CHAPTER II.

FUZZY IDEALS

2.1 FUZZY IDEALS :

Definition (2.1.1.) :

Let R be a ring. A function $S:R \longrightarrow I$ is called a fuzzy subring of R if,

(1) $S(x + y) \geqslant S(x) \wedge S(y)$

(2) $S(-x) \geqslant S(x)$

(3) $S(x.y) \geqslant S(x) \wedge S(y)$.

where I denotes the closed interval $[0,1]$.

Remark (2.1.2) :

(1) If $S:R \longrightarrow I$ is a characteristic function of a subring S of R, then S is a fuzzy subring of R.

(2) For $t \in I$, the set,

$S_t = \{x \in R / S(x) \geqslant t\}$ is a subring of R.

Definition (2.1.3) :

Let R be a ring. A function $J:R \longrightarrow I$ is called left fuzzy ideal if.

(1) $J(x + y) \geqslant J(x) \wedge J(y)$

(2) $J(-x) \geqslant J(x)$

(3) $J(x.y) \geqslant J(y)$

J is called a right fuzzy ideal if J satisfies (3)' instead of (3)

(3)' $J(x.y) \geqslant J(x)$.

Remark : (2.1.3)

(1) A characteristic function of a left (right) ideal of a ring R is a fuzzy left (right) ideal of R.

(2) If J is a fuzzy left (right) ideal of R then for $t \in I$ the set,

$$J_t = \{ x \in R \,/\, J(x) \geq t \}$$

is a left (right) ideal of R.

If left fuzzy ideal is same as right fuzzy ideal then in that case it is called as Fuzzy ideal. If R is a commutative ring then its left fuzzy ideal is same as right fuzzy ideal.

Here after unless stated otherwise R denotes commutative ring with unity and I denotes the closed interval [0, 1].

Definition (2.1.4) :

Let $(R, +, \cdot)$ be a ring. A fuzzy ideal J of R is a function $J : R \rightarrow I$, which satisfied

(J1) $J(x+y) \geq J(x) \wedge J(y)$

(J2) $J(-x) \geq J(x)$

(J3) $J(x.y) \geq J(x) \vee J(y)$.

Remark (2.1.5) :

If J is an ideal of a ring R, then characteristic function of J, $\chi_J : R \rightarrow I$ is a fuzzy ideal of R.

Moreover converse is also true.

<u>Proposition</u> (2.1.6) :

If T is a subset of R such that characteristic function of T satisfies (J1), (J2), (J3), then T is an ideal of R.

<u>Proof</u> :

$$x, y \in T \Rightarrow T(x) = T(y) = 1.$$

But $T(x+y) \geqslant T(x) \wedge T(y) = 1 \Rightarrow x + y \in T$.

Also $T(x) = T(-x) = 1$. Next, if $x \in T$ and $y \in R$ then

$$T(x.y) \geqslant T(x) \vee T(y) = 1 \Rightarrow T(x.y) = 1.$$

Hence T is an ideal of R.

<u>Remark</u> (2.1.7) :

If J is a fuzzy ideal then,

(1) $J(x) \neq J(1) \Rightarrow J(x-1) = J(1-x) = J(1+x) = J(1)$.

and $J(1+x) = J(1-x) \neq J(1) \Rightarrow J(x) = J(1)$.

(2) $\left\{ J(x^n) \right\}_{n=0}^{\infty}$ is a monotonic increasing sequence which is bounded above, by $J(0)$. Hence convergent.

(3) If $J(x) \neq J(1)$ then $J(1) = J(1+nx)$ for all n.

<u>Proposition</u> (2.1.8) :

Let $J : R \rightarrow I$ be a function. Then J is a fuzzy ideal of R if an only if $J(x-y) \geqslant J(x) \wedge J(y)$ and $J(x.y) \geqslant J(x) \vee J(y)$.

<u>Proof</u> :

Suppose J is a fuzzy ideal of R. Then

$J(x-y) = J(x+(-y)) \geqslant J(x) \wedge J(-y) = J(x) \wedge J(y)$ and

$J(x.y) \geqslant J(x) \vee J(y)$.

Conversely, suppose that conditions holds. Then,

$J(0) = J(x-x) \geqslant J(x) \wedge J(x) = J(x)$ for all $x \in R$

Therefore $J(-x) = J(0-x) \geqslant J(0) \wedge J(x) = J(x)$ and

$$J(x+y) = J(x-(-y)) \geqslant J(x) \wedge J(-y) \geqslant J(x) \wedge J(y)$$

Also, $J(x.y) \geqslant J(x) \vee J(y)$.

Hence J is a fuzzy ideal of R.

<u>Proposition</u> (2.1.9) :

Let R be a ring and J be a fuzzy ideal of R. Then for any $x \in R$, $J(x) = J(-x)$ and $J(0) \geqslant J(x) \geqslant J(1)$.

<u>Proof</u> :

$$J(x) = J(-(-x)) \geqslant J(-x) \geqslant J(x).$$

Hence $J(x) = J(-x)$

Next, $J(0) = J(x+(-x)) \geqslant J(x) \wedge J(-x) = J(x) \wedge J(x)$ $= J(x)$. Also, $J(x) = J(x \cdot 1) \geqslant J(x) \vee J(1) \geqslant J(1)$

Hence $J(0) \geqslant J(x) \geqslant J(1)$.

<u>Proposition</u> (2.1.10) :

If J is a fuzzy ideal of a ring R, then $J(x-y) = 0$ $\Rightarrow J(x) = J(y)$ for all $x, y \in R$.

<u>Proof</u> :

Since J is a fuzzy ideal of R,

$J(x) = J(x-y+y) \geqslant J(x-y) \wedge J(y) = J(0) \wedge J(y) = J(y)$ and

$J(y) = J(y-x+x) \geqslant J(y-x) \wedge J(x) = J(0) \wedge J(x) = J(x)$

Hence $J(x) = J(y)$.

Proposition (2.1.11)

The intersection of any set of fuzzy ideals of R is a fuzzy ideal of R.

Proof :

Let $\{Ji\}_{i\in I}$ be a family of fuzzy ideals of R and let $x\in R$, $y\in R$ be arbitrary, Then,

$$\bigcap_i J_i\ (x-y) = \bigwedge_i (J_i(x-y)) \geq \bigwedge_i (J_i(x) \wedge J_i(y))$$

$$= (\bigwedge_i J_i(x)) \wedge (\bigwedge_i J_i(y))$$

$$= (\bigcap_i J_i)(x) \wedge (\bigcap_i J_i)(y)$$

and

$$(\bigcap_i J_i)(x.y) = \bigwedge_i (J_i(x.y))$$

$$\geq \bigwedge_i (J_i(x) \vee J_i(y))$$

$$= (\bigwedge_i J_i(x)) \vee (\bigwedge_i J_i(y))$$

$$= (\bigcap_i J_i)(x) \vee (\bigcap_i J_i)(y)$$

Hence by (2.2.5) $\bigcap_i J_i$ is a fuzzy ideal of R.

Proposition (2.1.12)

Let J be a fuzzy ideal of R. Then nonempty level subsets J_t , $t\in I$ is an ideal of R.

Proof :

Let $x \in J_t$ and $y \in J_t$. Then $J(x-y) \geq J(x) \wedge J(y)$ $\geq t \wedge t = t$. Hence $x-y$ J_t.

Next, let $r \in R$ be arbitrary and $x \in J_t$. Consider,

$J(rx) \geq J(r) \vee J(x) \geq J(r) \vee t \geq t \Rightarrow rx \in J_t$

Therefore J_t is an ideal of R for all $t \in I$.

Corollary (2.1.13)

Let J be a fuzzy ideal of R. Then,

$\{x \in R / J(x) = J(o)\}$ is an ideal of R, where 0 is an additive identity of R.

Proof :

Since $J(0) \geq J(x)$ for all $x \in R$,

$\{x \in R / J(x) = J(0)\} = \{x \in R / J(x) \geq J(0)\} = J_{J(0)}$

Hence by proposition (2.1.12) it follows that

$\{x \in R / J(x) = J(0)\}$ is an ideal of R.

Definition (2.1.14)

Let R be a ring and J be a fuzzy ideal of R. Then the ideals J_t, $t \in I$ are called level ideals of R.

Remarks (2.1.15)

If J is a fuzzy ideal of a ring R and t_1, $t_2 \in I$ with $t_1 < t_2$ then $J_{t_2} \subseteq J_{t_1}$

For $x \in J_{t_2} \Rightarrow J(x) \geq t_2 > t_1 \Rightarrow x \in J_{t_1}$

Proposition (2.1.16)

Let R be a ring and J be a fuzzy set in R such that J_t is an ideal for all $t \in I$. Then J is a fuzzy ideal of R.

Proof

Let $x, y \in R$ be arbitrary and let $J(x) = t_1$ and $J(y) = t_2$. Since I is totally orderd, we assume that $t_1 < t_2$. Hence $y \in J_{t_2} \Rightarrow y \in J_{t_1}$. Also $x \in J_{t_1}$. Therefore, $(x - y) \in J_{t_1}$ i.e. $J(x-y) \geq t_1$, Hence

$$J(x - y) \geq t_1 \geq t_1 \wedge t_2 = J(x) \wedge J(y)$$

Next, for any $x.y \in R$, let $J(x) = t$ i.e. $x \in J_t$.

Since J_t is an ideal of R $x.y \in J_t \Rightarrow J(x.y) \geq t = J(x)$

Similarly we can show that $J(x.y) \geq J(y)$

Hence $J(x.y) \geq J(x) \vee J(y)$. This proves that J is a fuzzy ideal of R.

Proposition (2.1.17)

Let J be a fuzzy ideal of a ring R. Two level ideals J_{t_1}, J_{t_2} with $t_1 < t_2$, are equal if and only if there is no $x \in R$ such that $t_1 \leq J(x) < t_2$

Proof

Let $J_{t_1} = J_{t_2}$. If there is $x \in R$ such that $t_1 \leq J(x) < t_2$ then $x \in J_{t_1}$ but $x \notin J_{t_2}$ which is a contradiction. Conversely,

suppose that there is no x such that $t_1 \leq J(x) < t_2$.
Since $t_1 < t_2$, $J_{t_2} \subseteq J_{t_1}$ And if, $x \in J_{t_1}$ then $J(x) \geqslant t_1$. Hence by condition it follows that $J(x) \geqslant t_2$ i.e. $x \in J_{t_2}$. Therefore $J_{t_1} = J_{t_2}$.

Remark (2.1.18)

(1) If J is a fuzzy ideal then, $J(n.x) \geq J(x)$ and $J(x^n) \geq J(x)$ for all positive integers n.

(2) If J is a fuzzy ideal then $\{J_t\}_{t \in \mathrm{Im} J}$ is a set of ideals of R. Moreover,

$J_{J(0)} = \{ x \in R / J(x) = J(0) \}$ is the smallest element of this set and $J_{J(1)} = \{ x \in R / J(x) > J(1) \}$ is the greatest element of this set. ($J_{J(1)} = R$ when we consider $J(x) \geq J(1)$)

(3) If J is a fuzzy ideal of R, then J is constant on each coset of $J_{J(0)}$ in R. For, if $x \in R$ is an arbitrary element of R then $x + J_{J(0)}$ is the coset of $J_{J(0)}$. Let $u \in x + J_{J(0)}$ then $u = x + y$, where $y \in J_{J(0)}$. This implies

$J(u - x) = J(y) = J(0)$ Hence by proposition (2.2.7), $J(x) = J(u)$

Proposition (2.1.19)

A homomorphic image or preimage of a fuzzy ideal is a fuzzy ideal.

Proof

Let f be a homomorphism of ring R onto R'. Let J be a fuzzy ideal of R. Define,

$$K : R' \longrightarrow I$$

$$K(y) = \bigvee_{x \in f^{-1}(y)} J(x)$$

We prove that K is a fuzzy ideal of R'

Let $u \in R'$ and $v \in R'$. Since f is onto there exists $x \in R$, $y \in R$ such that $f(x) = u$ $f(y) = V$.

Hence $f(x-y) = u-v$

Thus,

$$K(u-v) = \bigvee_{z \in f^{-1}(u-v)} J(z) = \bigvee_{x-y \in f^{-1}(u-v)} J(x-y)$$

$$\geq \bigvee_{\substack{x \in f^{-1}(u) \\ y \in f^{-1}(v)}} \left(J(x) \wedge J(y)\right)$$

$$= \left(\bigvee_{x \in f^{-1}(u)} J(x)\right) \wedge \left(\bigvee_{y \in f^{-1}(v)} J(y)\right)$$

$$= K(u) \wedge K(v)$$

and

$$K(u.v) = \bigvee_{z \in f^{-1}(u.v)} J(z) \geq \bigvee_{x.y \in f^{-1}(u.v)} J(x.y)$$

$$\geq \bigvee_{\substack{x \in f^{-1}(u) \\ y \in f^{-1}(v)}} \left(J(x) \vee J(y)\right)$$

$$= \left(\bigvee_{x \in f^{-1}(u)} J(x) \right) \vee \left(\bigvee_{y \in f^{-1}(v)} J(y) \right)$$

$$= K(u) \vee K(v)$$

Therefore K is a fuzzy ideal of R'

On theother hand, if $f : R \longrightarrow R'$ is a homomorphism of a ring R into R' and if K is a fuzzy ideal of R' then, define,

$$J : R \longrightarrow I$$

$$J(x) = K \circ f(x)$$

J is a fuzzy ideal of R.

For

$$J(x + y) = K \circ f(x + y)$$

$$= K(f(x) + f(y))$$

$$\geq K(f(x)) \wedge K(f(y))$$

$$= K \circ f(x) \wedge K \circ f(y)$$

$$= J(x) \wedge J(y)$$

And , $J(x.y) = K \circ f(x.y) = K(f(x).f(y))$

$\geq K \circ f(x) \vee K \circ f(y) = J(x) \vee J(y)$

Remark (2.1.20)

If $f:R \longrightarrow R'$ is not an epimorphism then image of fuzzy ideal of R need not be a fuzzy ideal of R.

<u>Proposition</u> (2.1.21)

If $f : R \longrightarrow R'$ is an epimorphism of a ring R onto R' and if J_1 and J_2 are fuzzy ideals of R, then,

(1) $f(J_1 \cap J_2) \subseteq f(J_1) \cap f(J_2)$

(2) $f(J_1 \cup J_2) = f(J_1) \cup f(J_2)$

<u>Proof</u>

(1) $f(J_1)$ and $f(J_2)$ are fuzzy ideals of R'. Clearly $f(J_1) \cap f(J_2)$ is a fuzzy ideal of R'.

Let $x' \in R'$ be arbitrary. Then,

$$\begin{aligned} f(J_1) \cap f(J_2) \ (x') &= f(J_1)\ (x') \wedge f(J_2)\ (x') \\ &= \Big(\bigvee_{x \in f^{-1}(x')} J_1(x)\Big) \wedge \Big(\bigvee_{x \in f^{-1}(x')} J_2(x)\Big) \\ &\geqslant \bigvee_{x \in f^{-1}(x')} \big(J_1(x) \wedge J_2(x)\big) \\ &= \bigvee_{x \in f^{-1}(x')} (J_1 \cap J_2)(x) \\ &= f(J_1 \cap J_2)(x). \end{aligned}$$

Thus $f(J_1 \cap J_2) \subseteq f(J_1) \cap f(J_2)$

(2) Let $x' \in R'$ be arbitrary Then

$$\begin{aligned} \big(f(J_1) \cup f(J_2)\big)\ (x') &= f(J_1)(x') \vee f(J_2)\ (x') \\ &= \bigvee_{x \in f^{-1}(x')} \big(J_1(x) \vee J_2(x)\big) \\ &= f(J_1 \cup J_2)\ (x') \end{aligned}$$

Thus $f(J_1 \cup J_2) = f(J_1) \cup f(J_2)$.

Remark (2.1.22) :

The above proposition, is also true for more general case. If $\{J_i\}_{i\in I}$ is a family of Fuzzy sets in X and if $f:X\to Y$ is a function, then

$$f\left(\bigcap_{i\in I} J_i\right) \subseteq \bigcap_{i\in I} f(J_i)$$

and

$$f\left(\bigcup_{i\in I} J_i\right) = \bigcup_{i\in I} f(J_i)$$

Proposition (2.1.23)

Let $f:R \to R'$ be a homomorphism of rings R into R'. If J_1' and J_2' are fuzzy ideals of R' then,

(1) $f^{-1}(J_1' \cap J_2') = f^{-1}(J'_1) \cap f^{-1}(J'_2)$.

(2) $f^{-1}(J'_1 \cup J'_2) = f^{-1}(J'_1) \cup f^{-1}(J'_2)$.

Proof :

(1) Let $x \in R$ be arbitrary. Then,

$$\begin{aligned} f^{-1}(J'_1 \cap J'_2) &= (J'_1 \cap J'_2)\ (f(x)) \\ &= J'_1(f(x)) \wedge J'_2(f(x)). \\ &= f^{-1}(J'_1)(x) \wedge f^{-1}(J'_2)(x). \\ &= \left(f^{-1}(J'_1) \cap f^{-1}(J'_2)\right)(x). \end{aligned}$$

Thus, $f^{-1}(J'_1 \cap J'_2) = f^{-1}(J'_1) \cap f^{-1}(J'_2)$

(2) Let $x \in R$ be arbitrary. Then

$$\begin{aligned} f^{-1}(J'_1 \cup J'_2)(x) &= (J'_1 \cup J'_2)\ (f(x)). \\ &= J_1'(f(x)) \vee J_2'(f(x)) \\ &= f^{-1}(J_1')(x) \vee f^{-1}(J_2')(x) \\ &= \left(f^{-1}(J_1') \cup f^{-1}(J_2')\right)(x) \end{aligned}$$

Thus, $f^{-1}(J_1' \cup J_2') = f^{-1}(J_1') \cup f^{-1}(J_2')$.

Remark (2.1.24)

The above proposition is also true for more general case. If $\{J_i\}$ $i \in I$ is a family of fuzzy sets in X and if $f: X \longrightarrow Y$ is a function, then

$$f^{-1}\left(\bigcup_{i \in I} J_i\right) = \bigcup_{i \in J} f^{-1}(J_i) \text{ and } f^{-1}\left(\bigcap_{i \in I} J_i\right) = \bigcap_{i \in I} f^{-1}(J_i)$$

Proposition (2.1.25)

If $f: R \longrightarrow R'$ is an epimorphism of a ring R onto R' and if J_1 and J_2 are fuzzy ideals of R then,

$$J_1 \subseteq J_2 \Rightarrow f(J_1) \subseteq f(J_2)$$

Proof

Let $x' \in R'$ be arbitrary. Then,

$$f(J_1)(x') = \bigvee_{x \in f^{-1}(x')} J_1(x) \leqslant \bigvee_{x \in f^{-1}(x')} J_2(x) = f(J_2)(x')$$

Thus $f(J_1) \subseteq f(J_2)$

Proposition (2.1.26)

If $f: R \longrightarrow R'$ is a Homorphism of a ring R into R' and if J_1' and J_2' are fuzzy ideal of R', then

$$J_1' \subseteq J_2' \Rightarrow f^{-1}(J_1') \subseteq f^{-1}(J_2')$$

Proof :

Let $x \in R$ be arbitary. Then,

$$f^{-1}(J_1')(x) = J_1'(f(x)) \leqslant J_2'(f(x)) = f^{-1}(J_2')(x)$$

Hence $f^{-1}(J_1') \subseteq f^{-1}(J_2')$

Lemma (2.1.27)

Let $f: R \longrightarrow R'$ be an epimorphism of rings with Kernel K. Then $f^{-1}(f(J)) = J$ for every fuzzy ideal of R which is constant on K.

Proof

Let $x \in R$ be arbitrary. Then,

$$f^{-1}(f(J))(x) = f(J)\ (f(x))$$

$$= \bigvee_{y \in f^{-1}(f(x))} J(y)$$

$$= \bigvee_{y-x \in \mathrm{Ker} f} J(y)$$

$= J(x)$, since J is constant on Kernel K.

Therefore $f^{-1}(f(J)) = J$.

Remark (2.1.28)

If $f: R \longrightarrow R'$ is a homomorphism of rings and J' is a fuzzy ideal of R' then,

$$f\ (f^{-1}(J')) = J'$$

Theorem : (2.1.29)

If $f : R \longrightarrow R'$ is an epimorphism of rings with Kernel K, then there is a one to one correspondance between the set of fuzzy ideals of R' and the set of fuzzy ideals of R which are constant on K.

Proof :

Let J be a fuzzy ideal of R which is constant on K. Since f is an epimorphism, f(J) is a fuzzy ideal of R'. Thus there is a correspondance J $\longrightarrow$ f(J) from the set of fuzzy ideals of R which are constant on K into the set of fuzzyideals of R'. We show that this correspondance is bijective.

Let J' be a fuzzy ideal of R' then

$f^{-1}(J')$ is a fuzzy ideal of R and $f^{-1}(J)$ is constant on K. For, if $x \in K$ then $f(x) = 0$ and,

$f^{-1}(J')\ (x) = J'(f(x)) = J'(0).$

Therefore correspondance is onto.

Let J_1 and J_2 be fuzzy ideals of R which are constant on K and let $f(J_1) = f(J_2)$, then

$$f^{-1}(f(J_1)) = f^{-1}(f(J_2))$$

$$\Rightarrow J_1 = J_2$$

This shows that correspondance is one-one.

Corollary (2.1.30)

If R is a ring and Q is an ideal of R then there is a one-to-one correspondance between fuzzy ideals of R/Q and fuzzy ideals of R which are constant of Q.

Proof

Since there is a natural epimorphism F : R $\longrightarrow$ R/Q, from above Therem , the result follows.

Definition (2.1.31)

Let J be a fuzzy ideal of a ring R. For any $x \in R$, define

$$J_{(x)} : R \longrightarrow I$$

$$J_{(x)}(y) = J(y - x)$$

Then $J_{(x)}$ is called as fuzzy coset of a fuzzy ideal of R.

Remark (2.1.32)

(1) $J_{(0)} = J$

(2) For any $t \in I$, $(J_{(x)})_t$ is a coset of an ideal J_t of R.

i.e. $(J_{(x)})_t = J_t + x$

(3) If J is a characteristic function of an ideal Q of R. then $J_{(x)}$ is a characteristic function of coset Q + x of R.

Proposition (2.1.33)

Let J be a fuzzy ideal of a ring R and let $\mathcal{R}$ be a collection of all fuzzy cosets of J. Define

$$J_{(x)} + J_{(y)} = J_{(x+y)}$$

$$J_{(x)} \cdot J_{(y)} = J_{(x.y)} \quad \text{for all } x, y \in R.$$

Then $\mathcal{R}$ is a ring under these two operation.

Proof:

First we show that these two operations are well defined.

Let $J_{(x)} = J_{(x')}$ and $J_{(y)} = J_{(y')}$

Therefore $J(x - x') = J(0)$ and $J(y - y') = J(0)$

$\Rightarrow J(x) = J(x')$ and $J(y) = J(y')$.

Let $z \in R$ be arbitrary. Then,

$$
\begin{aligned}
J_{(x+y)}(z) &= J(z-x-y) \\
&= J(z-x'-y'+x'-x+y'-y) \\
&\geqslant J(z-x'-y') \wedge J(x'-x) \wedge J(y'-y) \\
&= J(z-x'-y') \quad \text{since } J(x'-x) = J(0) \\
&\qquad\qquad\qquad\qquad J(y'-y) = J(0) \\
&= J_{(x'+y')}(z)
\end{aligned}
$$

Similarly $J_{(x'+y')}(z) \geqslant J_{(x+y)}(z)$.

Hence $J_{(x+y)} = J_{(x'+y')}$

Next ,

$$
\begin{aligned}
J_{(x.y)}(z) &= J(z-xy) \\
&= J(z-x'y'+x'y'-xy) \\
&\geqslant J(z-x'y') \wedge J(x'y'-xy)
\end{aligned}
$$

But
$$
\begin{aligned}
J(x'y'-xy) &= J(x'y'-x'y+x'y-xy) \\
&= J\big(x'(y'-y)+(x'-x)y\big) \\
&\geqslant J\big(x'(y'-y)\big) \wedge J\big((x'-x)y\big) \\
&\geqslant \big(J(x') \vee J(y'-y)\big) \wedge \big(J(x'-x) \vee J(y)\big) \\
&= \big(J(x') \vee J(0)\big) \wedge \big(J(0) \vee J(y)\big)
\end{aligned}
$$

Thus $J(x'y'-xy) = J(0)$

Therefore,

$$
\begin{aligned}
J_{(xy)}(z) &\geqslant J(z-x'y') \wedge J(0) \\
&= J(z-x'y') \\
&= J_{(x'y')}(z)
\end{aligned}
$$

Similarly $J_{(x'y')}(z) \geqslant J_{(xy)}(z). \Rightarrow J_{(x \cdot y)} = J_{(x' \cdot y')}$

Thus + and . operations are well defined. , Further

$$J_{(x)} + (J_{(y)} + J_{(z)}) = (J_{(x)} + J_{(y)}) + J_{(z)} = J_{(x+y+z)}$$

$$J_{(x)} + J_{(-x)} = J_{(0)} = J$$

$$J_{(x)} + J_{(0)} = J_{(x)}$$

$$J_{(x)} \cdot (J_{(y)} \cdot J_{(z)}) = (J_{(x)} \cdot J_{(y)}) \cdot J_{(z)} = J_{(x.y.z)}$$

$$J_{(x)} \cdot J_{(1)} = J_{(x)} \text{ . and } J_{(x)} \cdot J_{(y)} = J_{(y)} \cdot J_{(x)} = J_{(x.y)}$$

Hence $\mathcal{R}$ is a commulative ring with unity.

Remark (2.1.34)

Define a function $\theta : R \longrightarrow \mathcal{R}$ by

$\theta(x) = J_{(x)}$ where J is a fuzzy ideal of R. Then,

$$\theta(x+y) = J_{(x+y)} = J_{(x)} + J_{(y)} = \theta(x) + \theta(y)$$

$$\theta(x \cdot y) = J_{(x.y)} = J_{(x)} \cdot J_{(y)} = \theta(x) \cdot \theta(y)$$

$$\theta(1) = J_{(1)}$$

And for any $J_{(x)} \in \mathcal{R}$, $\theta(x) = J_{(x)}$.

Hence θ is an epimorphism. Therefore by proposition (2.1.29) there is a one-to-one correspondance between fuzzy ideals of $\mathcal{R}$ and fuzzy ideals of R which are constant on Kernel of θ.

$$\begin{aligned} \text{But} \quad \text{Ker } \theta &= \{ x \in R / \theta(x) = J_{(0)} \} \\ &= \{ x \in R / J_{(x)} = J_{(0)} \} \\ &= \{ x \in R / J(x) = J(0) \} \quad \text{since} \end{aligned}$$

$J_{(x)} = J_{(0)} \Leftrightarrow J(X) = J(0)$.

Thus Ker $\theta = J_{J(0)}$. Clearly J is a fuzzy ideal of R which is constant on $J_{J(0)}$.

Definition (2.1.35)

Let R be a commutative ring with unity and let J be a fuzzy ideale of R. Then $\theta(j)$ is called fuzzy quotient ideal determined by J. Where $\theta : R \longrightarrow \mathcal{R}$ is an epimorphism from R onto a ring of fuzzy cosets of J.

$$\theta (J) (J_{(x)}) = \bigvee_{y \in \theta^{-1}(J_{(x)})} J(y)$$

$$= \bigvee_{J_{(y)} = J_{(x)}} J(y)$$

$$= J(x) \quad \text{since } J_{(y)} = J_{(x)}$$

$$\Rightarrow J(y) = J(X)$$

Proposition (2.1.36)

Let J be a fuzzy ideal of R and $\mathcal{R}$ Be a ring of fuzzy cosets of J. Then each fuzzy ideal of $\mathcal{R}$ corresponds to a fuzzy ideal of R which is constant on $J_{J(0)}$.

Proof :

Since $\theta : R \longrightarrow \mathcal{R}$ defined by $\theta(x) = J_{(x)}$ is an epimorphism, there is one-to-one correspondance between fuzzy ideals of $\mathcal{R}$ and fuzzy ideals of R which are constant an $J_{J(0)}$ In perticular if $\tilde{J}$ is a fuzzy ideal of $\mathcal{R}$ then $\theta^{-1}(\tilde{J})$ defined by,

$$\theta^{-1}(\tilde{J}) (x) = \tilde{J} (\theta(x)) = \tilde{J} (J_{(x)})$$

is a fuzzy ideal of R which is constant on $J_{J(0)}$ i.e. Kernel of θ.

2.2 PRIME FUZZY IDEALS

Definition (2.2.1)

Let R be a ring. A prime fuzzy ideal of R is a fuzzy ideal of R such that

$$J(x.y) = J(x) \quad \text{or } J(x.y) = J(y)$$

Remark (2.2.2)

(1) A fuzzy ideal J is a Prime fuzzy ideal if and only if $J(x.y) = J(x) \vee J(y)$

(2) For any lattice L with 0 and 1 if $P: R \rightarrow L$ is a prime fuzzy ideal of a ring R, then P(R) is totally ordered subset of L.

Proposition (2.2.3)

Characteristic function of a prime ideal of R is a prime fuzzy ideal of R.

Proof :

Let Q be a prime ideal of R. Define ,

$$J : R \longrightarrow I$$

$$J(x) = 1 \quad \text{if } x \in Q$$

$$= 0 \quad \text{if } x \notin Q$$

Then J is a fuzzy ideal of R. We prove that J is a prime fuzzy ideal of R. Let $x \in R$ and $y \in R$ be arbitrary.

If $x.y \in Q$, then since Q is a prime ideal of R, $x \in Q$ or $y \in Q$. Thus $J(x.y) = 1 \Rightarrow J(x) = 1$ or $J(y) = 1$ i.e. $J(x.y) = J(x)$ or $J(x.y) = J(y)$.

If $x.y \notin Q$, then $x \notin Q$ and $y \notin Q$.

Thus $J(x.y) = 0 = J(x) = 0$, $J(y) = 0$

i.e. $J(x.y) = J(x)$ or $J(x.y) = J(y)$.

Therefore $J(x.y) = J(x)$ or $J(x.y) = J(y)$ for all x, $y \in R$. .

Proposition (2.2.4)

If P is a Prime fuzzy ideal of R, then for all $t \in I$, $P_t = \{ x \in R / P(x) \geqslant t \}$ is a prime ideal of R.

Proof :

Since P is a fuzzy ideal of R, P_t is an ideal of R. We prove that P_t is a prime ideal of R.

Let $x.y \in P_t \Rightarrow P(x.y) \geqslant t$. But $P(x.y) = P(x)$ or $P(x.y) = P(y)$. Therefore $P(x) \geqslant t$ or $P(y) \geqslant t$.

Thus $x.y \in P_t \Rightarrow x \in P_t$ or $y \in P_t$.

Remark (2.2.5)

In other words above proposition means, ' A level subset of a Prime fuzzy ideal is a prime ideal of R'.

Remark (2.2.6)

Proposition (2.2.4) is true even if I is replaced by an aribitrary lattice with zero and one.

Corollary (2.2.7)

If $Q = \{ x \in R/P(x) = P(0), \}$ then Q is a prime ideal of R where P is a Prime fuzzy ideal of R.

Proposition (2.2.8)

Let $f: R \longrightarrow R'$ be a homomorphism of a ring R onto a ring R' with kernel K. If P is a fuzzy ideal of R which is constant on K, then P is Prime fuzzy ideal of R, if and only if f(p) is

Prime fuzzy ideal of R'.

Proof :

Suppose that P is a Prime fuzzy ideal of R. We show that homomorphic image f(P) is a prime fuzzy ideal of R'. Clearly f(P) is a fuzzy ideal of R'. Let $x', y' \in R'$ be arbitrary. Since f is onto, there is $x \in R$ and $y \in R$ such that $f(x) = x'$ and $f(y) = y'$, consider,

$$f(P)(x'.y') = \bigvee_{z \in f^{-1}(x'.y')} P(z)$$

$$= \bigvee_{\substack{f(z)=f(x.y) \\ y \in f^{-1}(y'),\, x \in f^{-1}(x')}} P(z) \quad \text{since f is a ring homomorphism}$$

$$= \bigvee_{\substack{z-x.y \in K \\ x \in f^{-1}(x') \\ y \in f^{-1}(y')}} P(z)$$

$$= \bigvee_{\substack{x \in f^{-1}(x') \\ y \in f^{-1}(y')}} \left(P(x \cdot y) \right) \quad \text{since P is constant on K}$$

$$= \bigvee_{\substack{x \in f^{-1}(x') \\ y \in f^{-1}(y')}} \left(P(x) \vee P(y) \right) \quad \text{since P is a Prime fuzzy ideal}$$

$$= \left(\bigvee_{x \in f^{-1}(x')} P(x) \right) \vee \left(\bigvee_{y \in f^{-1}(y')} P(y) \right)$$

$$= f(P)(x') \vee f(P)(y')$$

Thus f(P) is prime fuzzy ideal of R'. Conversely suppose that f(P) is a prime fuzzy ideal of R', where P is a fuzzy ideal of R which is constant on K. We show that P is a prime fuzzy ideal of R. Suppose that P is not a prime fuzzy ideal of R. Then there are $x \in R$ and $y \in R$ such that,

$$P(x.y) > P(x) \vee P(y)$$

Now $x, y \in R$, hence $f(x), f(y) \in R'$

consider,

$$f(P)(f(x).f(y)) = f(P)(f(x.y))$$

$$= \bigvee_{r \in f^{-1}(f(x.y))} P(r)$$

$$= \bigvee_{f(r) = f(x.y)} P(r)$$

$$= \bigvee_{r - x.y \in K} P(r)$$

$= P(x.y)$ since P is constant on K

$> P(x) \vee P(y)$ By hypothesis

on the other hand,

$$f(P)(f(x)) = \bigvee_{r \in f^{-1}(f(x))} P(r)$$

$$= \bigvee_{r - x \in K} P(r)$$

$$= P(x)$$

And, $f(P)\,(f(y)) = \bigvee_{r \in f^{-1}(f(y))} P(r)$

$= \bigvee_{r-y \in K} P(r)$

$= P(y)$ since P is constant on K

Thus,

$P(x \cdot y) > P(x) \vee P(y)$

$\Rightarrow f(P)\,(f(x).f(y)) > f(P)\,(f(x)) \vee f(P)\,(f(y))$

Which is a contradiction since f(P) is a Prime fuzzy ideal. Hence P is a Prime fuzzy ideal of R.

Proposition (2.2.9)

Let $f: R \longrightarrow R'$ be a homomorphism of a ring R onto a ring R'. If P' is a fuzzy ideal of R', then P' is a prime fuzzy ideal of R' if and only if $f^{-1}(P')$ is a prime fuzzy ideal of R.

Proof :

Suppose that P' is a a prime fuzzy ideal of R'. Clearly $f^{-1}(P')$ is a fuzzy ideal of R. Let $x, y \in R$ be arbitrary. Consider

$f^{-1}(P')\,(x.y) = P'\,(f(x.y))$

$= P'\,(f(x).f(y))$ since f is a ring homomorphism

$= P'\,(f(x)) \vee P'(f(y))$ since P' is a Prime fuzzy ideal

$= f^{-1}(P')\,(x) \vee f^{-1}(P')(y)$

Thus, it shows that $f^{-1}(P')$ is a prime fuzzy ideal of R. Conversely, suppose that $f^{-1}(P')$ is a Prime fuzzy ideal of R. Suppose that P' is not a Prime fuzzy ideal of R'. Therefore for some $x', y' \in R'$.

$$P'(x'.y') > P'(x') \vee P'(y')$$

Since f is onto, there is $x \in R$ and $y \in R$ such that $f(x) = x'$ and $f(y) = y'$, consider

$$\begin{aligned} f^{-1}(P')(x.y) &= P'(f(x.y)) \\ &= p'(f(x).f(y)) \\ &= P'(x'.y') \\ &> P'(x') \vee P'(y') \\ &= P'(f(x)) \vee P'(f(y)) \\ &= f^{-1}(P')(x) \vee f^{-1}(P')(y) \end{aligned}$$

Thus, $f^{-1}(P')(x.y) > f^{-1}(P')(x) \vee f^{-1}(P')(y)$

Which is a contradiction. Hence the proof.

2.3 PRIMARY FUZZY IDEALS

Definition (2.3.1)

If J is a fuzzy ideal of R, then fuzzy radical of J, denoted by $\bar{J}$, is given by

$$\bar{J}: R \longrightarrow I$$

$$\bar{J}(x) = \bigvee_{n} J(x^n), \quad n \in \mathbb{N}^+.$$

Proposition (2.3.2)

The radical of a fuzzy ideal J is a fuzzy ideal of R containing J.

Proof

Let $\bar{J}$ be a fuzzy radical of J and let $x, y \in R$ be arbitrary. Then,

$$\bar{J}(x+y) = \underset{n}{\vee} J((x+y)^n)$$

$$= \left(\underset{n}{\vee} J((x+y)^{2n}) \right) \vee \left(\underset{n}{\vee} J((x+y)^{2n+1}) \right)$$

Consider,

$$\underset{n}{\vee} J((x+y)^{2n}) = \underset{n}{\vee} J\left(\sum_{r=0}^{2n} C_r\, x^{2n-r} y^r \right)$$

Now,

$$J\left(\sum_{r=0}^{2n} C_r\, x^{2n-r} y^r \right) = J\left(x^{2n} + C_1 x^{2n-1} y + C_2 x^{2n-2} y^2 + \ldots \right.$$

$$\ldots + C_n x^n . y^n + C_{n+1} x^{n-1} y^{n+1} + \ldots\ldots$$

$$\left. \ldots + C_{2n-1} xy^{2n-1} + y^{2n} \right)$$

Where $C_o = C_{2n} = 1$

$$\geqslant J(x^{2n}) \wedge J(x^{2n-1}y) \wedge \ldots\ldots$$

$$\ldots \wedge J(x^n y^n) \wedge J(x^{n-1} y^{n+1}) \wedge \ldots$$

$$\ldots \wedge J(xy^{2n-1}) \wedge J(y^{2n}).$$

$$\geqslant J(x^{2n}) \wedge \left(J(x^{2n-1}) \vee J(y) \right) \wedge \ldots$$

$$\ldots \wedge \left(J(x^n) \vee J(y^n) \right) \wedge \left(J(x^{n-1}) \vee J(y^{n+1}) \right) \ldots$$

$$\ldots \wedge \left(J(x) \vee J(y^{2n-1}) \right) \wedge J(y^{2n})$$

$$\geqslant J(x^{2n}) \wedge J(x^{2n-1}) \wedge \cdots \wedge J(x^n) \wedge J(y^{n+1}) \wedge \cdots \wedge J(y^{2n-1}) \wedge J(y^{2n})$$

$$= J(x^n) \wedge J(y^{n+1})$$

Therefore,

$$\bigvee_n J((x+y)^{2n}) \geqslant \bigvee_n (J(x^n) \wedge J(y^{n+1}))$$

$$= (\bigvee_n J(x^n)) \wedge (\bigvee_n J(y^{n+1}))$$

$$= (\bigvee_n J(x^n)) \wedge (\bigvee_n J(y^n))$$

Similarly,

$$\bigvee_n (J(x+y)^{2n+1}) \geqslant (\bigvee_n J(x^n)) \wedge (\bigvee_n J(y^n))$$

Thus,

$$\bigvee_n (J(x+y)^n) \geqslant (\bigvee_n J(x^n)) \wedge (\bigvee_n J(y^n))$$

i.e. $\bar{J}(x+y) \geqslant \bar{J}(x) \wedge \bar{J}(y)$.

next ,

$$\bar{J}(-x) = \bigvee_n J(-x)^n$$

$$= \bigvee_n J(x^n)$$

$$= \bar{J}(x)$$

And lastly,

$$\bar{J}(x.y) = \bigvee_n J(x.y)^n$$

$$= \bigvee_n J(x^n . y^n)$$

$$\geqslant \bigvee_n (J(x^n) \vee J(y^n))$$

$$= (\bigvee_n J(x^n)) \vee (\bigvee_n J(y^n))$$

Hence,

$$\bar{J}(x.y) \geqslant \bar{J}(x) \vee \bar{J}(y)$$

This proves that $\bar{J}$ is a fuzzy ideal of R, And

$\bar{J}(x) = \bigvee_n J(x^n) \geqslant J(x)$ for all $x \in R$ i.e.

$$J \subseteq \bar{J}$$

Proposition (2.3.3)

If J_1 and J_2 are fuzzy ideals of R, then following holds

(1) If $J_1 \subseteq J_2$ then $\bar{J}_1 \subseteq \bar{J}_2$

(2) $\overline{J_1 \cap J_2} = \bar{J}_1 \cap \bar{J}_2$

(3) $\bar{\bar{J}}_1 = \bar{J}_1$

Proof :

(1) Let $J_1 \subseteq J_2$ and $x \in R$. Consider,

$$\bar{J}_1(x) = \bigvee_n J_1(x^n) \leq \bigvee_n J_2(x^n) = \bar{J}_2(x)$$

Hence, $\bar{J}_1 \subseteq \bar{J}_2$

(2) Let $x \in R$ consider

$$\overline{(J_1 \cap J_2)}(x) = \bigvee_n (J_1 \cap J_2)(x^n)$$

$$= \bigvee_n (J_1(x^n) \wedge J_2(x^n))$$

$$= (\bigvee_n J_1(x^n)) \wedge (\bigvee_n J_2(x^n))$$

$$= \bar{J}_1(x) \wedge \bar{J}_2(x)$$

$$= (\bar{J}_1 \cap \bar{J}_2)(x)$$

Thus, $\overline{J_1 \cap J_2} = \bar{J}_1 \cap \bar{J}_2$

(3) $$\bar{\bar{J}}_1(x) = \bigvee_n \bar{J}_1(x^n)$$

$$= \bigvee_n \bigvee_m J_1(x^{nm})$$

$$= \bigvee_n J_1(x^n)$$

$$= \bar{J}_1(x)$$

Thus, $\bar{\bar{J}}_1 = \bar{J}_1$.

Proposition (2.3.4)

Let J be a fuzzy ideal of a ring R. Then for each $t \in I$, $\bar{J}_t$ is a radical of J_t where

$$\bar{J}_t = \{ x \in R / \bar{J}(x) > t \}$$

Proof : Clearly J_t is an ideal of R. And,

$$\bar{J}_t = \{ x \in R / \bar{J}(x) > t \} = \{ x \in R / \bigvee_n J(x^n) > t \}$$

$$= \{ x \in R / J(x^n) > t \text{ for some } n \}$$

Now, if $x \in \bar{J}_t$ then $J(x^n) > t$ for some n. This shows that $\bar{J}_t$ is a set of all the elements x in R such that $x^n \in J_t$ for some n. Hence $\bar{J}_t$ is a radical of J_t.

Proposition (2.3.5)

Let J be a fuzzy ideal of a ring R. If $J_{t_1} = J_{t_2}$ then $\bar{J}_{t_1} = \bar{J}_{t_2}$

Proof :

If $t_1 < t_2$ and $J_{t_1} = J_{t_2}$, there is no $x \in R$ such that $t_1 \leqslant J(x) < t_2$

Now, if $x \in \bar{J}_{t_2}$ $\Rightarrow$ $\bar{J}(x) > t_2$

$\Rightarrow$ $\bar{J}(x) > t_1$

$\Rightarrow$ $x \in \bar{J}_{t_1}$

Hence

$$\bar{J}_{t_2} \subseteq \bar{J}_{t_1}$$

On the other hand, if $x \in \bar{J}_{t_1}$ then $J(x^n) > t_1$ for some n. Since $J(x^n) \geqslant J(x)$ and there is no $x \in R$ such that $t_1 \leq J(x) < t_2$

$t_2 \leqslant J(x^n)$ Hence $x \in \bar{J}_{t_2}$. Therefore $\bar{J}_{t_1} \subseteq \bar{J}_{t_2}$. This shows that $\bar{J}_{t_1} = \bar{J}_{t_2}$

Proposition (2.3.6)

If P is a prime fuzzy ideal of R , then $P = \overline{P}$.

Proof :

Let $x \in R$ be arbitrary. Since P is a prime fuzzy ideal,

$P(x^2) = P(x.x) = P(x) \vee P(x) = P(x)$,

$P(x^3) = P(x^2.x) = P(x^2) \vee P(x) = P(x) \vee P(x) = P(x)$

and so on,

Thus for all n, $P(x^n) = P(x)$. Therefore,

$$\overline{P}(x) = \bigvee_n P(x^n) = P(x).$$

Since x is arbitrary, $\overline{P} = P$.

Proposition (2.3.7)

Let $f:R \longrightarrow R'$ be a homomorphism of rings R onto R',

and let J be a fuzzy ideal of R.

Then $f(\overline{J}) \subseteq \overline{f(J)}$ and equality holds if J is constant on Kernel of f.

Proof :

Let $x' \in R'$ be arbitrary. Consider,

$$f(\overline{J})(x') = \bigvee_{x \in f^{-1}(x')} \overline{J}(x)$$

$$\leqslant \bigvee_{x^n \in f^{-1}(x'n)} \left(\bigvee_n J(x^n) \right)$$

$$= \bigvee_n \left(\bigvee_{x^n \in f^{-1}(x'n)} J(x^n) \right)$$

$$= \bigvee_n f(J)\left((x')^n \right)$$

$$= \overline{f(J)}(x')$$

Thus $f(\bar{J})\ (x') \leq f(J)(x')$. Since $x' \in R'$ is arbitrary, $f(\bar{J}) \subseteq \overline{f(J)}$.

Next, if J is a constant on Kernel of f., then for every $x \in f^{-1}(x'^n)$ there is $x_1 \in f^{-1}(x')$, such that $J(x) = J(x_1{}^n)$

Hence $\bigvee_{x \in f^{-1}(x')} J(x^n) = \bigvee_{x^n \in f^{-1}(x'^n)} J(x^n)$

Therefore,

$$\begin{aligned} f(\bar{J})\ (x') &= \bigvee_{x \in f^{-1}(x')} J\ (x) \\ &= \bigvee_n \bigvee_{x \in f^{-1}(x')} J(x^n) \\ &= \bigvee_n \bigvee_{x^n \in f^{-1}(x'^n)} J(x^n) \\ &= \bigvee_n f(J)\ (x'^n) \\ &= \overline{f(J)}\ (x') \end{aligned}$$

Since $x' \in R'$ is arbitrary, $f(\bar{J}) = \overline{f(J)}$

Proposition (2.3.8)

Let $f: R \longrightarrow R'$ be a homomorphism of rings R onto R' and let J' be a fuzzy ideal of R'. Then,

$$f^{-1}(\bar{J}) = \overline{f^{-1}(J')}$$

Proof: Let $x \in R$ be arbitrary. Then,

$$\begin{aligned} f^{-1}(\bar{J})(x) = \bar{J}(f(x)) &= \bigvee_n J(f(x)^n). \\ &= \bigvee_n J\ (f(x^n)\ ) \\ &= \bigvee_n f^{-1}(J)\ (x^n) \\ &= \overline{f^{-1}(J)}\ (x) \end{aligned}$$

Therefore $f^{-1}\ (\bar{J}) = \overline{f^{-1}(J)}$.

Definition (2.3.9)

Let R be a ring. A fuzzy ideal J of R is called a primary fuzzy ideal if,

$$J(x.y) = J(x^n) \text{ or } J(x.y) = J(y^n) \text{ for some } n \in Z+$$

Remark (2.3.10)

If P is a prime fuzzy ideal then P is a primary fuzzy ideal.

Proposition (2.3.11)

If P is a primary fuzzy ideal of a ring R, then fuzzy radical $\overline{P}$ is a prime fuzzy ideal.

Proof :

Since P is a fuzzy ideal of R, $\overline{P}$ is a fuzzy ideal of R. Let $x, y \in R$, then,

$$\begin{aligned}
\overline{P}(x.y) &= \bigvee_n P(x.y)^n \\
&= \bigvee_n P(x^n.y^n) \\
&= \bigvee_m P(x^{nm}) \text{ or } \bigvee_m P(y^{nm}) \\
&\leq \bigvee_n P(x^n) \text{ or } \bigvee_n P(y^n) \\
&= \overline{P}(x) \text{ or } \overline{P}(y)
\end{aligned}$$

Thus $\overline{P}(x.y) \leq \overline{P}(x) \vee \overline{P}(y)$. But since $\overline{P}$ is a fuzzy ideal, $\overline{P}(x.y) \geqslant \overline{P}(x) \vee \overline{P}(y)$. Hence $\overline{P}(x.y) = \overline{P}(x) \vee \overline{P}(y)$ which proves that P is a prime fuzzy ideal.

Proposition (2.3.12)

If P is a primary fuzzy ideal then for each $t \in I$, P_t is a primary ideal of R.

Proof :

P is a primary fuzzy ideal of R. For each $t \in I$,

$$P_t = \{ x \in R / P(x) \geqslant t \}$$

Clearly P_t is an ideal of R

Let $x.y \in P_t \Rightarrow P(x.y) \geqslant t$

$\Rightarrow P(x^n) \geqslant t$ or $P(y^n) \geqslant t$ for some $n \in Z+$

$\Rightarrow x^n \in P_t$ or $y^n \in P_t$ for some n.

Thus P_t is a primary ideal of R.

Proposition (2.3.13) If P is a Primary fuzzy ideal of R, then for each $t \in I$, $(\overline{P})_t$ is a radical of P_t.

Proof : Since P is a primary fuzzy ideal each P_t is a primary ideal of R for $t \in I$. $\overline{P}$ is a fuzzy radical of P. Hence $(\overline{P})_t$ is a radical of P_t for each t i.e. $(\overline{P})_t = \overline{P_t}$

Proposition (2.3.14)

Let $f: R \longrightarrow R'$ be an epimorphism of rings. If P is a primary fuzzy ideal of R which is constant on Kernel of f, then image f(P) of P is a primary fuzzy ideal of R'.

Proof :

Clearly f(P) is a fuzzy ideal of R' . Let $x', y' \in R'$, Then,

$$f(P)\ (x'.y') = \bigvee_{z \in f^{-1}(x'.y')} P(z)$$

$$= \bigvee_{\substack{x \in f^{-1}(x') \\ y \in f^{-1}(y')}} P(x.y)$$

$$= \bigvee_{x \in f^{-1}(x')} P(x^n) \text{ or } \bigvee_{y \in f^{-1}(y')} P(y^n)$$

$$= \bigvee_{x \in f^{-1}(x'^n)} P(x) \text{ or } \bigvee_{x \in f^{-1}(y'^n)} P(x)$$

$$= f(P)\ (x'^n) \text{ or } f(P)\ (y'^n)$$

Thus $f(P)\ (x'y') = f(P)\ (x'^n)$ or $f(P)\ (y'^n)$

Which shows that $f(P)$ is primary fuzzy ideal of R'.

<u>Proposition</u> (2.3.15)

Let $f: R \longrightarrow R'$ be an epimorphism of rings. If P' is a primary fuzzy ideal of R' then $f^{-1}(P)$ is a primary fuzzy ideal of R.

<u>Proof</u> :

Clearly $f^{-1}(P')$ is a fuzzy ideal. Let $x, y \in R$. Then,

$$f^{-1}(P')\ (x.y) = P'\ (f(x.y))$$

$$= P'\ (f(x).f(y))$$

$$= P'\left((f(x))^n\right) \text{ or } P'\left((f(y))^n\right)$$

$$= P'\left(f(x^n)\right) \text{ or } P'\ (f(y^n))$$

$$= f^{-1}(P')\ (x^n) \text{ or } f^{-1}(P')\ (y^n)$$

Therefore $f^{-1}(p')$ is a primary fuzzy ideal of R.

Proposition (2.3.16) :

Let P and J be two fuzzy ideals of R. Then P is primary and $\overline{P} = \overline{J}$ if.

i) $P \subseteq J$ i) $J(x) = P(x^m)$ for some m iii) $P(x.y)=J(x)$ or $J(y)$.

Proof :

Let $x, y \in R$. $P(x.y) = J(x)$ or $J(y)$. But $J(x) = P(x^m)$ for some m. Therefore $P(x.y) = p(x^m)$ or $p(y^m)$, For some $m \in z_+$

Hence P is primary fuzzy ideal. And,

$$J(x) = P(x^m) \leqslant \bigvee_m p(x^m) = \overline{P}(x).$$

Therefore $J \subseteq \overline{P}$. But $\overline{p} \subseteq \overline{J}$ because $P \subseteq J$.

Hence $\overline{J} \subseteq \overline{\overline{P}} = \overline{P} \subseteq \overline{J} \implies \overline{P} = \overline{J}$.

2.4 FINITE VALUED FUZZY IDEALS

Definition (2.4.1)

Let J be a fuzzy ideal of a ring R. If ImJ is finite then we say that J is a finite valued fuzzy ideal.

Remark (2.4.2)

The characteristic function $J:R \longrightarrow I$ of an ideal Q of R is a typical two valued fuzzy ideal. However if t_1, $t_2 \in I$ and $t_1 \geqslant t_2$, then

$$J_1 : R \longrightarrow I \text{ defined by,}$$

$$J_1(r) = t_1 \text{ if } r \in Q$$

$$= t_2 \text{ if } r \notin Q$$

Can be identified with characteristic function J of Q. Hence J_1 can be more appropriately described as $J_1(r) = J_1(0)$ if $x \in Q$ and $J_1(r) = J(1)$ if $x \notin Q$.

Proposition (2.4.3)

If R is a simple ring and $J : R \longrightarrow I$ is a fuzzy ideal of R, then for all $x \neq 0$, $J(x) = J(1), x \in R$.

Proof :

Let $Q = \{ x \in R / J(x) = J(0) \}$. If Q = R then there is nothing to prove. If $J(y) \neq (J(1)$ for some $0 \neq y \in R$, then,

$I = \{ x \in R/J(x) \geqslant J(y) \}$ is a proper ideal of :

which is a contradiction. Hence $J(x) = J(1)$ for all $0 \neq x \in R$.

Proposition (2.4.4)

If R is a division ring then $J(x) = J(1)$ for all $0 \neq x \in R$. Where J is a fuzzy ideal of R.

Proof :

Let $0 \neq x \in R$ be arbitrary. Then there is $x^{-1} \in R$ such that $x.x^{-1} = x^{-1}.x = 1$. Therefore,

$$J(1) = J(x.x^{-1}) \geqslant J(x) \vee J(x^{-1}) \geqslant J(x) \geqslant J(1).$$

Hence $J(x) = J(1)$ for all $0 \neq x \in R$.

Proposition (2.4.5)

If $J:R \longrightarrow I$ is a fuzzy ideal of R and $Q = \{x \in R/J(x) = J(0)\}$ is maximal then $J(x) = J(0)$ for $x \in Q$ and $J(x) = J(1)$ for $x \notin Q$

Proof:

Since Q is an ideal of R, R/Q is a ring. If $x+Q = y+Q$, then $J(x-y) = J(0)$, and hence $J(x) = J(y)$. Therefore fuzzy ideal $J:R \longrightarrow I$ induces a fuzzy ideal $J^{*}: R/Q \longrightarrow I$. defined by $J^{*}(x + Q) = J(x)$. Since Q is a maximal ideal of R, R/Q is a division ring. Hence by above proposition (2.4.4), $J^{*}(x + Q) = J(0)$ if $x \in Q$ and $J^{*}(x+Q) = J(1)$ if $x \notin Q$. i.e. $J(x) = J(0)$ if $x \in Q$ and $J(x) = J(1)$ if $x \notin Q$.

Proposition (2.4.6)

If R is one of the following rings

(i) Principal ideal domain,

(ii) Boolean ring,

(iii) Artinian ring,

and if $P : R \longrightarrow I$ is a prime fuzzy ideal such that $P(x) = P(0)$ for some $x \neq 0$, then either $P(x) = P(0)$ or $P(x) = P(1)$.

Proof :

Since P is a prime fuzzy ideal,

$$P_{P(0)} = \{ x \in R/P(x) = P(0) \}$$ is a prime ideal of R and since $P(x) = P(0)$ for some $0 \neq x \in R$, $P_{P(0)}$ is a nontrivial prime ideal of R. Moreover, if R is one of the above rings, then $P_{P(0)}$ is a maximal ideal of R. Hence by proposition (2.4.5), either $P(x) = P(0)$ or $P(x) = P(1)$ for all $x \in R$.

Remark (2.4.7)

The following example shows that the condition $P(x) = P(0)$ for some $0 \neq x \in R$ is necessary for P to be two valued function.

Let R be a principal ideal Domain and Q be a prime ideal of R.

Define $P : R \longrightarrow I$

$$P(0) = 1$$

$$P(x) = t \quad \text{if} \quad x \in Q - \{0\}$$

$$P(x) = 0 \quad \text{if} \quad x \notin Q$$

where $0 < t < 1$. We show that P is a prime fuzzy ideal of R.

Let $x, y \in R$ be arbitrary.

(a) i) If $x, y \in Q$ then $x - y \in Q$.

Hence $P(x-y) \geqslant P(x) \wedge P(y)$

ii) If $x \in Q$ and $y \notin Q$, then $x - y \notin Q$

$\Rightarrow P(x) = 1$ or t, $P(y) = 0$, $P(x-y) = 0$

Hence $P(x-y) \geqslant P(x) \wedge P(y)$.

iii) If $x \notin Q$, $y \notin Q$, then either $x - y \in Q$ or $x-y \notin Q$.

$\Rightarrow P(x) = P(y) = 0$, $P(x-y) = 0$ or $P(x-y) = 1$ or t.

Hence $P(x-y) \geqslant P(x) \wedge P(y)$.

(b) i) If $x \in Q$ and $y \in Q$ then $x.y \in Q$

$\Rightarrow P(x) = 1$ or t, $p(y) = 1$ or t and $P(x.y) = 1$ or t.

Hence $P(x.y) = P(x) \vee P(y)$.

ii) If $x \in Q$, $y \notin Q$, then $x.y \in Q$

$\Rightarrow P(x) = 1$ or t, $P(y) = 0$ and $P(x.y) = 1$ or t.

Hence $P(x.y) = P(x) \vee P(y)$

iii) If $x \notin Q$, $y \notin Q$, then $x.y \notin Q$, Since Q is a prime ideal $\Rightarrow P(x) = P(y) = P(x.y) = 0$

Hence $P(x.y) = P(x) \vee P(y)$

Thus P is a prime fuzzy ideal of R and clearly P is not two valued.

Remark (2.4.8)

If R is a Boolean ring, then the condition $P(x) = P(0)$ for some $x \neq 0$ is satisfied trivially whenever P is prime fuzzy ideal of R. For, if there is no $x \neq 0$, such that $P(x) = P(0)$, then $P_{P(0)} = \{0\}$ And since P is a prime fuzzy ideal of R, $\{0\}$ is a prime ideal of R. This shows that R is an integral domain which is a contradiction.

Remark (2.4.9)

If R is an Artinian ring then thecondition $P(x)= P(0)$ for some. $0 \neq X \in R$ is necessary. If R is not a domain then for any prime fuzzy ideal of R, the condition is satisfied, and if R is a domain then any for any prime ideal Q of R, a function $P:R \longrightarrow I$ defined by,

$$\begin{aligned} P(0) &= 1 \\ P(x) &= t \quad \text{if } x \in Q - \{0\} \\ &= 0 \text{ if } x \notin Q \end{aligned}$$

is a prime fuzzy ideal of R. which is not two valued.

The above discussion leads to the question " Under what condition any fuzzy ideal of a ring R assumes finitely many values ? "

Proposition (2.4.10)

If R is any ring and L is a lattice of finite length with 0 and 1, then any prime fuzzy ideal $P:R \longrightarrow L$ assumes finitely many values.

Proof :

The result follows from the fact that any totally orderd subset of L is finite. In perticular P(R) is finite.

Proposition (2.4.11)

If R is both Noetherian and Artinian then every fuzzy ideal $J : R \longrightarrow I$ has only finitely many distinct values.

Proof :

If P(R) is not a finite set, then since P(R) is totally orderd , it contains

either, (i) an increasing sequence of distinct elements,

$a_1 < a_2 < a_3 < \ldots\ldots \ldots\ldots < a_n < \ldots\ldots$ with

$a_i = J(x_i)$ for some $x_i \in R$, or,

(ii) a decreasing sequence of distinct elements

$b_1 > b_2 > b_3 > \ldots\ldots > b_n > \ldots\ldots$ with

$b_i = J(y_i)$ for some $y_i \in R$

If (i) holds then we have an ascending chain of dinstinct ideals.

$$J_{a_1} \subset J_{a_2} \subset J_{a_3} \subset J_{a_4} \subset \ldots\ldots \subset J_{a_n} \subset \ldots\ldots$$

This contradicts the fact that R is Noetherian.

If(ii) holds then we can get a descending chain of distinct ideals

$$J_{b_1} \supset J_{b_2} \supset J_{b_3} \supset \ldots\ldots \supset J_{b_n} \supset \ldots\ldots$$

This contradicts fhe fact that R is Artinian.

Hence P(R) is finite.

Remark (2.4.12)

Following examples shows that it is necessary to assume that R is Noetherian and Artinian.

Example-1

Let R be a ring which is not Noetherian. Then there is an ascending chain of ideals of R,

$$J_o \subset J_1 \subset J_2 \subset J_3 \subset J_4 \subset \ldots\ldots \subset J_n \subset \ldots.$$

Define $J : R \longrightarrow I$

$$J(x) = 1 \quad \text{if} \quad x \in J_o$$

$$= \frac{1}{n+1} \text{ if } x \in J_n - J_{n-1} \qquad n \in N .$$

We show that J is a fuzzy ideal of R.

Let x, y ∈ R. be arbitrary. Let m and n be least numbers (m > n) such that $x \in J_m$ and $y \in J_n$. Then $J(x) = \frac{1}{m+1}$ and $J(y) = \frac{1}{n+1}$

Since m > n , $J_n \subseteq J_m$. Hence $x, y \in J_m \Rightarrow x-y \in J_m$ $\Rightarrow J(x-y) \geq \frac{1}{m+1}$. Also $\frac{1}{n+1} > \frac{1}{m+1}$ since m > n

Therefore $J(x-y) \geq J(x) \wedge J(y)$.

Next $x \in J_m$, $y \in J_n$ and $J_n \subset J_m$

$\Rightarrow x.y \in J_n$ since J_n is an ideal of R.

Hence $J(x.y) \geq \frac{1}{n+1}$ Also $\frac{1}{n+1} > \frac{1}{m+1}$

Therefore $J(x.y) \geq J(x) \vee J(y)$.

Thus $J : R \longrightarrow I$ is a fuzzy ideal of R. and clearly J(R) is not finite.

Example-II

Let R be a ring which is not Artinian. In perticular, we take R = Z. Then we have descending chain of ideals of Z,

$$\langle 2\rangle \supset \langle 2^2\rangle \supset \langle 2^3\rangle \supset \ldots\ldots \supset \langle 2^n\rangle \supset \ldots\ldots$$

Define,

$$J : Z \longrightarrow I$$

$$J(x) = \frac{n}{n+1} \text{ is } x \in \langle 2^n\rangle - \langle 2^{n+1}\rangle$$

$$= 0 \quad \text{if } x \notin \langle 2\rangle$$

$$= 1 \quad \text{if } x = 0$$

Then we show that J is a fuzzy ideal of R.

Let $x, y \in Z$ be arbitrary.

(a) i) if $x = y = 0$ then $x-y = 0$. Hence $J(x-y) \geqslant J(x) \wedge J(y$

ii) If $x = 0$ and $y \neq 0$ $y \in \langle 2\rangle$. Then $J(y) = \frac{n}{n+1}$ for some n. Hence $J(x-y) = J(y) = \frac{n}{n+1} \geqslant J(x) \wedge J(y)$.

iii) $x \neq 0$, $y \neq 0$ and $x, y \in \langle 2\rangle$. Let $J(x) = \frac{n}{n+1}$ and $J(y) = \frac{m}{m+1}$, $m > n$ Then $x-y \in \langle 2^n\rangle$ since $\langle 2^m\rangle \subset \langle 2^n\rangle$. Therefore $J(x-y) \geqslant \frac{n}{n+1} = \frac{n}{n+1} \wedge \frac{m}{m+1}$

$= J(x) \wedge J(y)$

iv) If $x \in \langle 2\rangle$ and $y \notin \langle 2\rangle$ then $x-y \notin \langle 2\rangle$.
Therefore $J(x-y) \geqslant J(x) \wedge J(y)$

v) If $x \notin \langle 2\rangle$, $y \notin \langle 2\rangle$. Then $x-y \notin \langle 2\rangle$ or $x-y \in \langle 2\rangle$
Hence $J(x-y) \geqslant J(x) \wedge J(y)$.

(b) (i) If $x = y = 0$ then $x \cdot y = 0$ Hence $J(x.y) \geqslant J(x) \vee J(y)$

(ii) If $x = 0, y \neq 0$ then $y = 0$, Hence $J(x.y) \geqslant j(x) \vee J(y)$.

(iii) If $x \neq 0$, $y \neq 0$, $x, y \in \langle 2 \rangle$ Let $J(x) = \frac{n}{n+1}$, $J(y) = \frac{m}{m+1}$,

$m > n$, Then $x.y \in \langle 2^m \rangle$

Therefore $J(x.y) \geqslant \frac{m}{m+1} = \frac{m}{m+1} \vee \frac{n}{n+1} = J(x) \vee J(y)$

(iv) $0 \neq x \in \langle 2 \rangle$, $y \notin \langle 2 \rangle$ Let $J(x) = \frac{n}{n+1}$ for some n. Then $x.y \in \langle 2^n \rangle$ Hence $J(x.y) \geqslant \frac{n}{n+1} = \frac{n}{n+1} \vee 0 = J(x) \vee J(y)$

(v) $x \notin \langle 2 \rangle$ $y \notin \langle 2 \rangle$ Then $x.y \notin \langle 2 \rangle$. Hence $J(x.y) \geqslant J(x) \vee J(y)$.

This shows that J is a fuzzy ideal of $\mathbb{Z}$ and clearly J is not finite valued.

<u>Definition</u> (2.4.13)

Let $J : R \longrightarrow I$ be a fuzzy set. A smallest fuzzy ideal containing J is called a fuzzy ideal generated by J and it is denoted by $\langle J \rangle$

<u>Remark</u> (2.4.14)

Let R be a ring and let $\tilde{L}$ be a set of all fuzzy ideals of R. Then $(\tilde{L}, \wedge, \vee)$ is a lattice where

$$J_1 \vee J_2 = \langle J_1 \cup J_2 \rangle$$

$$J_1 \wedge J_2 = J_1 \cap J_2 \quad \text{for all } J_1, J_2 \in \tilde{L}$$

<u>Proposition</u> (2.4.15)

If $f : R \longrightarrow R'$ is an epimorphism of ring R onto R' and if K is a fuzzy set in R, then

$$f(\langle K \rangle) = \langle f(K) \rangle$$

Proof

Clearly $f(\langle K\rangle)$ is a fuzzy ideal of R' since $\langle K\rangle$ is a fuzzy ideal of R.

Since $\langle K\rangle$ is fuzzy ideal generated by K, $K \subseteq \langle K\rangle$. Hence $f(K) \subseteq f(\langle K\rangle)$. Let J' be a fuzzy ideal of R' such that $f(K) \subseteq J'$ We show that $f(\langle K\rangle) \subseteq J'$

Now $f(K) \subseteq J' \Rightarrow f^{-1}(f(K)) \subseteq f^{-1}(J')$

$$\Rightarrow K \subseteq f^{-1}(J')$$

since $K \subseteq f^{-1}(f(K))$

Since $\langle K\rangle$ is generated by K, and $f^{-1}(J')$ is a fuzzy ideal containing K,

$$\langle K\rangle \subseteq f^{-1}(J')$$

$$\Rightarrow f(\langle K\rangle) \subseteq f(f^{-1}(J'))$$

$$\Rightarrow f(\langle K\rangle) \subseteq J'$$

Thus $f(\langle K\rangle)$ is the smallest fuzzy ideal containing $f(K)$.

Hence $\langle f(K)\rangle = f(\langle K\rangle)$.

Proposition (2.4.16)

Let $f: R \longrightarrow R'$ be an epimorphism and let $K' : R' \longrightarrow I$ be a fuzzy set in R'. Then

$$f^{-1}(\langle K'\rangle) = \langle f^{-1}(K')\rangle \text{ if either}$$

$K'(0) = 1$ or f is one-one.

Proof :

Since $\langle K'\rangle$ is a fuzzy ideal of R' generated by K', $K' \subseteq \langle K'\rangle$

$$\Rightarrow f^{-1}(K') \subseteq f^{-1}(\langle K'\rangle)$$

Let J be a fuzzy ideal of R such that $f^{-1}(K') \subseteq J$. We show that $f^{-1}(\langle K' \rangle) \subseteq J$, Now $f^{-1}(K') \subseteq J \Rightarrow$

$$f(f^{-1}(K')) \subseteq f(J)$$

$$\Rightarrow K' \subseteq f(J)$$

$$\Rightarrow \langle K' \rangle \subseteq f(J)$$ since $\langle K' \rangle$ is generated by K'

$$\Rightarrow f^{-1}(\langle K' \rangle) \subseteq f^{-1}(f(J))$$

If $K'(0) = 1$ then for $x \in \text{Ker } f$, $f^{-1}(K')(x) = K'(f(x)) = K'(0) = 1$ and since $f^{-1}(K') \subseteq J$, $J(x) = 1$. Thus if $K'(0) = 1$, then J is constant on Kernel of f. And when J is constant on Kernel of f, $f^{-1}(f(J)) = J$, Next if f is one-one then $\text{Ker} f = \{0\}$ And hence J is constant on Kernel,

Therefore,

$$f^{-1}(K') \subseteq J \Rightarrow f^{-1}(\langle K' \rangle) \subseteq J$$

which proves that $f^{-1}(\langle K' \rangle)$ is the smallest fuzzy ideal containing $f^{-1}(K')$.

i.e. $\langle f^{-1}(K') \rangle = f^{-1}(\langle K' \rangle)$

<u>Proposition</u> (2.4.17)

If $f: R \longrightarrow R'$ is an epimorphism of rings and if J_1 and J_2 are fuzzy ideals of R, then

$$f(\langle J_1 \cup J_2 \rangle) = \langle f(J_1) \cup f(J_2) \rangle$$

<u>Proof</u>:

by proposition (2.4.15)

$$f(\langle J_1 \cup J_2 \rangle) = \langle f(J_1 \cup J_2) \rangle$$
$$= \langle f(J_1) \cup f(J_2) \rangle$$

Hence, $f(\langle J_1 \cup J_2 \rangle) = \langle f(J_1) \cup f(J_2) \rangle$

Proposition (2.4.18):

Let $f : R \longrightarrow R'$ be an epimorphism of rings and let J_1' and J_2' be fuzzy ideals of R' such that $J_1' \cup J_2'$ (0) = 1 Then $f^{-1}(\langle J_1' \cup J_2' \rangle) = \langle f^{-1}(J_1') \cup f^{-1}(J_2') \rangle$

Proof :

Since $(J_1' \cup J_2')(0) = 1$, by proposition (2.4.16)

$$f^{-1}(\langle J_1' \cup J_2' \rangle) = \langle f^{-1}(J_1' \cup J_2') \rangle$$

$$= \langle f^{-1}(J_1') \cup f^{-1}(J_2') \rangle$$

Proposition (2.4.19):

A fuzzy ideal of R is finite valued if and only if it is generated by finite valued fuzzy set in R.

Proof :

Let $J : R \longrightarrow I$ be a finite valued fuzzy ideal of R. Let $\text{Im}J = \{ t_1, t_2, t_3 \; \ldots\ldots \; t_n \}$

$$t_1 \geqslant t_2 \geqslant \ldots\ldots \geqslant t_n.$$

Then we have a chain of ideals of R,

$$J_{t_1} \subseteq J_{t_2} \subseteq \ldots\ldots \subseteq J_{t_n}$$

Let $S_1, S_2 \ldots\ldots S_n$ be the sets such that S_i is a set of generators of J_{t_i} for $i = 1, 2, \ldots\ldots n$.

Define $K : R \longrightarrow I$

$$K(x) = t_1 \quad \text{if} \quad x \in S_1$$

$$= t_i \text{ if } x \in S_i - S_{i-1}$$

$$= 0 \quad \text{otherwise}$$

If $x \in R$ and $K(x) = t_i$ then $x \in S_i - S_{i-1}$

Hence $x \in J_{t_i}$ since S_i generates J_{t_i}.

$\Rightarrow$ $J(x) \geq t_i$. Thus $K(x) \leq J(x)$. Since $x \in R$ is arbitrary $K \subseteq J$.

Let $\bar{J}$ be any other fuzzy ideal of R , such that $K \subseteq \bar{J}$.

Therefore $K_{t_i} \subseteq \bar{J}_{t_i}$ for each i

$\Rightarrow$ $S_i \subseteq \bar{J}_{t_i}$

$\Rightarrow$ $J_{t_i} \subseteq \bar{J}_{t_i}$ for each i

$\Rightarrow$ $J \subseteq \bar{J}$.

Thus J is the smallest fuzzy ideal containing K i.e. $\langle K \rangle = J$. And clearly K is finite valued fuzzy set in R.

Conversely let $K : R \longrightarrow I$ be a finite valued fuzzy set in R.

Let $ImK = \{ t_1, t_2 \ldots\ldots\ldots t_n \}$ $t_1 > t_2 \ldots > t_n$

Let $S_i = K^{-1}(t_i)$ $i = 1, 2, 3 \ldots\ldots n$

Let $J_1 = \langle S_1 \rangle$, $J_2 = \langle S_1 \cup S_2 \rangle$ . . .

$$J_i = \left\langle \bigcup_{j=1}^{i} S_j \right\rangle \quad i = 1, 2 \ldots . n$$

Define $J : R \longrightarrow I$ by

$$J(x) = t_1 \quad \text{if } x \in J_1$$
$$= t_i \quad \text{if } x \in J_i - J_{i-1}$$

Then J is a fuzzy ideal of R. Let $x \in R$ and let $K(x) = t_i$

$\Rightarrow$ $x \in S_i$ $\Rightarrow$ $x \in J_r$ where r is the smallest index $r \leq i$

$\Rightarrow$ $J(x) = t_r > t_i$ $\Rightarrow$ $J(x) > t_i$

Hence $K \subseteq J$. Let $\bar{J}$ be any other fuzzy ideal of R such that $K \subseteq \bar{J} \Rightarrow K_{t_i} \subseteq \bar{J}_{t_i}$ for each i = 1,2. . . n

But $K_{t_i} = \bigcup_{j=1}^{i} S_j$ which generates $J_i = J_{t_i}$

Hence $J_{t_i} \subseteq \bar{J}_{t_i}$, i = 1,2. . . .n

$\Rightarrow J \subseteq \bar{J}$

Thus J is a smallest fuzzy ideal of R containing K i.e. $J = \langle K \rangle$ And clearly J is finite valued fuzzy ideal of R.

Proposition (2.4.20)

For an exact sequence,

$$0 \longrightarrow R' \xrightarrow{\alpha} R \xrightarrow{\beta} R'' \longrightarrow 0$$ of rings, every fuzzy ideal of R is finite valued if and only if every fuzzy ideal of R' and R'' is finite valued.

Proof :

First suppose that every fuzzy ideal of R' and R'' is finite valued. Let $J : R \longrightarrow I$ be a fuzzy ideal of R. Then $\alpha^{-1}(J)$ and $\beta(J)$ are fuzzy ideals of R' and R'' respectively and hence finite valued. Thus, since Im $\alpha^{-1}(J)$ and Im$\beta(J)$ are finite, ImJ is finite. i.e. J is finite valued fuzzy ideal. Conversely, suppose that every fuzzy, ideal of R is finite valued.

Let J'' be a fuzzy ideal of R''. Then $\beta^{-1}(J'')$ is a fuzzy ideal of R and hence Im $\beta^{-1}(J'')$ is finite i.e. ImJ'' is finite.

Next, let J' be a fuzzy ideal of R' . Then α(J') is a fuzzy set in R. Let $\langle \alpha(J') \rangle$ be a fuzzy ideal generated by α(J') . Then by hypothesis $\langle \alpha(J') \rangle$ is finite valued. By proposition (2·4·19) this implies α(J') is finite valued. Therefore ImJ' is finite since α is monomorphism. Hence the proof.