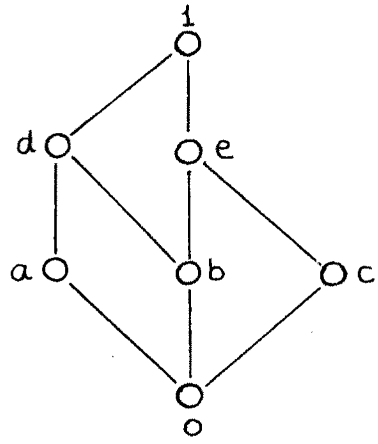
Fig·10

In fig.10 $L = \langle \{0,a,b,c,d,e,f,g,h,1\} ; \wedge , \vee \rangle$ is a lattice. Consider $I = \{0, a,b,c,d,e\}$. Clearly I is 2-ideal, but it is not an ideal since $c \vee d \notin I$ though $c \in I$ and $d \in I$.

We know that L is 0-distributive if and only if $\{a\}^* = \{x \in L \,/\, x \wedge a = 0\}$ is an ideal for all $a \in L$ [9] . As 2-ideal is a generalization of an ideal, in an attempt to characterize those lattices for which $\{a\}^*$ is 2-ideal for every $a \in L$ we define

<u>Def.3.3</u> : <u>Strongly 0-distributive lattice</u> : L is said to be strongly 0-distributive if $a \wedge b = 0$, $a \wedge c = 0$ ($a, b, c \in L$) and if $t \succ b, c \Rightarrow a \wedge t = 0$

Example 3.4 : Example of a lattice which is strongly 0-distributive.



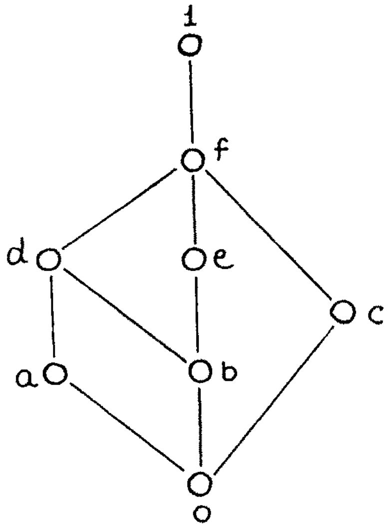
Fig·11

The lattice shown in figure 11 is strongly 0-distributive.

Remark 3.5 : Not every finite lattice with 0 is strongly 0-distributive.

For this consider the following :

Example 3.6 : Example of a lattice which is not strongly 0-distributive.



Fig·12

In the lattice shown in figure 12 $c \wedge a = 0$, $e \wedge a = 0$ and $f \succ c, e$ but $f \wedge a = a \neq 0$.

Next we have following characterization of strongly 0-distributive lattice.

<u>Result 3.7</u> : L is strongly 0-distributive if and only if $\{a\}^* = \{x \in L \,/\, x \wedge a = 0\}$ is a 2-ideal for any $a \in L$

<u>Proof</u> : Let L be strongly 0--distributive

i) Let $y \in \{a\}^*$ and $x \in L$ such that $x \leq y$

Now $x \leq y \Rightarrow x \wedge a \leq y \wedge a = 0 \Rightarrow x \wedge a = 0 \Rightarrow x \in \{a\}^*$

ii) Let $x, y \in \{a\}^*$ such that $x \neq y$ and suppose $t \in L$ such that $t \succ x, y$ as $x, y \in \{a\}^*$, $x \wedge a = 0$ and $y \wedge a = 0$ since L is strongly 0-distributive we must have $t \wedge a = 0 \Rightarrow t \in \{a\}^*$.

From i) and ii) it follows that $\{a\}^*$ is a 2-ideal.

Conversely let $\{a\}^*$ be a 2-ideal. To prove that L is strongly 0-distributive, let $x \wedge a = 0$, $y \wedge a = 0$ ($x, y, a \in L$) and suppose $\exists$ $t \in L$ such that $t \succ x, y$. As $\{a\}^*$ is 2-ideal we get that $t \in \{a\}^*$. But this implies that $t \wedge a = 0$, proving that L is strongly 0-distributive. ▮

As every ideal is a 2-ideal we get

Result 3.8 : Every finite 0-distributive lattice is strongly 0-distributive.

Proof : L is 0-distributive

$\Rightarrow$ $\{a\}^*$ is an ideal for every $a \in L$

$\Rightarrow$ $\{a\}^*$ is 2-ideal for every $a \in L$

$\Rightarrow$ L is strongly 0-distributive. ∎

Remark 3.9 : Converse of the Result 3.8 need not be true. For this we provide following :

Example 3.10 : Example of strongly 0-distributive lattice which is not 0-distributive.

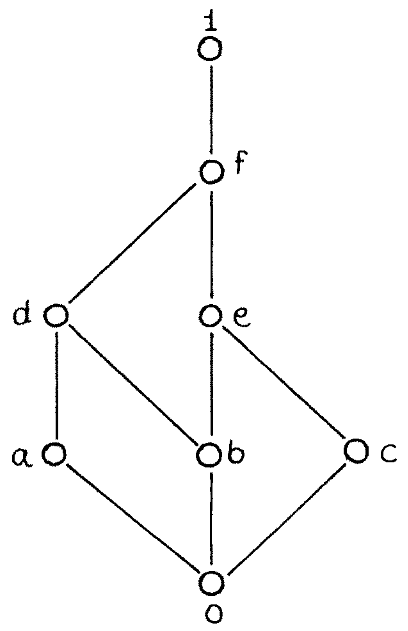


Fig 13

The lattice shown in figure 13 is strongly 0-distributive. Note that it is not 0-distributive since $a \wedge b = 0$, $c \wedge b = 0$

do not imply $b \wedge (a \vee c) = 0$.

Further we have following

<u>Result 3.11</u> : If L is distributive then L is strongly 0-distributive.

<u>Remark 3.12</u> : The converse of the Result 3.11 need not be true. For this consider the following

<u>Example 3.13</u> : Example of a lattice which is strongly 0-distributive which is not distributive.

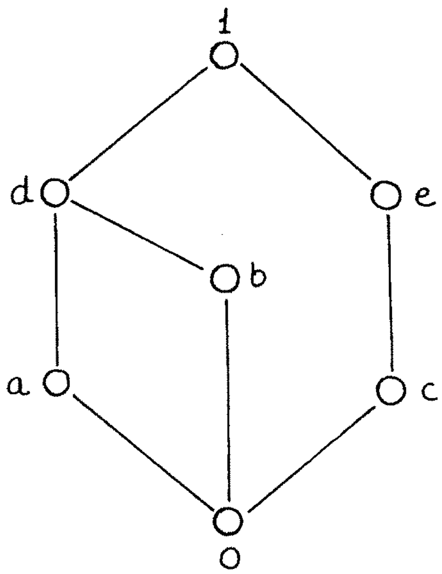


Fig. 14

The lattice shown in the figure 14 is strongly 0-distributive but it is not distributive since $a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$.

As every pseudocomplemented lattice is 0-distributive we have by Result 3.8 the following,

<u>Result 3.14</u> : Every pseudocomplemented lattice is strongly 0-distributive.

Note that every strongly 0-distributive lattice need not be pseudocomplemented.

For any subset $A \subseteq L$ define

$$A^* = \{x \in L \,/\, x \wedge a = 0, \forall\, a \in A\}$$

Then obviously,

$$A^* = \bigcap \{\{a\}^* \,/\, a \in A\} .$$

One more characterization of a strongly 0-distributive lattice is obtained in the following,

<u>Result 3.15</u> : L is strongly 0-distributive if and only if A^* is a 2-ideal for any $A \subseteq L$.

<u>Proof</u> : If L is strongly 0-distributive then $\{a\}^*$ is an ideal for each $a \in L$ (Result 3.7) . As arbitrary intersection of 2-ideals is a 2-ideal $A^* = \bigcap \{\{a\}^*/a \in A\}$ is a 2-ideal.

Conversely, if A^* is an ideal for any $A \subseteq L$ then in particular, $\{a\}^*$ is an ideal, $a \in L$. But this implies that L is strongly O-distributive (Result 3.7). ∎

We denote the lattice of 2-ideals of L by $I_2(L)$.

Distributivity and modularity of $I_2(L)$ has been characterized by Nieminen & Juhani [7] . In the following result we characterize pseudocomplementedness in $I_2(L)$.

Result 3.16 : $I_2(L)$ is pseudocomplemented if and only if L is strongly O-distributive.

Proof : Let L be strongly O-distributive. For any $A \in I_2(L)$, $A^* \in I_2(L)$ (Result 3.15) : Further $A \cap A^* = \{0\}$. If $A \cap B = \{0\}$ for any $B \in I_2(L)$ we have $B \subseteq A^*$. This proves that A^* is the pseudocomplement of $A \in I_2(L)$. Hence $I_2(L)$ is pseudocomplemented.

Conversely, if $I_2(L)$ is pseudocomplemented $(a] \in I_2(L) \Rightarrow (a]^* \in I_2(L)$. As $(a]^* = \{a\}^*$ by Result 3.7 we get L is strongly O-distributive. ∎

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