##

C H A P T E R I

INTRODUCTION

##

CHAPTER I

1.1 : Preliminary Remarks :

The theory of Integral Transforms is a classical subject in mathematics whose literature can be traced back through atleast one and half century. On the other hand, the theory of generalized functions is of recent origin. The concept of generalized integral transformations is confluence of these two mathematical streams. In this chapter we give a brief account of the elementary concepts that are required for the development of the work of dissertation.

1.2 : Integral Transforms :

Both pure and applied mathematics widely consist of use of Integral Transformations. They are used in solving some boundary value problems and integral equations. A function F(s), where s is real or complex, expressed in the form of integrals

$$F(s) = \int_{0 \text{ or } -\infty}^{\infty} k(s,x)f(x)dx \qquad \ldots(1.2.1)$$

is called integral transform of function f(x). Function k(s,x) in the integrand is called kernel of the transformation. It is assumed that integral on RHS of (1.2.1) is convergent. Different forms of kernel $k(s,x)$ and the range of integration, give rise to different integral transformations; such as Fourier, Laplace, Mellin, Hankel transformations.

The problems involving several variables can be solved by applying integral transformations successively with regard to several variables.

1.3 : Mellin Transform :

The transform theory provides a powerful technique to solve an ordinary and partial differential equation in a direct and systematic manner. A suitable integral transform is required according to the conditions of the problem. When the kernel $k(s,x)$ in (1.2.1) is x^{s-1} and the range of integration is o to ∞ we get Mellin transform. The Mellin transform and its inverse are defined by the relations,

$$F(s) = \int_0^\infty x^{s-1} f(x)\,dx \qquad \ldots(1.3.1)$$

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} F(s)\,ds \qquad \ldots(1.3.2)$$

$$s = \sigma + i\tau$$

The idea of such a reciprocity occures in Riemann's [17] famous memoir on prime numbers. It was formulated explicitely by Cahen [3], and the first accurate discussion was given by Mellin [12,13].

The integral (1.3.1) can be derived from two-sided Laplace transformation by the change of variable $x = e^{-t}$. Therefore many theorems bearing on Mellin transform can be deduced from the corresponding theorems for Laplace transform. The basic results of the theory of the Mellin transform are given by Titchmarsh [21]. Doetsch [6] has proved an interesting theorem on the analytic continuation of the Mellin transform for the product of two functions satisfying particular asymptotic relations at zero and infinity. Mikolas [14] has applied the Mellin transform to the generalized Riemann ζ-function

$$\zeta(s,u) = \sum_{n=0}^{\infty} \frac{1}{(u+n)^s}$$

to obtain a relation extending the functional equation for the Riemann ζ-function. Integral transforms of the Mellin-type, suitable for the solution of boundary value problems for the Laplace equation in polar coordinates on the plane and in spherical coordinates in space, have been analyzed by Naylor [15]. Barrucand [1] has used the formula for the inverse Mellin transform to solve integral equations. Lemon [11] has applied the Mellin transform to the solution of boundary value problems in mathematical physics. The Mellin transform and the formula for its inverse have been used by Tranter [22] to find the stress distribution in an infinite wedge, C. Fox [7,8] used them to analyze iterative integral transforms and to solve integral equations. Several problems in the theory of elasticity have been solved by means of the Mellin transform in the book of Uflyand [23]. Two dimensional Mellin transforms have been considered by Reed [16].

Following section gives inversion theorem and convolution theorems as discussed by Delavault [4].

1.4 Mellin Transform of two variables

Let $f(x,y)$ be a function of real variables x and y, where $0 < x < \infty$, $0 < y < \infty$. If for a domain in the plane of complex numbers s, t the double integral

$$F(s,t) = \int_0^\infty \int_0^\infty x^{s-1} y^{t-1} f(x,y)\, dxdy \qquad \ldots (1.4.1)$$

exists then $F(s,t)$ is called Mellin transform of $f(x,y)$ and we write in symbol

$$m\,[f(x,y)] = F(s,t)$$

We shall state the following inversion theorem due to Reed [16].

Theorem : 1.4-1

(i) If $F(s,t)$ is a holomorphic function of s and t in the domain

$$\alpha_1 < \text{Re}\,s = \alpha < \alpha_2; \qquad \gamma_1 < \text{Re}\,t = \gamma < \gamma_2$$

(ii) if $F(s,t) \longrightarrow o$ when $\sqrt{\beta^2 + \delta^2} \longrightarrow \infty$,

where $\beta = J_s$, $\delta = J_t$

(iii) if $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| F(\alpha + i\beta, \gamma + i\delta) \right| d\beta d\delta$ converges

absolutely, then $F(s,t)$ is the Mellin transform (in the direction of the principal value of the integral) of the function

$$f(x,y) = \frac{1}{(2\pi i)^2} \int_{\alpha-i\infty}^{\alpha+i\infty} \int_{\gamma-i\infty}^{\gamma+i\infty} x^{-s} y^{-t} F(s,t) ds dt \quad \ldots (1.4.2)$$

One of the important aspects of integral transformation is the transform of convolution of two functions f and g. If $m\,[f(x,y)] = F(s,t)$ $m\,[g(x,y)] = G(s,t)$ and the domains of absolute convergence have common part then we have

$$\int_0^{\infty} \int_0^{\infty} x^{s-1}\, y^{t-1}\, f(x,y) g(x,y) dx\, dy$$

$$= \frac{1}{(2\pi i)^2} \int_{\alpha-i\infty}^{\alpha+i\infty} \int_{\gamma-i\infty}^{\gamma+i\infty} F(s_1,t_1) G(s-s_1, t-t_1) ds_1 dt_1 \quad \ldots (1.4.3)$$

For $s = t = 1$, this result becomes

$$\int_0^{\infty} \int_0^{\infty} f(x,y) g(x,y) dx dy =$$

$$\frac{1}{(2\pi i)^2} \int_{\alpha-i\infty}^{\alpha+i\infty} \int_{\gamma-i\infty}^{\gamma+i\infty} F(s,t) G(1-s, 1-t) ds\, dt \quad \ldots (1.4.4)$$

Using inversion formula for Mellin transform following theorem is obtained.

Theorem 1.4-2

If $m\,[f(x,y)] = F(s,t)$, $m\,[g(x,y)] = G(s,t)$,

as before then $m^{-1}\,[F(s,t)\,G(s,t)]$

$$= \int_0^\infty \int_0^\infty g\,(\xi,\eta)\; f\,\left(\frac{x}{\xi}, \frac{y}{\eta}\right) \frac{d\xi}{\xi}\,\frac{d\eta}{\eta}$$

That is, $F(s,t)\,G(s,t) = m\left[\int_0^\infty \int_0^\infty g(\xi,\eta)\; f\left(\frac{x}{\xi}, \frac{y}{\eta}\right) \frac{d\xi}{\xi}\,\frac{d\eta}{\eta}\right]$

$$= m\left[\int_0^\infty \int_0^\infty g\left(\frac{x}{\xi},\frac{y}{\eta}\right) f(\xi,\eta)\,\frac{d\xi}{\xi}\,\frac{d\eta}{\eta}\right] \qquad \ldots(1.4.5)$$

Since $F(1-s,t) = m\left[\frac{1}{x}\; f\left(\frac{1}{x}, y\right)\right]$

we get another result

$$F(s,t)\,G(1-s,\,1-t) = m\left[\int_0^\infty \int_0^\infty g(\xi,\eta)\,f(x\xi\,,\,y\eta)\,d\xi\,d\eta\right]$$

$$\ldots\,(1.4.6)$$

1.5 : Generalized Functions and Distributions :

P. Dirac [5] introduced delta function in 1947. The idea of specifying a function not by its value but by its behaviour as a functional on some space of testing functions was a new concept. This new mode of thinking gave rise to the theory of generalized functions. As an effect the wheels of research in several branches of mathematics were put in rapid motion.

The impact of generalized functions on the integral transforms has recently revolutionised the theory of integral transformations. The foundations of the theory of generalized functions were laid by Bochner [2] and Sobolev [19]. But the work of Laurent Schwartz [18] was a systematic construction of theory of generalized functions on firm foundation (1950-51).

Generalized Functions :

Let I be an open subset of R^n or C^n; where C^n is the complex n-dimensional euclidean space. A set V(I) is said to be a testing-function space on I if the following conditions are satisfied.

1) V(I) consists entirely of smooth complex-valued functions defined on I.

2) V(I) is either a complete countably multinormed space or a complete countable-union space.

3) If sequence $\{\phi_\nu\}_{\nu=1}^{\infty}$ converges in V(I) to zero then for every non-negative integer k in R^n, $\{D^k \phi_\nu\}_{\nu=1}^{\infty}$ converges to the zero function uniformly on every compact subset of I.

A generalized function on I is any continuous linear functional on any testing function space on I. Thus f is called a generalized function, if it is a member of the dual-space V'(I) of some testing-function space V(I).

Distributions :

Let I be a nonempty openset in R^n and K be a compact subset of I. $D_K(I)$ is the set of all complex-valued smooth functions defined on I which vanish at those points of I, that are not in K, $D_K(I)$ is a linearspace under the usual definitions of addition of functions and their multiplication by complex numbers. The zero element in $D_K(I)$ is the identically, zero function on I. For each non-negative integer k in R^n define γ_k by

$$\gamma_k(\phi) = \sup_{t \in I} \left| D^k \phi(t) \right| ; \quad \phi \in D_K(I) \qquad \ldots (1.5.1)$$

Then $\{\gamma_k\}$ is a countable multinorm on $D_K(I)$. We assign to $D_K(I)$ the topology generated by $\{\gamma_k\}$ and thus $D_K(I)$ is a countably multinormed space. Moreover $D_K(I)$ is complete and hence a Frechet space. Let $\{K_m\}_{m=1}^{\infty}$ be a sequence of compact subsets of I with properties.

1) $K_1 \subset K_2 \subset K_3 \subset \ldots$

2) each compact subset of I is contained in one of the K_m.

Then $I = \bigcup_{m=1}^{\infty} K_m$ and $D_{K_m}(I) \subset D_{K_{m+1}}(I)$ and topology of $D_{K_m}(I)$ is stronger than the topology induced on it by $D_{K_{m+1}}(I)$. Now countable-union space D(I) is defined by

$$D(I) = \bigcup_{m=1}^{\infty} D_{K_m}(I) \qquad \ldots(1.5.2)$$

Its dualspace is denoted by D'(I). Members of D'(I) are called distributions on I. Thus a distribution is a continuous linear functional on space D(I).

Main advantage of generalized functions and distributions is that by widening the class of functions, many theorems and operations are freed from tedious restrictions. Generalized functions in mathematical physics are discussed by Vladimirov [24]. A treatise by Gelfand and Shilov [10] is also available.

1.6 : Mellin transform of Distributions :

The first and important achievement to the theory of generalized functions is the extension of Fourier transformation to the generalized functions. Schwartz [18] extended Laplace-transform to generalized functions in 1952. Fung Kang [9] is supposed to be the first one to discuss the generalized Mellin transformation.

A classical integral transform can be extended to generalized functions, in mainly three ways. In each case, a complete topological vectorspace of infinitely differentiable testing functions is constructed. In our discussion, the method of extension requires a testing function space V(I) containing the kernel k(s,x) and its dualspace V'(I). Then we define an integral transform F(s) of generalized functions as the application of a generalized function to the kernel function. Thus if $f \in V'(I)$ and $k(s,x) \in V(I)$ then

$$F(s) = \langle f(x), \quad k(s,x) \rangle \qquad \ldots(1.6.1)$$

According to this approach Mellin transformation of a certain type of generalized function f(x) on $o < x < \infty$ can be defined by

$$mf \triangleq F(s) \triangleq \langle f(x), \quad x^{s-1} \rangle \qquad \ldots(1.6.2)$$

Mellin transformation generates operational calculus for differential equations of the form $P(xDx)\,\mathcal{U}(x) = g(x)$; where P is a polynomial.

Srivastav and Parihar [20] have applied the generalized Mellin transformation to the theory of dual integral equations. Generalized Mellin transformation can be extended to the n-dimensional case where $x \in R^n$ and $x > o$. This is discussed by Zemanian [26]. Mellin transform of distribution f(x,y) on R^2_+ is defined by

$$(mf(x,y))\,(u,v) \triangleq F(u,v) \triangleq \langle f(x,y),\, x^{u-1}y^{v-1} \rangle \ldots(1.6.3)$$

for suitably restricted (u,v).

1.7 : Notations and Terminology :

Notations and Terminology in the present work follow from that of [25].

R^n and C^n denote the real and complex n-dimensional euclidean spaces respectively. By a compact set in R^n we mean a closed and bounded set in R^n. A conventional function is a function whose domain is contained in R^n or C^n and whose range is in either R^1 or C^1. A conventional function is said to be smooth if all its derivatives of all orders are continuous at all points of its domain.

The support of a continuous function f(t) defined on open set Ω in R^n, is the closure with respect to Ω of the set of points t where $f(t) \neq 0$. Whenever a certain equation is a definition, the symbol $\triangleq$ is used for equality. When f is a generalized function in R^2 and $(x,y) \in R^2$, the notation f(x,y) is used merely to indicate that the testing functions on which f is defined have (x,y) as their independent variable. It doesn't mean f is a function of (x,y). The symbol $\langle f, \theta \rangle$ denotes a unique complex number, assigned by the functional f in dual space to some element θ of a testing function space. An operator $\mathcal{L}$ from a linearspace $\mathcal{U}$ into linearspace V is called linear if domain of $\mathcal{L}$ is a linearspace and for every $\theta_1, \theta_2 \in \text{dom}\, \mathcal{L}$ we have for $\alpha, \beta \in C'$,

$$\mathcal{L} (\alpha\theta_1 + \beta\theta_2) = \alpha \mathcal{L} (\theta_1) + \beta \mathcal{L} (\theta_2)$$

Consequently a linear operator $\mathcal{L}$ from $\mathcal{U}$ into $\mathcal{V}$ maps the origin of $\mathcal{U}$ into the origin of $\mathcal{V}$.

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