CHAPTER II

MELLIN-TYPE CONVOLUTION AND ITS PROPERTIES

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2.1 Introduction

Zemanian [2,3] has derived properties of distributional Mellin-type convolution by using isomorphism between spaces $L_{a,b}$ and $M_{a,b}$. But in this chapter we have derived these properties of distributional Mellin-type convolution independently, that is, without taking into consideration the space $L_{a,b}$.

2.2 The Testing-function space $M_{a_1,a_2;\ b_1,b_2}$:

Let R_+^2 denote the open domain of points (x,y), $o < x < \infty$; $o < y < \infty$, in R^2. For $a = (a_1, a_2)$, $b = (b_1, b_2)$ in R^2 with $a_1 < b_1$ and $a_2 < b_2$ we define

$$\zeta_{a,b}(x,y) = \zeta_{a_1,a_2;\ b_1,b_2}(x,y)$$

$$= \begin{cases} x^{-a_1} y^{-a_2} & o < x \leqslant 1 \quad o < y \leqslant 1 \\ x^{-a_1} y^{-b_2} & o < x \leqslant 1 \quad 1 < y < \infty \\ x^{-b_1} y^{-a_2} & 1 < x < \infty \quad o < y \leqslant 1 \\ x^{-b_1} y^{-b_2} & 1 < x < \infty \quad 1 < y < \infty \end{cases}$$

$M_{a_1,a_2;\ b_1,b_2}$ is a linearspace of all smooth functions $\theta(x,y)$ defined on R_+^2 with values in C. For every non-negative integer $k = (k_1, k_2)$ we define functionals γ_k by

$$\gamma_{k_1,k_2}(\theta) \triangleq \gamma_{a_1,a_2;\ b_1,b_2;\ k_1,k_2}(\theta)$$

$$\triangleq \sup_{\substack{0<x<\infty \\ 0<y<\infty}} \left| \zeta_{a,b}(x,y)\,[x^{k_1+1}\, y^{k_2+1}]\ \frac{\partial^{k_1+k_2}}{\partial x^{k_1}\,\partial y^{k_2}}\,\theta(x,y) \right| < \infty \qquad \dots (2.2.1)$$

The γ_k are seminorms on $M_{a_1,a_2;\ b_1,b_2}$ and γ_0 is a norm. Therefore $\{\gamma_k\}_{k \geq 0}$ is a multinorm on $M_{a_1,a_2;\ b_1,b_2}$. We assign to $M_{a_1,a_2;\ b_1,b_2}$ the topology generated by this multinorm $\{\gamma_k\}_{k \geq 0}$. Any smooth function whose support is contained in R^2_+ is in $M_{a_1,a_2;\ b_1,b_2}$. Other members of the space are $x^{u-1}.\ y^{v-1}$ for $a_1 \leqslant \mathrm{Re}\,u \leqslant b_1$; $a_2 \leqslant \mathrm{Re}\,v \leqslant b_2$ and $(\log x)^{k_1}$, $(\log y)^{k_2}$, x^{u-1}, y^{v-1} for $a_1 < \mathrm{Re}\,u < b_1$; $a_2 < \mathrm{Re}\,v < b_2$.

A sequence $\{\theta_\nu\}_{\nu=1}^{\infty}$ is a Cauchy-sequence in $M_{a_1,a_2;\ b_1,b_2}$ if and only if each θ_ν is in $M_{a_1,a_2;\ b_1,b_2}$ and for each fixed non-negative integer $k = (k_1, k_2)$ the functions

$$\zeta_{a,b}(x,y)\ x^{k_1+1}\ y^{k_2+1}\ \frac{\partial^{k_1+k_2}}{\partial x^{k_1}\,\partial y^{k_2}}\ \theta_\nu(x,y)$$

Converge uniformly on R^2_+ as $\nu \to \infty$.

We shall call this type of convergence as "convergence in $M_{a_1,a_2;\ b_1,b_2}$". Here afterwards the space $M_{a_1,a_2;\ b_1,b_2}$ will be denoted simply by $M_{a,b}$; unless it is specifically mentioned. It is clear that $M_{a,b}$ is sequentially complete and hence is a Frechet-space.

2.3 The Dualspace $M'_{a,b}$ of $M_{a,b}$

The dualspace of $M_{a,b}$ is denoted by $M'_{a,b}$. Thus it consists of continuous linear functionals f on $M_{a,b}$. Equality addition and multiplication by a complex number are defined in the usual way. $M'_{a,b}$ is a linearspace over C. $\langle f, \theta \rangle$ denotes the number that $f \in M'_{a,b}$ assigns with $\theta \in M_{a,b}$.

If the support of distribution f is contained in a compact subset of R^2_+, then f is in $M'_{a,b}$. On the other hand every member of $M'_{a,b}$ is a distribution on R^2_+. We define a topology for $M'_{a,b}$ by using following set of seminorms. For each $\theta \in M_{a,b}$ we define a seminorm by

$$\gamma_\theta (f) = \left| \langle f, \theta \rangle \right| ; \quad f \in M'_{a,b} \qquad \ldots(2.3.1)$$

It follows that a sequence $\left\{ f_\nu \right\}_{\nu=1}^{\infty}$ is a Cauchy-sequence in $M'_{a,b}$ if and only if for every $\theta \in M_{a,b}$ the numerical sequence

$$\left\{ \langle f_\nu , \theta \rangle \right\}_{\nu=1}^{\infty} \text{ converges.}$$

This type of convergence we shall refer as "convergence in $M'_{a,b}$". It is clear that $M'_{a,b}$ is sequentially complete.

2.4 The Distributional Mellin Transform m :

Now we turn to the definition of distributional Mellin-transform m.

We shall say that a distribution f is m-transformable if there exists at least one pair of points $a = (a_1, a_2)$; $b = (b_1, b_2)$ in R^2 with $a_1 < b_1$; $a_2 < b_2$ such that $f \in M'_{a,b}$. For each such f there exists a unique set Ω_f in c^2 which is defined as follows.

A point $(u,v) \in C^2$ is in Ω_f if and only if there exist two points $a, b \in R^2$ with $a_1 < b_1$, $a_2 < b_2$ such that $a_1 < \text{Re}u < b_1$; $a_2 < \text{Re}v < b_2$ and $f \in M'_{a,b}$.
We assert that Ω_f is a tube. Because if $\sigma_1 + iw_1 \in \Omega_f$ for some fixed σ_1 and w_1 then $\sigma_1 + iw \in \Omega_f$ for all w. It is also clear that Ω_f is an openset.

If $x = (x_1, x_2)$; $y = (y_1, y_2) \in R^2$ such that $x_1 < y_1$; $x_2 < y_2$ and the tube $x_1 \leqslant \text{Re}u \leqslant y_1$; $x_2 \leqslant \text{Re}v \leqslant y_2$ is contained in Ω_f then $f \in M'_{x,y}$.

Let f be a Mellin transformable distribution. Its Mellin transform mf is defined as the function F(u,v) from subset Ω_f of C^2 into C^1 by

$$(mf)\ (u,v) \triangleq F(u,v) \triangleq \langle f(x,y),\ x^{u-1}y^{v-1} \rangle \qquad \ldots(2.4.1)$$

for $(u,v) \in \Omega_f$

We shall state following useful result.

Theorem : (uniqueness)

If (i) $mf = F(u,v)$; $(u,v) \in \Omega_f$, $mh = H(u,v)$ for $(u,v) \in \Omega_h$ and $\Omega_f \cap \Omega_h$ is not empty

(ii) $F(u,v) = H(u,v)$ for $(u,v) \in \Omega_f \cap \Omega_h$,

then $f = h$ in the sense of equality in $M'_{a,b}$.

2.5 Mellin-Type Convolution 'V' :

Convolution of generalized functions arises in a variety of mathematical problems and is related to the behaviour of many physical systems. There are two Mellin-type convolutions which we shall denote by 'V' and 'Λ'. The section is devoted to discussion of convolution 'V', where the generalized function are members of $M'_{a,b}$; $a \leqslant b$. Before coming to the definition of this convolution we prove the following Lemmas.

Lemma 2.5-1 :

Assume that $g \in M'_{a,b}$; $\theta \in M_{a,b}$ and $\Psi(x,y)$ is defined by

$$\Psi(x,y) = \langle g(\xi, \eta);\ \theta(x\xi, y\eta) \rangle$$

Then Ψ is smooth function and

$$\frac{\partial^{k_1+k_2}}{\partial x^{k_1}.\partial y^{k_2}} \psi(x,y) = \left\langle g(\xi,\eta); \frac{\partial^{k_1+k_2}}{\partial x^{k_1} \partial y^{k_2}} \theta(x\xi, y\eta) \right\rangle$$

<u>Proof</u> : Let $k = (k_1,k_2)$ be a non-negative integer. We shall consider two cases.

(i) one of the k_1, k_2 is zero.

(ii) both k_1, k_2 are non-zero.

For case (i) first we shall consider $k_1 = 1$, $k_2 = 0$; then increasing k_1 case (i) will be proved.

We have $\psi(x,y) = \langle g(\xi,\eta), \theta(x\xi, y\eta) \rangle$

Therefore for fixed x and $\Delta x \neq 0$ we have

$$\frac{1}{\Delta x}[\psi(x+\Delta x,y)-\psi(x,y)] - \left\langle g(\xi,\eta), \frac{\partial\theta}{\partial x}(x\xi, y\eta) \right\rangle$$

$$= \frac{1}{\Delta x}[\langle g(\xi,\eta), \theta((x+\Delta x)\xi, y\eta)\rangle - \langle g(\xi,\eta), \theta(x\xi, y\eta)\rangle]$$

$$- \left\langle g(\xi,\eta), \frac{\partial\theta}{\partial x}(x\xi, y\eta) \right\rangle \qquad \ldots(2.5.1)$$

$$= \left\langle g(\xi,\eta); \frac{1}{\Delta x}[\theta((x+\Delta x)\xi, y\eta) - \theta(x\xi, y\eta)] - \frac{\partial\theta}{\partial x}(x\xi, y\eta) \right\rangle.$$

$$= \langle g(\xi,\eta); \Phi_{\Delta x}(\xi,\eta) \rangle \quad \text{say.}$$

where $\Phi_{\Delta x}(\xi,\eta)$

$$= \frac{1}{\Delta x}[\theta((x+\Delta x)\xi, y\eta) - \theta(x\xi,y\eta)] - \frac{\partial\theta}{\partial x}(x\xi, y\eta)$$

By Taylor's theorem [1, p.43] we have

$$\theta((x+\Delta x)\xi, y\eta) = \theta(x\xi, y\eta) + \Delta x \frac{\partial\theta}{\partial x}(x\xi, y\eta)$$

$$+ \int_0^{\Delta x} (\Delta x - z) \frac{\partial^2}{\partial (x+z)^2} \theta((x+z)\xi, y\eta)\,dz$$

Therefore $\frac{1}{\Delta x}\left[\theta((x+\Delta x)\xi, y\eta) - \theta(x\xi, y\eta)\right]$

$$= \frac{\partial\theta}{\partial x}(x\xi, y\eta) + \frac{1}{\Delta x}\int_0^{\Delta x} (\Delta x - z)\frac{\partial^2}{\partial (x+z)^2}\theta((x+z)\xi, y\eta)\,dz$$

Thus $\Phi_{\Delta x}(\xi,\eta) = \frac{1}{\Delta x}\int_0^{\Delta x} (\Delta x - z)\frac{\partial^2}{\partial (x+z)^2}\theta((x+z)\xi, y\eta)\,dz$

Hence $\frac{\partial^{p_1+p_2}}{\partial\xi^{p_1}\,\partial\eta^{p_2}}\Phi_{\Delta x}(\xi,\eta)$

$$= \frac{1}{\Delta x}\int_0^{\Delta x} (\Delta x - z)\frac{\partial^{p_1+p_2}}{\partial\xi^{p_1}\,\partial\eta^{p_2}}\frac{\partial^2}{\partial (x+z)^2}\theta((x+z)\xi, y\eta)\,dz$$

But for $0 < \Delta x < 1$,

$$\zeta_{a,b}(\xi,\eta) \sup_{0<z<\Delta x}\left|\frac{\partial^{p_1+p_2}}{\partial\xi^{p_1}\,\partial\eta^{p_2}}\frac{\partial^2}{\partial (x+z)^2}\theta((x+z)\xi, y\eta)\right|$$

is bounded therefore we get

$$\left|\zeta_{a,b}(\xi,\eta)\frac{\partial^{p_1+p_2}}{\partial\xi^{p_1}\,\partial\eta^{p_2}}\Phi_{\Delta x}(\xi,\eta)\right| \leq \frac{B}{|\Delta x|}\int_0^{\Delta x}(\Delta x - z)\,dz$$

$$= \frac{B}{|\Delta x|}\cdot\frac{|\Delta x|^2}{2} = \frac{B}{2}|\Delta x|$$

Hence $\Phi_{\Delta x}(\xi,\eta)$ converges in $M_{a,b}$ to 0 as $\Delta x \to 0$. That is LHS of (2.5.1) tends to 0 as $x \to 0$. Thus

$$\frac{\partial}{\partial x}\Psi(x,y) - \langle g(\xi,\eta); \frac{\partial \theta}{\partial x}(x\xi , y\eta) \rangle = 0$$

or

$$\frac{\partial}{\partial x}\Psi (x,y) = \langle g(\xi ,\eta); \frac{\partial \theta}{\partial x}(x\xi , y\eta) \rangle$$

This means result is established for $k_1 = 1$, $k_2 = 0$ since $M_{a,b}$ is closed under differentation we may apply this result for higher values of k_1, to get

$$\frac{\partial^{k_1}}{\partial x^{k_1}}\Psi(x,y) = \langle g(\xi,\eta), \frac{\partial^{k_1}}{\partial x^{k_1}}\theta (x\xi,y\eta) \rangle$$

Case(ii) Let k_1, k_2 be both non-zero; and

$$\chi (x\xi ,y\eta) = \frac{\partial^{k_1}}{\partial x^{k_1}} \theta (x\xi , y\eta)$$

Now $\theta \in M_{a,b}$, $\chi \in M_{a,b}$ therefore using the method of Case (i) we obtain

$$\frac{\partial^{k_2}}{\partial y^{k_2}}(\chi (x\xi,y\eta)) = \frac{\partial^{k_1+k_2}}{\partial y^{k_2}\partial x^{k_1}}\theta (x\xi ,y\eta)$$

That is, $$\frac{\partial^{k_1+k_2}}{\partial y^{k_2}\partial x^{k_1}}\Psi(x,y) = \langle g(\xi,\eta), \frac{\partial^{k_1+k_2}}{\partial y^{k_2}\partial x^{k_1}}\theta(x\xi,y\eta) \rangle$$

We conclude that Ψ is a smooth function.

Lemma : 2.5-2 :

In addition to the hypothesis of lemma 2.5-1 assume that $a \leq b$ then $\Psi \in M_{a,b}$ where

$$\Psi(x,y) = \langle g(\xi,\eta), \theta (x\xi , y\eta) \rangle$$

Proof : In view of prece_ding lemma, we need merely to show that for each $k = (k_1, k_2)$;

$$\zeta_{a,b}(x,y)\,[x^{k_1+1}\,y^{k_2+1}]\,\frac{\partial^{k_1+k_2}}{\partial x^{k_1}\partial y^{k_2}}\,\Psi(x,y)$$

is bounded. We have

$$\left|\zeta_{a,b}(x,y)\,[x^{k_1+1}\,y^{k_2+1}]\,\frac{\partial^{k_1+k_2}}{\partial x^{k_1}\partial y^{k_2}}\,\Psi(x,y)\right| =$$

$$= \left|\zeta_{a,b}(x,y)[x^{k_1+1}\,y^{k_2+1}]\left\langle g(\xi,\eta);\ \frac{\partial^{k_1+k_2}}{\partial x^{k_1}\,\partial y^{k_2}}\,\theta(x\xi,y\eta)\right\rangle\right|$$

$$\leqslant C \max_{\substack{0\leqslant p_1\leqslant r\\ 0\leqslant p_2\leqslant r}}\ \sup_{\substack{0<\xi<\infty\\ 0<\eta<\infty}}\Bigg|\zeta_{a,b}(x,y).\ \zeta_{a,b}(\xi,\eta)\ .$$

$$.\ x^{k_1+1}.\ y^{k_2+1}\ .\ \xi^{p_1+1}.\ \eta^{p_2+1}\ \frac{\partial^{k_1+k_2+p_1+p_2}}{\partial\xi^{p_1}\partial\eta^{p_2}\partial x^{k_1}\partial y^{k_2}}\,\theta(x\xi,y\eta)\Bigg|$$

$$= C \max_{\substack{0\leqslant p_1\leqslant r\\ 0\leqslant p_2\leqslant r}}\ \sup_{\substack{0<\xi<\infty\\ 0<\eta<\infty}}\Bigg|\frac{\zeta_{a,b}(x,y)\ \zeta_{a,b}(\xi,\eta)}{\zeta_{a,b}(x\xi,y\eta)}\,\zeta_{a,b}(x\xi,y\eta)\ .$$

$$.\ x^{k_1+1}\,y^{k_2+1}\ \xi^{p_1+1}\,\eta^{p_2+1}\ \frac{\partial^{k_1+k_2+p_1+p_2}}{\partial\xi^{p_1}\partial\eta^{p_2}\partial x^{k_1}\partial y^{k_2}}\,\theta(x\xi,y\eta)\Bigg|$$

$$= C \left(\sup_{\substack{0<\xi<\infty \\ 0<\eta<\infty}} \left| \frac{\zeta_{a,b}(x,y)\ \zeta_{a,b}(\xi,\eta)}{\zeta_{a,b}(x\xi, y\eta)} \right| \right) .$$

$$\max_{\substack{0\leq p_1\leq r \\ 0\leq p_2\leq r}} [\ \zeta_{a,b}(x\xi,y\eta)\ x^{k_1+1}\ y^{k_2+1}\ \xi^{p_1+1}\ \eta^{p_2+1} \ .$$

$$. \frac{\partial^{k_1+k_2+p_1+p_2}}{\partial\xi^{p_1}\ \partial\eta^{p_2}\partial x^{k_1}\partial y^{k_2}}\ \theta\ (x\xi,\ y\eta)\quad]$$

But $\dfrac{\partial^{k_1+k_2+p_1+p_2}}{\partial\xi^{p_1}\partial\eta^{p_2}\partial x^{k_1}\partial y^{k_2}}\ \theta\ (x\xi\ ,y\eta)$

$$= \frac{\partial^{p_1+p_2}}{\partial\xi^{p_1}.\partial\eta^{p_2}} \left[\frac{\partial^{k_1+k_2}}{\partial x^{k_1}\partial y^{k_2}}\ \theta\ (x\xi,y\eta) \right]$$

$$= \frac{\partial^{p_1+p_2}}{\partial\xi^{p_1}\partial\eta^{p_2}} \left[\frac{\partial^{k_1+k_2}}{\partial(x\xi)^{k_1}\partial(y\eta)^{k_2}}\ \theta(x\xi,y\eta)\ \xi^{k_1}\ \eta^{k_2} \right]$$

$$= \sum_{i=0}^{p_1} \sum_{j=0}^{p_2} \binom{p_1}{i} \binom{p_2}{j} \frac{\partial^{k_1+i+k_2+j}}{\partial(x\xi)^{k_1+i}\partial(y\eta)^{k_2+j}}\ \theta\ (x\xi,y\eta)\ .$$

$$.\ x^i\ y^j\ c_{i,k_1}\ \xi^{k_1-p_1+i}\ c_{j,k_2}\ \eta^{k_2-p_2+j}$$

And $\dfrac{\zeta_{a,b}(x,y)\ \zeta_{a,b}(\xi,\eta)}{\zeta_{a,b}(x\xi\ ,y\eta)}$ is bounded by 1

Therefore

$$\sup_{\substack{0<x<\infty \\ 0<y<\infty}} \left| \zeta_{a,b}(x,y) x^{k_1+1} y^{k_2+1} \frac{\partial^{k_1+k_2}}{\partial x^{k_1} \partial y^{k_2}} \psi(x,y) \right|$$

$$\leqslant C_1 \max_{\substack{0\leqslant p_1\leqslant r \\ 0\leqslant p_2\leqslant r}} \sum_{i=0}^{p_1} \sum_{j=0}^{p_2} a_{k_1,i} \cdot b_{k_2,j} \cdot$$

$$\sup_{\substack{0<x<\infty \;\; 0<y<\infty \\ 0<\xi<\infty,\; 0<\eta<\infty}} \left| \zeta_{a,b}(x\xi, y\eta)(x\xi)^{k_1+i+1} (y\eta)^{k_2+j+1} \frac{\partial^{k_1+i+k_2+j} \theta}{\partial(x\xi)^{k_1+i} \partial(y\eta)^{k_2+j}} \right|$$

Thus $\gamma_{k_1,k_2}(\psi) \leqslant C_2 \max_{\substack{0\leqslant p_1\leqslant r \\ 0\leqslant p_2\leqslant r}} \sum_{i=0}^{p_1} \sum_{j=0}^{p_2} \gamma_{k_1+i,k_2+j}(\theta)$

$< \infty$, as $\theta \in M_{a,b}$.

Hence $\psi(x,y) \in M_{a,b}$, for $a_1 \leqslant b_1$; $a_2 \leqslant b_2$, and proof is complete.

Lemma 2.5-3 :

Let $a = (a_1, a_2)$ $b = (b_1, b_2)$ be points in R^2.

If (i) $a \leqslant b$ and $g \in M'_{a,b}$

(ii) sequence $\{\theta_\nu\}_{\nu=1}^{\infty}$ converges in $M_{a,b}$ to 0

(iii) $\psi_\nu(x,y) \triangleq \langle g(\xi,\eta), \theta_\nu(x\xi, y\eta) \rangle$

then the sequence $\{\psi_\nu\}_{\nu=1}^{\infty}$ converges in $M_{a,b}$ to zero.

Proof : Since $K((x,y),(\xi,\eta)) = \dfrac{\zeta_{a,b}(x,y)\,\zeta_{a,b}(\xi,\eta)}{\zeta_{a,b}(x\xi,\, y\eta)}$

is a bounded function from lemma 2.5-2 we have,

$$\sup_{\substack{0<x<\infty \\ 0<y<\infty}} \left| \zeta_{a,b}(x,y)\,[x^{k_1+1}\, y^{k_2+1}]\, \frac{\partial^{k_1+k_2}}{\partial x^{k_1}\partial y^{k_2}} \psi_\nu(x,y) \right|$$

$$\leqslant C_2 \max_{\substack{0\leqslant p_1\leqslant r \\ 0\leqslant p_2\leqslant r}} \sum_{i=0}^{p_1} \sum_{j=0}^{p_2} \sup_{\substack{0<x<\infty,\ 0<y<\infty \\ 0<\xi<\infty,\ 0<\eta<\infty}}$$

$$\left| \zeta_{a,b}(x\xi,y\eta)(x\xi)^{k_1+i+1}(y\eta)^{k_2+j+1} \frac{\partial^{k_1+i+k_2+j}}{\partial(x\xi)^{k_1+i}\,\partial(y\eta)^{k_2+j}} \theta_\nu(x\xi,y\eta) \right|$$

This means for every ν,

$$\gamma_{k_1,k_2}(\psi_\nu) \leqslant C_2 \max \sum_i \sum_j \gamma_{k_1+i,\, k_2+j}(\theta_\nu)$$

But by hypothesis (ii) $\theta_\nu \to 0$ as $\nu \to \infty$.

Therefore $\psi_\nu \to 0$ as $\nu \to \infty$.

Thus sequence $\{\psi_\nu\}_{\nu=1}^{\infty}$ also converges in $M_{a,b}$ to zero.

We are finally ready to define convolution.

Definition of Mellin-type convolution 'V'

Let f and g be two generalized functions in $M'_{a,b}$ and $a\leqslant b$. Then convolution of f and g denoted by $f v g$, is defined as a functional on $M_{a,b}$ by

$$\langle fvg, \theta \rangle = \langle f(x,y), \langle g(\xi,\eta), \theta(x\xi, y\eta)\rangle\rangle \qquad ...(2.5.2)$$

for $\theta \in M_{a,b}$.

RHS of (2.5.2) has a sense, since $\langle g(\xi,\eta), \theta(x\xi, y\eta)\rangle$ is a member of $M_{a,b}$ whenever $\theta \in M_{a,b}$; by lemma 2.5-2. Moreover this functional is linear on $M_{a,b}$ and its continuity follows from lemma 2.5-3.

Theorem 2.5-1

Let $a = (a_1, a_2)$ $b = (b_1, b_2) \in R^2$; $a_1 < b_1$; $a_2 < b_2$.

If f and g are members of $M'_{a,b}$ then fvg defined by (2.5.2) is also a member of $M'_{a,b}$.

Proof : By lemma 2.5-2; $\langle g(\xi,\eta), \theta(x\xi, y\eta)\rangle$ is a member of $M_{a,b}$ whenever θ is a member of $M_{a,b}$.

Now for $\theta_1, \theta_2 \in M_{a,b}$ we have

$$\begin{aligned}\langle fvg, \alpha\theta_1 + \beta\theta_2\rangle &= \langle f(x,y), \langle g(\xi,\eta), \alpha\theta_1 + \beta\theta_2\rangle\rangle \\ &= \langle f(x,y); \alpha\langle g, \theta_1\rangle + \beta\langle g, \theta_2\rangle\rangle \\ &= \alpha\langle f(x,y), \langle g(\xi,\eta), \theta_1(x\xi, y\eta)\rangle\rangle \\ &\quad + \beta\langle f(x,y), \langle g(\xi,\eta), \theta_2(x\xi, y\eta)\rangle\rangle \\ &= \alpha\langle fvg, \theta_1\rangle + \beta\langle fvg, \theta_2\rangle;\ \alpha, \beta \in C\end{aligned}$$

which shows that fvg is linear on $M_{a,b}$.

If $\psi_\nu(x,y) = \langle g(\xi,\eta),\ \theta_\nu(x\xi, y\eta)\rangle$; by lemma 2.5-3 the sequence $\{\psi_\nu\}_{\nu=1}^{\infty}$ converges to zero function in $M_{a,b}$; whenever $\theta_\nu \to o$ in $M_{a,b}$. Since f is continuous and linear $\langle f,\ \psi_\nu\rangle \to o$ as $\nu \to \infty$.

Thus $\langle fvg,\ \theta_\nu\rangle \to o$ whenever $\theta_\nu \to o$ in $M_{a,b}$ and hence fvg is continuous on $M_{a,b}$. Now fvg, being a continuous linear functional on $M_{a,b}$, is a member of $M'_{a,b}$

Theorem : 2.5-2

If $f \in M'_{a,b}$ and $g \in D(I)$ where $I = R^2_+$, then in the sense of equality in $D'(I)$; fvg is equal to a smooth function on I namely

$$(fvg)(\xi,\eta) = \left\langle f(x,y);\ \frac{1}{x}\frac{1}{y}\, g\left(\frac{\xi}{x}, \frac{\eta}{y}\right)\right\rangle$$

where $(\xi,\eta) \in I$.

Proof : We shall prove that $\left\langle f(x,y),\ \frac{1}{x}\frac{1}{y}\, g\left(\frac{\xi}{x}, \frac{\eta}{y}\right)\right\rangle$ is a smooth function. Let $k = (k_1, k_2)$ be a non-negative integer. First suppose that $k_1 = 1$, $k_2 = 0$, and

$$\psi(\xi,\eta) = \left\langle f(x,y),\ \frac{1}{x}\frac{1}{y}\, g\left(\frac{\xi}{x}, \frac{\eta}{y}\right)\right\rangle$$

for ξ fixed and $\Delta\xi \neq 0$ we have

$$\frac{1}{\Delta\xi}\left[\psi(\xi+\Delta\xi,\eta) - \psi(\xi,\eta)\right] - \left\langle f(x,y),\ \frac{1}{x}\frac{1}{y}\frac{\partial g}{\partial\xi}\left(\frac{\xi}{x}, \frac{\eta}{y}\right)\right\rangle$$

$$= \frac{1}{\Delta\xi}\left[\left\langle f(x,y);\ \frac{1}{x}\frac{1}{y}\, g\left(\frac{\xi+\Delta\xi}{x}, \frac{\eta}{y}\right)\right\rangle - \left\langle f(x,y);\ \frac{1}{x}\frac{1}{y}\, g\left(\frac{\xi}{x}, \frac{\eta}{y}\right)\right\rangle\right]$$

$$- \left\langle f(x,y);\ \frac{1}{x}\frac{1}{y}\ \frac{\partial g}{\partial \xi}\left(\frac{\xi}{x}, \frac{\eta}{y}\right)\right\rangle$$

$$= \left\langle f(x,y),\ \frac{1}{\Delta\xi}\ \frac{1}{x}\frac{1}{y}\left[g\left(\frac{\xi+\Delta\xi}{x}, \frac{\eta}{y}\right) - g\left(\frac{\xi}{x}, \frac{\eta}{y}\right)\right] - \frac{1}{x}\frac{1}{y}\ \frac{\partial g}{\partial \xi}\left(\frac{\xi}{x}, \frac{\eta}{y}\right)\right\rangle$$

$$= \left\langle f(x,y);\ \Phi_{\Delta\xi}(x,y)\right\rangle; \quad \text{say.}$$

By Taylor's theorem we have

$$g\left(\frac{\xi+\Delta\xi}{x}, \frac{\eta}{y}\right) = g\left(\frac{\xi}{x}, \frac{\eta}{y}\right) + \Delta\xi \frac{\partial g}{\partial \xi}\left(\frac{\xi}{x}, \frac{\eta}{y}\right) + \int_0^{\Delta\xi} (\Delta\xi - z)\ \frac{\partial^2}{\partial(\xi+z)^2}\, g\left(\frac{\xi+z}{x}, \frac{\eta}{y}\right) dz$$

Therefore $\frac{1}{x}\frac{1}{y}\left[g\left(\frac{\xi+\Delta\xi}{x}, \frac{\eta}{y}\right) - g\left(\frac{\xi}{x}, \frac{\eta}{y}\right)\right]$

$$= \frac{\Delta\xi}{xy}\ \frac{\partial g}{\partial \xi}\left(\frac{\xi}{x}, \frac{\eta}{y}\right) + \frac{1}{xy}\int_0^{\Delta\xi} (\Delta\xi - z)\ \frac{\partial^2}{\partial(\xi+z)^2}\, g\left(\frac{\xi+z}{x}, \frac{\eta}{y}\right) dz$$

or $\frac{1}{\Delta\xi}\left(\frac{1}{x}\frac{1}{y}\right)\left[g\left(\frac{\xi+\Delta\xi}{x}, \frac{\eta}{y}\right) - g\left(\frac{\xi}{x}, \frac{\eta}{y}\right)\right]$

$$= \frac{1}{xy}\ \frac{\partial g}{\partial \xi}\left(\frac{\xi}{x}, \frac{\eta}{y}\right) + \frac{1}{xy\,\Delta\xi}\int_0^{\Delta\xi} (\Delta\xi - z)\frac{\partial^2}{\partial(\xi+z)^2}\, g\left(\frac{\xi+z}{x}, \frac{\eta}{y}\right) dz$$

Hence $\Phi_{\Delta\xi}(x,y) = \frac{1}{xy\,\Delta\xi}\int_0^{\Delta\xi} (\Delta\xi - z)\ \frac{\partial^2}{\partial(\xi+z)^2}\, g\left(\frac{\xi+z}{x}, \frac{\eta}{y}\right) dz$

consequently $\frac{\partial^{p_1+p_2}}{\partial x^{p_1}\partial y^{p_2}} \Phi_{\Delta\xi}(x,y)$

$$= \frac{\partial^{p_1+p_2}}{\partial x^{p_1}\partial y^{p_2}} \left[\frac{1}{xy\,\Delta\xi} \int_{o}^{\Delta\xi} (\Delta\xi - z) \frac{\partial^2}{\partial(\xi+z)^2} g\left(\frac{\xi+z}{x}, \frac{\eta}{y}\right) dz \right]$$

$$= \frac{1}{\Delta\xi} \int_{o}^{\Delta\xi} (\Delta\xi - z) \frac{\partial^{p_1+p_2}}{\partial x^{p_1}\partial y^{p_2}} \left[\frac{1}{x}\frac{1}{y} \frac{\partial^2}{\partial(\xi+z)^2} g\left(\frac{\xi+z}{x}, \frac{\eta}{y}\right) \right] dz$$

But for $o < \Delta\xi < 1$,

$$\zeta_{a,b}(x,y) \sup_{o<z<\Delta\xi} \left| \frac{\partial^{p_1+p_2}}{\partial x^{p_1}\,\partial y^{p_2}} \left[\frac{1}{xy} \frac{\partial^2}{\partial(\xi+z)^2} g\left(\frac{\xi+z}{x}, \frac{\eta}{y}\right) \right] \right|$$

is bounded and therefore we get

$$\left| \zeta_{a,b}(x,y) \frac{\partial^{p_1+p_2}}{\partial x^{p_1}\partial y^{p_2}} \Phi_{\Delta\xi}(x,y) \right|$$

$$\leqslant \frac{B}{|\Delta\xi|} \int_{o}^{\Delta\xi} (\Delta\xi - z)\, dz = \frac{B}{2} |\Delta\xi|$$

This shows that $\Phi_{\Delta\xi}(x,y)$ converges to o in D(I) as $\Delta\xi \to o$.

$$\Rightarrow \frac{\partial}{\partial\xi}\Psi(\xi,\eta) - \left\langle f(x,y), \frac{1}{xy} \frac{\partial g}{\partial\xi}\left(\frac{\xi}{x}, \frac{\eta}{y}\right) \right\rangle$$

tends to o as $\Delta\xi \to o$.

$$\Rightarrow \frac{\partial}{\partial\xi}\Psi(\xi,\eta) = \left\langle f(x,y), \frac{1}{xy} \frac{\partial g}{\partial\xi}\left(\frac{\xi}{x}, \frac{\eta}{y}\right) \right\rangle$$

Thus result is established for $k_1 = 1$, $k_2 = 0$. Since D(I) is closed under differentiation, we can apply repeatedly this result for higher values of k_1; to have

$$\frac{\partial^{k_1}}{\partial \xi^{k_1}} \Psi(\xi,\eta) = \left\langle f(x,y), \frac{1}{xy} \frac{\partial^{k_1}}{\partial \xi^{k_1}} g\left(\frac{\xi}{x}, \frac{\eta}{y}\right)\right\rangle$$

For $k_2 \neq 0$, $k_1 \neq 0$ we take

$$\chi\left(\frac{\xi}{x}, \frac{\eta}{y}\right) = \frac{\partial^{k_1}}{\partial \xi^{k_1}} g\left(\frac{\xi}{x}, \frac{\eta}{y}\right)$$

Since $g \in D(I)$, $\chi \in D(I)$ using the same method as above we obtain

$$\frac{\partial^{k_2}}{\partial \eta^{k_2}} \chi\left(\frac{\xi}{x}, \frac{\eta}{y}\right) = \frac{\partial^{k_1+k_2}}{\partial \eta^{k_2} \partial \xi^{k_1}} g\left(\frac{\xi}{x}, \frac{\eta}{y}\right)$$

That is $\dfrac{\partial^{k_1+k_2}}{\partial \eta^{k_2} \partial \xi^{k_1}} \Psi(\xi,\eta) =$

$$\left\langle f(x,y), \frac{\partial^{k_1+k_2}}{\partial_\eta^{k_2} \partial \xi^{k_1}} g\left(\frac{\xi}{x}, \frac{\eta}{y}\right)\right\rangle$$

We conclude that Ψ is a smooth function,

2.6 Mellin Transform of convolution 'V'

The following theorem states the exchange formula for Mellin transformation and two properties of Mellin-type convolution that can be derived from the exchange formula.

Theorem 2.6-1

If $mf(x,y) = F(u,v)$ for $(u,v) \in \Omega_f$, $mg(x,y) = G(u,v)$ for $(u,v) \in \Omega_g$ and $\Omega_f \cap \Omega_g$ is nonempty then $f \vee g$ exists in the sense of Mellin-type convolution in $M'_{a,b}$ and $m(f \vee g) = F(u,v)\,G(u,v)$

Moreover (i) $f \vee g = g \vee f$ (commutativity)

(ii) if $mh = H(u,v)$ for $(u,v) \in \Omega_h$

and $\Omega_f \cap \Omega_g \cap \Omega_h$ is nonempty then

$f \vee (g \vee h) = (f \vee g) \vee h$ (associativity)

Proof :

By Lemma 2.5-2 if $a \leqslant b$ then

$\Psi(x,y) = \langle g(\xi,\eta);\ \theta(x\xi, y\eta) \rangle$ is a member of $M_{a,b}$.

Then by theorem 2.5-1, $f \vee g$ defined by

$$\langle f \vee g, \theta \rangle = \langle f(x,y)\ ;\ \langle g(\xi,\eta),\ \theta(x\xi, y\eta) \rangle\rangle$$

is a member of $M'_{a,b}$.

$$\text{Now } m(f \vee g) = \langle f(x,y), \langle g(\xi,\eta)\ (x\xi)^{u-1}(y\eta)^{v-1} \rangle\rangle$$

$$= \langle f(x,y),\ x^{u-1}y^{v-1} \rangle \langle g(\xi,\eta),\ \xi^{u-1}\eta^{v-1} \rangle$$

$$= F(u,v)\ G(u,v)$$

$$\text{Also } m(g \vee f) = \langle g(\xi,\eta), \langle f(x,y),\ (x\xi)^{u-1}(y\eta)^{v-1} \rangle\rangle$$

$$= \langle g(\xi,\eta),\ \xi^{u-1}\eta^{v-1} \rangle \langle f(x,y),\ x^{u-1}y^{v-1} \rangle$$

$$= G(u,v)\ F(u,v)$$

Therefore by uniqueness theorem we get

$f \vee g = g \vee f$; which proves result (i) .

Now choose $(u,v) \in \Omega_f \cap \Omega_g \cap \Omega_h$ then as above we get

$$m\,[fv(gvh)] = F(u,v)\; G(u,v)\; H(u,v)$$
$$= m\,[(fvg)\; vh]$$

The uniqueness theorem implies again that

$$fv\,(gvh) = (fvg)\; vh \quad \text{in} \quad M'_{a,b}\,.$$

2.7 A Second Mellin-type convolution '$\wedge$'

Before defining this Mellin-type convolution we prove the following lemma

Lemma 2.7-1 : If $f(x,y) \in M'_{a,b}$ then the function $\frac{1}{x}\frac{1}{y} f\,(\frac{1}{x}, \frac{1}{y}) \in M'_{1-b,\,1-a}$.

Proof : We have $\left\langle \frac{1}{x}\frac{1}{y} f\,(\frac{1}{x}\frac{1}{y}),\; \phi\,(x,y) \right\rangle$

$$= \left\langle f(x,y),\; \frac{1}{x}\frac{1}{y}\; \phi\,(\frac{1}{x}, \frac{1}{y}) \right\rangle$$

Therefore we require to prove that

$$\phi\,(x,y) \in M_{a,b} \Rightarrow \frac{1}{x}\frac{1}{y}\phi\,(\frac{1}{x}\frac{1}{y}) \in M_{1-b,a-a}.$$

Consider

$$\gamma_{1-b,1-a,k_1,k_2}\,\left(\frac{1}{x}\frac{1}{y}\phi\,(\frac{1}{x},\frac{1}{y})\right)$$

$$= \sup \left| \zeta_{1-b,1-a}(x,y)\; x^{k_1+1}\, y^{k_2+1} \frac{\partial^{k_1+k_2}}{\partial_x^{k_1}\,\partial y^{k_2}} \left(\frac{1}{x}\frac{1}{y}\,\phi(\frac{1}{x},\frac{1}{y})\right) \right| \qquad \ldots(2.7.1)$$

For $k_1 \neq 0$, and $k_2 = 0$

$$x^{k_1+1} y^{k_2+1} \frac{\partial^{k_1+k_2}}{\partial x^{k_1} \partial y^{k_2}} \left[\frac{1}{x} \frac{1}{y} \phi \left(\frac{1}{x} \frac{1}{y}\right) \right]$$

$$= x^{k_1+1} . y \frac{\partial^{k_1}}{\partial x^{k_1}} \left[\frac{1}{x} \frac{1}{y} \phi \left(\frac{1}{x}, \frac{1}{y} \right) \right]$$

$$= x^{k_1+1} \frac{\partial^{k_1}}{\partial x^{k_1}} \left(\frac{1}{x} \phi \left(\frac{1}{x}, \frac{1}{y}\right) \right)$$

$$= x^{k_1+1} \left[(-1)^{k_1} \frac{1}{x^{k_1+1}} \sum_{i=o}^{k_1} \frac{1}{x^i} C_i \frac{\partial^i}{\partial (\frac{1}{x})^i} \phi\left(\frac{1}{x}, \frac{1}{y}\right) \right]$$

$$= (-1)^{k_1} \sum_{i=o}^{k_1} \frac{1}{x^i} C_i \frac{\partial^i}{\partial (\frac{1}{x})^i} \phi \left(\frac{1}{x}, \frac{1}{y}\right)$$

Therefore for $k_1 \neq o$ and $k_2 \neq o$ we obtain

$$x^{k_1+1} . y^{k_2+1} \frac{\partial^{k_1+k_2}}{\partial x^{k_1} \partial y^{k_2}} \left[\frac{1}{x} \frac{1}{y} \phi \left(\frac{1}{x}, \frac{1}{y}\right) \right]$$

$$= (-1)^{k_1+k_2} \sum_{i=o}^{k_1} \sum_{j=o}^{k_2} C_{ij} \frac{1}{x^i} \frac{1}{y^i} \frac{\partial^{i+j}}{\partial (\frac{1}{x})^i \partial (\frac{1}{y})^j} \phi \left(\frac{1}{x}, \frac{1}{y}\right)$$

Also, $\zeta_{1-b,\ 1-a}(x,y) = \frac{1}{x} \frac{1}{y} \zeta_{a,b} \left(\frac{1}{x} , \frac{1}{y} \right)$

Using these results in (2.7.1) we get

$$\gamma_{1-b,\ 1-a,\ k_1,k_2} \left(\frac{1}{x} \frac{1}{y} \phi \left(\frac{1}{x}, \frac{1}{y}\right) \right) =$$

$$= \operatorname{Sup} \left| \zeta_{a,b}\left(\frac{1}{x},\frac{1}{y}\right) \sum_{i=o}^{k_1} \sum_{j=o}^{k_2} C_{ij} \frac{1}{x^{i+1}} \frac{1}{y^{j+1}} \frac{\partial^{i+j}}{\partial(\frac{1}{x})^i \partial(\frac{1}{y})^j} \phi\left(\frac{1}{x}\frac{1}{y}\right) \right|$$

$$= \operatorname{Sup} \sum_{i=o}^{k_1} \sum_{j=o}^{k_2} C_{ij} \left| \zeta_{a,b}\left(\frac{1}{x},\frac{1}{y}\right) \left(\frac{1}{x}\right)^{i+1}\left(\frac{1}{y}\right)^{j+1} \frac{\partial^{i+j}}{\partial(\frac{1}{x})^i \partial(\frac{1}{y})^j} \phi\left(\frac{1}{x}, \frac{1}{y}\right) \right|$$

$$= \sum_{i=o}^{k_1} \sum_{j=o}^{k_2} C_{ij} \gamma_{a,b,i,j}(\phi)$$

$< \infty$ as $\phi \in M_{a,b}$.

Hence $\frac{1}{x}\frac{1}{y}\phi\left(\frac{1}{x}, \frac{1}{y}\right) \in M_{1-b,1-a}$.

It follows that $f \in M'_{a,b}$ implies that

$$\frac{1}{x}\frac{1}{y} f\left(\frac{1}{x}, \frac{1}{y}\right) \in M'_{1-b,\, 1-a}.$$

Remark : It is clear that $f(x,y) \in M'_{1-b,1-a}$

$\Longrightarrow \frac{1}{x}\frac{1}{y}f\left(\frac{1}{x}, \frac{1}{y}\right) \in M'_{a,b}$.

Definition of second Mellin-type convolution '$\wedge$'

Let $a = (a_1, a_2)$, $b = (b_1, b_2)$ be in R^2 such that $a_1 < b_1$, $a_2 < b_2$. If $f \in M'_{1-b,\, 1-a}$ and $g \in M'_{a,b}$; we define second Mellin-type convolution product $f \wedge g$ of f and g by

$$(f \wedge g)(x,y) \triangleq \left[\frac{1}{x}\frac{1}{y} f\left(\frac{1}{x},\frac{1}{y}\right)\right] V\ g(x,y) \qquad \ldots(2.7.2)$$

That is for any $\theta \in M_{a,b}$,

$$\langle f\wedge g,\ \theta \rangle \doteq \langle \frac{1}{x}\frac{1}{y} f(\frac{1}{x},\frac{1}{y}),\ \langle g(\xi,\eta),\ \theta(x\xi, y\eta)\rangle\rangle$$

$$= \langle f(x,y),\ \langle g(\xi,\eta),\ \frac{1}{x}\frac{1}{y}\theta(\frac{\xi}{x},\frac{\eta}{y})\rangle\rangle$$

Right hand side of (2.7.2) has a sense because

$\frac{1}{x}\frac{1}{y} f(\frac{1}{x},\frac{1}{y}) \in M'_{a,b}$ when $f(x,y) \in M'_{1-b,1-a}$; by lemma 2.7-1, and $\langle g(\xi,\eta), \frac{1}{x}\frac{1}{y}\theta(\frac{\xi}{x},\frac{\eta}{y})\rangle$ is a member of $M_{1-b,1-a}$.

The next theorem gives two results about second Mellin-type convolution ' $\wedge$ '.

<u>Theorem : 2.7-1</u> :

(i) Convolution product $f \wedge g$ defined by (2.7.2) is a member of $M'_{a,b}$.

(ii) If $mf = F(u,v)$ for atleast $1-b_1 < Re u < 1-a_1$, $1-b_2 < Re v < 1-a_2$ and $mg = G(u,v)$ for atleast $a_1 < Re u < b_1$; $a_2 < Re v < b_2$ then

$$m(f\wedge g) = F(1-u,\ 1-v)G(u,v)$$

<u>Proof</u> : (i) Since $f \in M'_{1-b;\,1-a,}$ by lemma 2.7-1 $\frac{1}{x}\frac{1}{y} f(\frac{1}{x},\frac{1}{y}) \in M'_{a,b}$. Then by using defn. (2.7.2) and theorem 2.5-1 we conclude that $f\wedge g$ is a member of $M'_{a,b}$.

(ii) By lemma 2.5-2 for $g \in M'_{a,b}$,

$$\Psi(x,y) = \langle g(\xi,\eta), \theta(x\xi, y\eta) \rangle$$

is a member of $M_{a,b}$.

Hence $\frac{1}{x}\frac{1}{y}\Psi(\frac{1}{x}\frac{1}{y}) \in M_{1-b,1-a}$ by lemma 2.7-1.

That is, $\langle g(\xi, \eta), \frac{1}{x}\frac{1}{y}\theta(\frac{\xi}{x}, \frac{\eta}{y}) \rangle$

is a member of $M_{1-b, 1-a}$.

By part (i) above $f \wedge g$ defined by

$$\langle f \wedge g, \theta \rangle = \langle f(x,y), \langle g(\xi,\eta), \frac{1}{x}\frac{1}{y}\theta(\frac{\xi}{x}, \frac{\eta}{y}) \rangle\rangle$$

is a member of $M'_{a,b}$.

Now consider $m(f \wedge g)(u,v)$

$$= \langle (f \wedge g)(x,y), x^{u-1} y^{v-1} \rangle$$

$$= \langle f(x,y), \langle g(\xi,\eta), \frac{1}{x}\frac{1}{y}(\frac{\xi}{x})^{u-1}(\frac{\eta}{y})^{v-1} \rangle\rangle$$

$$= \langle f(x,y), \langle g(\xi,\eta), \frac{\xi^{u-1}\eta^{v-1}}{x^u y^v} \rangle\rangle$$

$$= \langle f(x,y), x^{-u}y^{-v} \rangle \langle g(\xi,\eta) \quad \xi^{u-1}\eta^{v-1} \rangle$$

$$= F(1-u, 1-v)\, G(u,v)$$

Thus we have proved

$$m(f \wedge g)(u,v) = F(1-u, 1-v)\, G(u,v)$$

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