CHAPTER III

SUBSTITUTION THEOREMS FOR DISTRIBUTIONAL MELLIN TRANSFORMATION

CHAPTER III

3.1 Introduction :

In this chapter we shall prove substitution theorems for distributional Mellin Transformation with two variables. These theorems are similar to those given by Buschman [1]. For this, first we shall define space $L_{a,b}$ and establish isomorphism between spaces $M_{a,b}$ and $L_{a,b}$. Then using the isomorphism between spaces $M_{a,b}$ and $L_{a,b}$; we derive the results.

3.2 Testing function space $L_{a,b}$ and its Dual :

Suppose $a = (a_1, a_2)$, $b = (b_1, b_2)$, $(s,t) \in R^2$; $a < b$. we define a function

$$K_{a,b}(s,t) = K_{a_1,a_2;\, b_1,b_2}(s,t)$$

$$= \begin{cases} e^{a_1 s} \cdot e^{a_2 t} & 0 \leqslant s < \infty, \quad 0 \leqslant t < \infty \\ e^{a_1 s} \cdot e^{b_2 t} & 0 \leqslant s < \infty, \quad -\infty < t < 0 \\ e^{b_1 s} \cdot e^{a_2 t} & -\infty < s < 0, \quad 0 \leqslant t < \infty \\ e^{b_1 s} \cdot e^{b_2 t} & -\infty < s < 0, \quad -\infty < t < 0 \end{cases}$$

Let $L_{a_1,a_2;\, b_1,b_2}$ denote the linear space of all complex-valued smooth functions $\phi(s,t)$ on R^2; such that for non-negative integer $k = (k_1, k_2)$ the functionals λ_k satisfy

$$\lambda_k(\phi) \triangleq \lambda_{a_1, a_2; b_1, b_2; k_1, k_2}(\phi)$$

$$\triangleq \sup_{\substack{-\infty < s < \infty \\ -\infty < t < \infty}} \left| K_{a,b}(s,t) \frac{\partial^{k_1+k_2}}{\partial s^{k_1} \partial t^{k_2}} \phi(s,t) \right| < \infty \quad \ldots (3.2.1)$$

The collection $\{\lambda_k\}_{k \geq 0}$ is a collection of seminorms, and further a multinorm on $L_{a_1, a_2; b_1, b_2}$. If we assign to $L_{a_1,a_2; b_1,b_2}$ the topology generated by this multinorm then it becomes a multinormed space. $e^{-su} e^{-tv}$ is a member of $L_{a_1,a_2; b_1,b_2}$ if and only if $a_1 \leqslant Re u \leqslant b_1$; $a_2 \leqslant Re v \leqslant b_2$.

A sequence $\{\phi_\nu\}_{\nu=1}^{\infty}$ is a Cauchy-sequence in $L_{a_1,a_2;b_1,b_2}$ if and only if each ϕ_ν is in $L_{a_1,a_2; b_1,b_2}$ and for each fixed non-negative integer $K = (k_1,k_2)$ the functions

$$K_{a,b}(s,t) \frac{\partial^{k_1+k_2}}{\partial s^{k_1} \partial t^{k_2}} \phi(s,t)$$

converge uniformly on R^2 as $\nu \longrightarrow \infty$. This type of convergence we shall refer as convergence in $L_{a_1,a_2; b_1,b_2}$. Hereafterwards the space $L_{a_1,a_2;b_1,b_2}$ will be denoted simply by $L_{a,b}$, unless it is specifically mentioned. It is clear that $L_{a,b}$ is sequentially complete and hence a Frechet space.

Dual space of $L_{a,b}$ is denoted by $L'_{a,b}$. Thus $L'_{a,b}$ consists of continuous linear functionals on $L_{a,b}$. Since $L_{a,b}$

is a testing-function space, $L'_{a,b}$ is space of generalized functions. We assign to $L'_{a,b}$ its customary (weak) topology. It follows that $L'_{a,b}$ is also complete.

3.3 Laplace Transformation $\mathcal{L}$:

Now we shall define Laplace transformation $\mathcal{L}$ of a distribution f. We shall say that a distribution f is $\mathcal{L}$-transformable if there exists at least one pair of points $a = (a_1, a_2)$, $b = (b_1, b_2)$ in R^2 with $a_1 < b_1$; $a_2 < b_2$ such that $f \in L'_{a,b}$. For each such f there exists a unique set Ω_f in c^2 which is defined as follows.

A point $(u,v) \in c^2$ is in Ω_f if and only if there exist two points $a = (a_1, a_2)$, $b = (b_1, b_2)$ in R^2 with $a_1 < b_1$; $a_2 < b_2$ such that

$a_1 < \text{Re}u < b_1$; $a_2 < \text{Re}v < b_2$ and $f \in L'_{a,b}$. Ω_f is a tube, since if $\sigma_1 + iw_1 \in \Omega_f$ for some fixed σ_1 and w_1, then $\sigma_1 + iw \in \Omega_f$ for all w. Moreover Ω_f is an open set.

Note that if $x = (x_1, x_2)$, $y = (y_1, y_2) \in R^2$ such that the tube $x_1 \leqslant \text{Re}u \leqslant y_1$, $x_2 \leqslant \text{Re}v \leqslant y_2$ is contained in Ω_f, then $f \in L'_{x,y}$.

Let $f(s,t)$ be a $\mathcal{L}$-transformable distribution. Its Laplace transform $\mathcal{L}f$ is the function $F(u,v)$ from Ω_f into c^1 defined by

$$(\mathcal{L}f)(u,v) \triangleq F(u,v) \triangleq \langle f(s,t), e^{-us} e^{-vt} \rangle \qquad \dots (3.3.1)$$

The RHS of (3.3.1) has a sense as the application of $f \in L'_{a,b}$ to $e^{-us} e^{-vt} \in L_{a,b}$ so long as $(s,t) \in \Omega_f$. Note that this definition is independent of the choices of a and b.

Lemma 3.3.1 :

Let $\mathcal{L}[f(s,t)] = F(u,v)$, for $a_1 < Re u < b_1$ and $a_2 < Re v < b_2$ and $\Psi(u,v;s,t) \in L_{a,b}$ such that

$$\Psi(u,v;s,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi(u,v;\xi,\eta)\ e^{-\xi s} e^{-\eta t}\, d\xi\, d\eta$$

then $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\langle f(s,t),\ e^{-\xi s} e^{-\eta t} \right\rangle \psi(s,t;\xi,\eta)\, d\xi\, d\eta$$

$$= \left\langle f(s,t);\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi(s,t;\xi,\eta)\ e^{-\xi s} e^{-\eta t} d\xi\, d\eta \right\rangle \quad \ldots(3.3.2)$$

Proof : Since $\Psi(u,v,;s,t) \in L_{a,b}$ is of rapid descent by similar arguments as that for lemma 3.5-1 [2, p.64] we have for fixed reals r and r'

$$\int_{-r}^{r}\int_{-r'}^{r'} \left\langle f(s,t),\ e^{-\xi s} e^{-\eta t} \right\rangle \psi(u,v;\xi,\eta) d\xi d\eta$$

$$= \left\langle f(s,t),\ \int_{-r}^{r}\int_{-r'}^{r'} \psi(u,v;\xi,\eta)\ e^{-\xi s} e^{-\eta t}\, d\xi\, d\eta \right\rangle \ldots \quad (3.3.3)$$

where $r = (r_1, r_2)$ and $r' = (r'_1\ r'_2)$ satisfy $a_1 < r_1 < b_1$, $a_2 < r_2 < b_2$; $a_1 < r'_1 < b_1$; $a_2 < r'_2 < b_2$. Moreover since $\psi(s,t;\xi,\eta)$ is of rapid descent while $\left\langle f(s,t),\ e^{-\xi s} e^{-\eta t} \right\rangle$

is bounded by a polynomial in $|\xi||\eta|$, the integral on LHS of (3.3.3) converges to the integral on LHS of (3.3.2) as $r \to \infty$, $r' \to \infty$.

Also

$$\left| K_{a,b}(s,t) \frac{\partial^{k_1+k_2}}{\partial s^{k_1} \partial t^{k_2}} \left[\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} - \int_{-r}^{r}\int_{-r'}^{r'} \right) \psi(u,v;\xi,\eta)\, e^{-\xi s} e^{-\eta t}\, d\xi d\eta \right| $$

$$= \left| K_{a,b}(s,t) \frac{\partial^{k_1+k_2}}{\partial s^{k_1} \partial t^{k_2}} \left[\left(\int_{-\infty}^{\infty}\int_{r'}^{\infty} + \int_{-\infty}^{\infty}\int_{-\infty}^{-r'} + \int_{r}^{\infty}\int_{-r'}^{r'} + \int_{-\infty}^{-r}\int_{-r'}^{r'} \right) \psi(u,v,\xi,\eta)\, e^{-\xi s} e^{-\eta t}\, d\xi\, d\eta \right] \right|$$

$$\leqslant \left| K_{a,b}(s,t) \frac{\partial^{k_1+k_2}}{\partial s^{k_1} \partial t^{k_2}} \int_{-\infty}^{\infty}\int_{r'}^{\infty} \psi(u,v;\xi,\eta)\, e^{-s\xi} e^{-t\eta}\, d\xi\, d\eta \right|$$

$$+ \left| K_{a,b}(s,t) \frac{\partial^{k_1+k_2}}{\partial s^{k_1} \partial t^{k_2}} \int_{-\infty}^{\infty}\int_{-\infty}^{-r'} \psi(u,v;\xi,\eta)\, e^{-s\xi} e^{-t\eta} d\xi d\eta \right|$$

$$+ \left| K_{a,b}(s,t) \frac{\partial^{k_1+k_2}}{\partial s^{k_1} \partial t^{k_2}} \int_{r}^{\infty}\int_{-r'}^{r} \psi(u,v;\xi,\eta)\, e^{-s\xi} e^{-t\eta}\, d\xi\, d\eta \right|$$

$$+ \left| K_{a,b}(s,t) \frac{\partial^{k_1+k_2}}{\partial s^{k_1} \partial t^{k_2}} \int_{-\infty}^{-r}\int_{-r'}^{r} \psi(u,v;\xi,\eta) e^{-s\xi} e^{-t\eta}\, d\xi d\eta \right| \qquad \ldots(3.3.4)$$

Consider first term on RHS of this inequality (3.3.4).

$$\left| K_{a,b}(s,t) \frac{\partial^{k_1+k_2}}{\partial s^{k_1} \partial t^{k_2}} \int_{-\infty}^{\infty} \int_{r'}^{\infty} \psi(u,v,\xi,\eta)\, e^{-s\xi} e^{-t\eta} d\xi d\eta \right|$$

$$= \left| \int_{-\infty}^{\infty} \int_{r'}^{\infty} K_{a,b}(s,t) \psi(u,v;\xi,\eta)\, (-1)^{k_1+k_2} \xi^{k_1} \eta^{k_2} e^{-s\xi} e^{-t\eta} d\xi d\eta \right|$$

$$\leqslant \int_{-\infty}^{\infty} \int_{r'}^{\infty} B_{k_1,k_2} \left| \psi(u,v;\ \xi,\eta)\ \xi^{k_1} \eta^{k_2} \right| d\xi d\eta$$

where B_{k_1,k_2} are constants.

Since $\psi(u,v,\xi,\eta)$ is of rapid descent, the above integral, which is independent of s and t, vanishes as $r' \to \infty$.

Arguing similarly for other terms on RHS of (3.3.4) we finally obtain as $r \to \infty$, $r' \to \infty$

$$\int_{-r}^{r} \int_{-r'}^{r'} \psi(u,v;\ \xi,\eta)\ e^{-\xi s} e^{-\eta t} d\xi d\eta$$

$$\longrightarrow \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(u,v;\xi,\eta)\ e^{-\xi s} e^{-\eta t}\ d\xi d\eta$$

in the space $L_{a,b}$. Proof of the lemma is now complete.

Isomorphism between spaces $M_{a,b}$ and $L_{a,b}$ is established in the following theorem.

Theorem 3.3-1

Let $(x,y) \in R^2_+$ and $(s,t) \in R^2$ be related by $x = e^{-s}$, $y = e^{-t}$. Then the mapping

$$\theta(x,y) \longleftrightarrow e^{-s}e^{-t}\,\theta(e^{-s},e^{-t}) \triangleq \phi(s,t) \qquad \ldots(3.4.5)$$

is an isomorphism of $M_{a,b}$ onto $L_{a,b}$.

The inverse mapping is given by

$$\phi(s,t) \longleftrightarrow x^{-1}y^{-1}\,\phi(-\log x, -\log y) = \theta(x,y) \qquad \ldots(3.4.6)$$

Proof : Let $\eta_1 : \theta(x,y) \longleftrightarrow e^{-s}e^{-t}\,\theta(e^{-s},e^{-t})$ and

$\eta_2 : \phi(s,t) \longleftrightarrow x^{-1}y^{-1}\,\phi(-\log x, -\log y)$.

Then $(\eta_2 \circ \eta_1)(\theta) = \eta_2[\eta_1(\theta)]$

$= \eta_2[e^{-s}e^{-t}\,\theta(e^{-s},e^{-t})]$

$= x^{-1}y^{-1}\Psi(-\log x, -\log y)$ where

$\Psi(s,t) = e^{-s}e^{-t}\,\theta(e^{-s},e^{-t})$

Therefore $(\eta_2 \circ \eta_1)(\theta) = x^{-1}y^{-1}e^{\log x}e^{\log y}\,\theta(e^{\log x}, e^{\log y})$

$= x^{-1}y^{-1}xy\,\theta(x,y)$

$= \theta(x,y)$.

Similarly it can be shown that $\eta_1 \circ \eta_2(\phi) = \phi$.

Thus η_1 and η_2 are inverses of each other. Hence they are one-to-one and onto. Further it is clear that these mappings are linear. What remains to prove is continuity of η_1 and η_2.

Let $\theta(x,y) \in M_{a,b}$. If we compute

$$\frac{\partial^{k_1+k_2}}{\partial s^{k_1} \partial t^{k_2}} \left[e^{-s} e^{-t} \theta(e^{-s}, e^{-t}) \right] \text{ then we get}$$

$$\sum_{p_1} \sum_{p_2} a_p \, x^{p_1+1} y^{p_2+1} \frac{\partial^{p_1+p_2}}{\partial x^{p_1} \partial y^{p_2}} \theta(x,y)$$

where a_p are constants and $p = (p_1, p_2)$.

Therefore $K_{a,b}(s,t) \dfrac{\partial^{k_1+k_2}}{\partial s^{k_1} \partial t^{k_2}} \left[e^{-s} e^{-t} \theta(e^{-s}, e^{-t}) \right]$

$$= \sum_{p_1} \sum_{p_2} a_p \, \zeta_{a,b}(x,y) \, x^{p_1+1} y^{p_2+1} \frac{\partial^{p_1+p_2}}{\partial x^{p_1} \partial y^{p_2}} \theta(x,y)$$

Hence $\lambda_{a_1,a_2;\, b_1,b_2;\, k_1,k_2}(\phi)$

$$= \lambda_{a_1,a_2;\, b_1,b_2;\, k_1,k_2} \left[e^{-s} e^{-t} \theta(e^{-s}, e^{-t}) \right]$$

$$\leqslant \sum_{p_1} \sum_{p_2} |a_p| \, \gamma_{a_1,a_2;\, b_1,b_2;\, p_1,p_2}(\theta)$$

Consequently η_1 is a continuous mapping of $M_{a,b}$ onto $L_{a,b}$

Now assume $\phi(s,t) \in L_{a,b}$. We have

$$x^{k_1+1} y^{k_2+1} \frac{\partial^{k_1+k_2}}{\partial x^{k_1} \partial y^{k_2}} \left[x^{-1} y^{-1} \phi(-\log x, -\log y) \right]$$

$$= \sum_{p_1} \sum_{p_2} b_p \frac{\partial^{p_1+p_2}}{\partial s^{p_1} \partial t^{p_2}} \phi(s,t)$$

Therefore $\zeta_{a,b}(x,y)x^{k_1+1}y^{k_2+1}\frac{\partial^{k_1+k_2}}{\partial x^{k_1}\partial y^{k_2}}[x^{-1}y^{-1}\phi(-\log x,-\log y)]$

$$= \sum_{p_1}\sum_{p_2} b_p K_{a,b}(s,t)\frac{\partial^{p_1+p_2}}{\partial s^{p_1}\partial t^{p_2}}\phi(s,t) .$$

Consequently $\gamma_{a_1,a_2;\, b_1,b_2;\, k_1k_2}(\theta)$

$$\leqslant \sum_{p_1}\sum_{p_2}|b_p|\ \lambda_{a_1,a_2;\, b_1,b_2;\, p_1,p_2}(\phi),$$

where b_p are constants. This proves continuity of η_2 from $L_{a,b}$ onto $M_{a,b}$. It follows that space $M_{a,b}$ is isomorphic to space $L_{a,b}$, and proof of theorem is complete.

The next theorem relates dual spaces $M'_{a,b}$ and $L'_{a,b}$.

Theorem 3.3-2 :

The mapping $f(x,y) \longmapsto f(e^{-s}, e^{-t})$ defined by

$$\langle f(e^{-s}, e^{-t}), \phi(s,t)\rangle \triangleq \langle f(x,y), \theta(x,y)\rangle$$

is an isomorphism from $M'_{a,b}$ onto $L'_{a,b}$. The inverse mapping is given by

$$\langle g(-\log x, -\log y), \theta(x,y)\rangle \triangleq \langle g(s,t), \phi(s,t)\rangle$$

Proof : For each $f(x,y) \in M'_{a,b}$ we can associate a functional $f(e^{-s},e^{-t})$ on $L_{a,b}$ such that

$$\langle f(e^{-s}, e^{-t}); \phi(s,t)\rangle = \langle f(x,y), \theta(x,y)\rangle$$

This means the mapping $f(x,y) \longleftrightarrow f(e^{-s},e^{-t})$ is the adjoint mapping of $\emptyset(s,t) \longleftrightarrow \theta(x,y)$. It follows that $f(x,y) \longleftrightarrow f(e^{-s},e^{-t})$ is an isomorphism from $M'_{a,b}$ onto $L'_{a,b}$.

Now suppose $g(s,t) \in L'_{a,b}$. We associate functional $g(-\log x, -\log y)$ on $M_{a,b}$ with $g(s,t)$ by

$$\langle g(-\log x, -\log y), \theta(x,y) \rangle = \langle g(s,t), \emptyset(s,t) \rangle$$

Therefore the mapping $g(s,t) \longleftrightarrow g(-\log x, -\log y)$ is the inverse mapping of $f(x,y) \longleftrightarrow f(e^{-s}, e^{-t})$

It is isomorphism from $L'_{a,b}$ onto $M'_{a,b}$.

Under the mapping defined in theorem 3.3-1 the functions e^{-su}, e^{-tv} correspond to the functions x^{u-1}, y^{v-1} respectively. Therefore by using theorem 3.3-2, we get following theorem.

<u>Theorem 3.3-3</u> :

The distribution $f(x,y)$ is m-transformable if and only if $f(e^{-s}, e^{-t})$ is £-transformable. In this case the two tubes of definition of $m[f(x,y)]$ and £ $[f(e^{-s},e^{-t})]$ coincide and $m[f(x,y)] = F(u,v) =$ £ $[f(e^{-s}, e^{-t})]$ for $(u,v) \in \Omega_f$.

3.4 Substitution Theorems :

Let A, K be single valued analytic functions real on R^2_+, and G, H, $G^{-1} = g$, $H^{-1} = h$ be single valued analytic functions real on (o,∞) such that

$$G(0) = 0, \quad H(0) = 0 \quad \text{and} \quad G(\infty) = \infty, \quad H(\infty) = \infty$$

$$(\text{ or } G(0) = \infty, \; H(0) = \infty, \text{ and } G(\infty) = 0, \quad H(\infty) = 0)$$

Let $m\,[f(x,y)] = F(u,v)$, $m[A(x,y)f(x,y)] = F^*(u,v)$

for $a_1 < \text{Re}\,u < b_1$; $a_2 < \text{Re}\,v < b_2$

Also suppose that

$$\pounds\,[\psi(u,v;\,\xi,\eta)\,] = \Psi\,(u,v;\,p,q) \quad \text{and}$$

$$\pounds\,[\,\psi^*(u,v;\,\xi,\eta)] = \Psi^*(u,v;\,p,q) \quad \text{where}$$

$$\Psi\,(u,v;\,-\log x,-\log y) = [g(x)]^{u-1}[h(y)]^{v-1}K(g(x),h(y)).$$

$$|g'(x)|\,|h'(y)|\,xy$$

and $\Psi^*\,(u,v;\,-\log x,\,-\log y)$

$$= [g(x)]^{u-1}[h(y)]^{v-1}K(g(x),h(y))\,|g'(x)|\,|h'(y)|\,xy\,[A(x,y)]^{-1}.$$

The following theorem is obtained from lemma 3.3-1 by using isomorphism between $M_{a,b}$ and $L_{a,b}$.

Theorem 3.4-1 :

Let $m\,[f(x,y)] = f(u,v)$ for $a_1 < \text{Re}\,u < b_1$ and $a_2 < \text{Re}\,v < b_2$. Let $\Psi\,(u,v;\,p,q)$ be in $L_{a,b}$ such that

$\Psi(u,v;\,p,q) =$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi(u,v;\ \xi,\eta)\ e^{-s\xi}\ e^{-t\eta}\ d\xi d\eta\ ,$$

then

$$\int_{0}^{\infty} \int_{0}^{\infty} \left\langle f(x,y),\ x^{\xi-1}\ y^{\eta-1} \right\rangle\ \Psi(u,v;\ \xi,\eta) d\xi d\eta$$

$$= \left\langle f(x,y),\ \int_{0}^{\infty} \int_{0}^{\infty} \Psi(u,v;\ \xi,\eta)\ x^{\xi-1}\ y^{\eta-1}\ d\xi d\eta \right\rangle$$

Now we shall prove our main result.

Theorem 3.4-2 :

Let $m\,[f(x,y)] = F(u,v)$ for $a_1 < \text{Re}u < b_1$ and $a_2 < \text{Re}v < b_2$. Let $K(x,y)$ be a suitably chosen single-valued analytic function on R^2_{+}, and G, H, $G^{-1} = g$, $H^{-1} = h$ be single valued analytic functions real on (o, ∞) such that

$G(O) = O$, $H(O) = O$; and $G(\infty) = \infty$, $H(\infty) = \infty$

(or $G(O) = \infty$, $H(O) = \infty$ and $G(\infty) = O$, $H(\infty) = O$)

then $m\,[K(x,y)\ f(G(x),\ H(y))\ ]$

$$= \int_{0}^{\infty} \int_{0}^{\infty} \left\langle f(x,y);\ x^{\xi-1} y^{\eta-1} \right\rangle \Psi(u,v;\xi,\eta)\ d\xi d\eta$$

where $\pounds\,[\Psi(u,v;\xi,\eta)] = \Psi(u,v;\ p,q)$

and $\Psi(u,v;\ -\log x,\ -\log y)$

$$= [g(x)]^{u-1}\ [h(y)]^{v-1}\ K(g(x),h(y)).\ |g'(x)|\ |h'(y)|\ xy$$

Proof : Let $c = (c_1, c_2)$, $d = (d_1, d_2)$, $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$ be in R^2. Let $\theta(x,y)$ be a member of $M_{c,d}$. where $a < c$, $d < b$. Then the mapping $\theta(x,y) \longleftrightarrow \phi(s,t)$ is an isomorphism of $M_{c,d}$ onto $L_{c,d}$ by theorem 3.3-1. The mapping $\phi(s,t) \longleftrightarrow K(s,t)\phi(s,t)$ is an isomorphism from $L_{c,d}$ onto $L_{\alpha,\beta}$ where $\alpha < c$, $d < \beta$. Again applying theorem 3.3-1, we see that $K(x,y)\ \theta(x,y) \longleftrightarrow K(s,t)\ \phi(s,t)$ is an isomorphism from $M_{\alpha,\beta}$ onto $L_{\alpha,\beta}$. Hence for suitably chosen $K(x,y)$ the mapping $\theta(x,y) \longleftrightarrow K(x,y)\ \theta(x,y)$ is an isomorphism from $M_{c,d}$, onto $M_{\alpha,\beta}$ for $\alpha < c$, $d < \beta$. Further in accordance with section 2.5 [2] it follows that

$f(x,y) \longleftrightarrow K(x,y)\ f(x,y)$ is an isomorphism from $M'_{\alpha,\beta}$ onto $M'_{c,d}$ and we have

$$\langle K(x,y) f(x,y),\ \theta(x,y) \rangle = \langle f(x,y),\ K(x,y)\ \theta(x,y) \rangle$$

Therefore if $mf = F(u,v)$; $\alpha_1 < \text{Re}u < \beta_1$, $\alpha_2 < \text{Re}v < \beta_2$ the equation

$$\langle K(x,y) f(x,y),\ x^{u-1}y^{v-1} \rangle = \langle f(x,y),\ K(x,y)\ x^{u-1}y^{v-1} \rangle$$

has sense. Indeed we have

$f(x,y) \in M'_{\alpha,\beta}$, $K(x,y)\ x^{u-1}y^{v-1} \in M_{\alpha,\beta}$,

$k(x,y)\ f(x,y) \in M'_{c,d}$ and $x^{u-1}y^{v-1} \in M_{c,d}$

If $\chi(x,y) = f(G(x), H(y)) \in M'_{\alpha,\beta}$ then $\chi(x,y) \longleftrightarrow K(x,y)\chi(x,y)$ is an isomorphism from $M'_{\alpha,\beta}$ onto $M'_{c,d}$ and we can write,

$$\left\langle K(x,y)\ f(G(x),\ H(y)),\ \ x^{u-1}\ y^{v-1}\right\rangle$$

$$= \left\langle f(G(x),\ H(y)),\ k(x,y)x^{u-1}y^{v-1}\right\rangle \qquad \ldots(3.4.1)$$

Here $f(G(x),\ H(y)) \in M'_{\alpha,\beta}$, $K(x,y)x^{u-1}y^{v-1} \in M_{\alpha,\beta}$ $k(x,y)\ f(G(x),\ H(y)) \in M'_{c,d}$ and $x^{u-1}y^{v-1} \in M_{c,d}$.

Let $K(x,y)\ \Theta(x,y) = \eta_1(x,y)$ be an arbitrary member of $M_{\alpha,\beta}$. Choose $a,b \in R^2$, $a < \alpha$, $\beta < b$ such that

$$\eta_1\ [g(x),\ h(y)\ ]\ |g'(x)|\ |h'(y)| \in M_{a,b}\ .$$

Let $K(s,t)\ \phi(s,t) = \eta_2(s,t) \in L_{\alpha,\beta}$. By theorem 3.3-1 the mapping $\eta_1(x,y) \longleftrightarrow \eta_2(s,t)$ is an isomorphism from $M_{\alpha,\beta}$ onto $L_{\alpha,\beta}$. Then the mapping $\eta_2(s,t) \longleftrightarrow [g(s)][h(t)]$. $|g'(s)|\,|h'(t)|$ is an isomorphism from $L_{\alpha,\beta}$ onto $L_{a,b}$. Again applying theorem 3.3-1 we see that the mapping

$$\eta_1\ [g(x),\ h(y)\ ]\ |g'(x)|\,|h'(y)| \longleftrightarrow$$

$$\eta_2\ [g(s),\ h(t)]\ |g'(s)|\,|h'(t)|$$ is an isomorphism from $M_{a,b}$ onto $L_{a,b}$. Hence the mapping

$$\eta_1(x,y) \longleftrightarrow \eta_1(g(x),\ h(y))\ |g'(x)|\,|h'(y)|$$

is an isomorphism from $M_{\alpha,\beta}$ onto $M_{a,b}$. We denote the adjoint mapping

$$\eta_1(x,y) \longleftrightarrow \eta_1\ [g(x),\ h(y)\ ]\ |g'(x)|\,|h'(y)|$$ by

$$f(x,y) \longleftrightarrow f(G(x),\ H(y)\ )$$

Then we can write $\left\langle f(G(x),\ H(y)),\ \ \eta_1(x,y)\right\rangle$

$$= \left\langle f(x,y),\ \ \eta_1[g(x),\ h(y)\ ]\ |g'(x)|\,|h'(y)|\right\rangle$$

By similar arguments to theorem 1.10-2 [2] the mapping $f(x,y) \longleftrightarrow f(G(x), H(y))$ is an isomorphism from $M'_{a,b}$ onto $M'_{\alpha,\beta}$.

Therefore if $m[f(x,y)] = F(u,v)$, $a_1 < \text{Re}u < b_1$ $a_2 < \text{Re}v < b_2$, then the equation

$$\langle f(G(x), H(y)), k(x,y)\, x^{u-1} y^{v-1} \rangle$$

$$= \langle f(x,y); K(g(x), h(y)) [g(x)]^{u-1} [h(y)]^{v-1} |g'(x)|\, |h'(y)| \rangle$$

...(3.4.2)

has sense. Indeed we have

$$f(x,y) \in M'_{a,b}, \quad K(g(x), h(y)) [g(x)]^{u-1} |g'(x)| [h(y)]^{v-1} |h'(y)| \in M_{a,b},$$

$f(G(x), H(y)) \in M'_{\alpha,\beta}$ and $K(x,y)\, x^{u-1} y^{v-1} \in M_{\alpha,\beta}$.

From (3.4.1) and (3.4.2) we conclude that

$f(x,y) \longleftrightarrow K(x,y) f(G(x), H(y))$ is an isomorphism from $M'_{a,b}$ onto $M'_{c,d}$ where $a < c$ and $d < b$ and we write

$$\langle K(x,y) f(G(x), H(y)), x^{u-1} y^{v-1} \rangle$$

$$= \langle f(x,y), K(g(x), h(y) [g(x)^{u-1} [h(y)]^{v-1} |g'(x)|\, |h'(y)| \rangle$$

... (3.4.3)

Indeed we have $f(x,y) \in M'_{a,b}$,

$K(g(x), h(y)) [g(x)]^{u-1} [h(y)]^{v-1} |g'(x)|\, |h'(y)| \in M_{a,b}$

$K(x,y)\, f(G(x), H(y)) \in M'_{c,d}, \quad x^{u-1} y^{v-1} \in M_{c,d}$

The equation (3.4.3) further can be written as

$$m\,[\, K(x,y)\, f(G(x), H(y))\,]$$

$$= \left\langle f(x,y),\ K(g(x),h(y))\ [g(x)]^{u-1}[h(y)]^{v-1}\ |g'(x)|\,|h'(y)|\ xy\ \frac{1}{xy} \right\rangle$$

$$= \left\langle f(x,y);\ \Psi(u,v;\ -\log x,\ -\log y)\ \frac{1}{xy} \right\rangle$$

$$= \left\langle f(x,y),\ \int_0^\infty \int_0^\infty \Psi(u,v,\xi,\eta).\ e^{-\xi \log(1/x)}\ e^{-\eta \log(1/y)} \frac{1}{xy}\, dxdy \right\rangle$$

$$= \left\langle f(x,y),\ \int_0^\infty \int_0^\infty \Psi(u,v;\xi,\eta)\ x^{\xi-1}\, y^{\eta-1}\, d\xi d\eta \right\rangle$$

$$= \int_0^\infty \int_0^\infty \left\langle f(x,y),\ x^{\xi-1}\, y^{\eta-1} \right\rangle \Psi(u,v;\ \xi,\eta)\ d\xi d\eta$$

by using theorem 3.4-1.

Theorem 3.4-3 :

Let $m\,[A(x,y)\ f(x,y)\,] = F^*(u,v)$ where $a_1 < \mathrm{Re}\,u < b_1$, $a_2 < \mathrm{Re}\,v < b_2$ then

$$m\ [K(x,y)\ f(G(x), H(y)\ ]$$

$$= \int_0^\infty \int_0^\infty \left\langle f(x,y);\ x^{\xi-1}\, y^{\eta-1} \right\rangle \Psi^*(u,v;\ \xi,\eta)\ d\xi d\eta$$

where A, K are single-valued analytic functions real on R^2_+ and G, H, $G^{-1} = g$, $H^{-1} = h$ are single-valued analytic functions real on $(0,\infty)$ such that

$G(0) = 0$, $H(0) = 0$; and $G(\infty) = \infty$, $H(\infty) = \infty$

(or $G(0) = \infty$, $H(0) = \infty$, and $G(\infty) = 0$, $H(\infty) = 0$)

and $\mathcal{L} [\psi^* (u,v; \xi,\eta) = \Psi^* (u,v; p, q)$

where $\Psi^* (u,v, -\log x, -\log y)$

$$= [g(x)]^{u-1} [h(y)]^{v-1} K(g(x), h(y)) \left|g'(x)\right|\left|h'(y)\right| xy[A(x,y)]^{-1}.$$

Proof : The proof of this theorem follows on the same lines as that of theorem 3.4-2.

Theorem 3.4-4 :

Let $m [f(x,y)] = F(u,v)$, $a_1 < \mathrm{Re}\, u < b_1$ and $a_2 < \mathrm{Re}\, v < b_2$; then

$$m^{-1} [K(u,v) F(G(u), H(v))]$$

$$= \int_0^\infty \int_0^\infty f(\xi ,\eta) \,\theta\, (x,y; \xi,\eta) \,d\xi d\eta \,.$$

where K, G, H are analytic functions and

$$m [\theta (x,y; \xi,\eta)] = K (u,v) \,\xi^{G(u)-1} . \eta^{H(v)-1}$$

$$= K (u,v) \,\Phi\, [G(u), H(v); \xi,\eta]$$

Proof :

$$F(P,Q) = \langle f(\xi ,\eta), \xi^{P-1} \eta^{Q-1} \rangle$$

$$= \langle f(\xi ,\eta) ; \Phi (P, Q; \xi, \eta) \rangle$$

Therefore $F(G(u), H(v)) =$

$$\left\langle f(\xi,\eta),\ \Phi\ (G(u), H(v);\ \xi, \eta) \right\rangle$$

Therefore $K(u,v)\ F\ (G(u), H(v)\ )$

$$= \left\langle f(\xi,\eta),\ K(u,v)\ \Phi\ (G(u), H(v); \xi, \eta) \right\rangle$$

$$= \left\langle f(\xi,\eta),\ \int_0^\infty \int_0^\infty \Theta\ (x,y;\ \xi,\eta)\ x^{u-1}y^{v-1}\ dxdy \right\rangle$$

$$= \left\langle \int_0^\infty \int_0^\infty f(\xi,\eta)\ \Theta\ (x,y;\ \xi,\eta) d\xi\, d\eta,\ x^{u-1}y^{v-1} \right\rangle$$

by using theorem 3.4-1.

$$= m\ [\ \int_0^\infty \int_0^\infty f(\xi,\eta)\ \Theta\ (x,y;\ \xi,\eta)\ d\xi\ d\eta\ ]$$

Consequently

$$m^{-1}\ [K(u,v)\ F(G(u), H(v))\ ]$$
$$= \int_0^\infty \int_0^\infty f(\xi,\eta)\ \Theta\ (x,y;\ \xi,\eta)\ d\xi d\eta\ .$$

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