CHAPTER ONE

INTRODUCTION

1.1 The special functions of mathematical physics arise in the solution of partial differential equations governing the behavi--our of certain physical quantities. Probably the most frequently occuring equation of this type in physical sciences is laplace's equation

$$\nabla^2 \Psi = 0 \qquad (1.1.1)$$

satisfied by a certain function Ψ describing a physical situation under discussion. This was mentioned by Laplace, the great French mathematician, in his memoir, published in 1785. Legendre, to whom Laplce had communicated his famous potential theorem, investigated the expansion of the simple term of potential in the form of an infinite series and was thus led some time in 1784, to the discovery of Legendre Coefficients which were later on known as Legendre polynomials.

Bessel functions were first introduced by the mathema--tical astronomer, F.W.Bessel in 1824, in the investigation of a perturbative function in dynamical astronomy. Thereafter, the fun--ctions appeared in physical sciences almost as frequently as the circular functions.

It is the hypergeometric series

$$1 + \frac{a.b}{1.c} x + \frac{a\,(a+1)\quad b(b+1)}{1.2\ c(c+1)} x^2 + \frac{a(a+1)\ (a+2)\ b(b+1)(b+2)}{1.2.3.c\ (c+1)(c+2)} x^3 + \cdots \qquad (1.1.2)$$

which defines the pypergeometric function ${}_2F_1$ (a,b;c;x). It was Gauss [17] who made a detailed study of this function and published the work in 1876. The famous Gauss' Theorem gives us the value of the function when $x = 1$ in terms of Gamma functions. The Gauss' hyper--geometric series has been generalized in different ways by various mathematicians from time to time.

1.2 In this section we list some of the important and well known polynomials, in the field of special functions, which have been studied by the mathematicians like R.P.Agarwal, W.A.Al-Salam, H.Bateman , P.E.Bedient, B.R.Bhonsale, R.P.Boas, Jr. and R.C.Buck, Fred Brafman, L.Carlitz, D.J.Dickinson, Fasenmyer, E.D.Rainville, S.O.Rice, P.R. Khandekar, K.M.Pradhan, M.T.Shah, N.K.Thakare and B.K.Karande and several other workers.

The Legendre polynomials defined by

$$W(x) = P_n(X) = \sum_{K=0}^{[n/2]} \frac{(-1)^K \; (\tfrac{1}{2})_{n-k} \; (2X)^{n-2k}}{K! \; (n-2k)!} \qquad (1.2.1)$$

is a solution of the differential equation

$$(1-x^2)\, W''(x) - 2x\, W'(x) + n(n+1)\, W(x) = 0. \qquad (1.2.2)$$

The hypergeometric forms of Pn (X) are

$$Pn\,(X) = {}_2F_1\left[\begin{matrix} -n, & n+1; \\ 1; & \frac{1-X}{2} \end{matrix}\right] \qquad (1.2.3)$$

$$= (-1)^n \, {}_2F_1\left[\begin{matrix} -n, & n+1; \\ 1; & \frac{1+X}{2} \end{matrix}\right] \qquad (1.2.4)$$

The Hermite polynomials Hn (X) defined by the relation

$$W(X) = Hn\,(x) = \sum_{K=0}^{[n/2]} \frac{(-1)^k \; n! \, (2X)^{n-2k}}{K! \; (n-2k)!} \qquad (1.2.5)$$

satisfy the differential equation

$$W''(x) - 2X\, W'(X) + 2n\, W(X) = 0. \qquad (1.2.6)$$

The Hermite polynomials can also be defined as

$$H_n(X) = (2X)^n \; {}_2F_0\left[\begin{matrix} -\tfrac{1}{2}n, \; -\tfrac{1}{2}n+\tfrac{1}{2}; \\ \text{———} \;\; ; \end{matrix} \; -\frac{1}{X^2}\right]. \quad (1.2.7)$$

The generalized Laguerre or Sonine polynomials defined for n a non-negative integer as

$$W(x) = L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!} \; {}_1F_1\left[\begin{matrix} -n \; ; \\ 1+\alpha \; ; \end{matrix} \; x\right] \quad (1.2.8)$$

satisfy the following differential equation

$$XW''(X) + (1+\alpha - X)\, W'(X) + n\, W(X) = 0. \quad (1.2.9)$$

When $\alpha = 0$ we get Laguerre or simple Laguerre polynomials

$$L_n(X) = L_n^{(0)}(X) = {}_1F_1\left[\begin{matrix} -n; \\ 1; \end{matrix} \; X\right] \quad (1.2.10)$$

The Jacobi polynomials $P_n^{(\alpha,\beta)}(X)$ may be defined by

$$W(X) = P_n^{(\alpha,\beta)}(X) = \frac{(1+\alpha)_n}{n!} \; {}_2F_1\left[\begin{matrix} -n, \; 1+\alpha+\beta+n; \\ 1+\alpha; \end{matrix} \; \frac{1-X}{2}\right] \quad (1.2.11)$$

which satisfy the differential equation

$$(1-X^2)\, W''(X) + \left[\beta - \alpha - (2+\alpha+\beta)X\right] W'(X) + n(1+\alpha+\beta+n)\, W(X) = 0. \quad (1.2.12)$$

When $\alpha = \beta = 0$ the above polynomials reduce to Legendre polynomials (1.2.3). When $\alpha = \beta = \nu - \tfrac{1}{2}$ the Jacobi polynomials (1.2.11) reduce to the Gegenbauer polynomials $C_n^{\nu}(X)$ defined by

$$C_n^{\nu}(X) = \frac{(2\nu)_n}{n!} \; {}_2F_1\left[\begin{matrix} -n, \; 2\nu+n \; ; \\ \nu+\tfrac{1}{2}, \end{matrix} \; \frac{1-X}{2}\right]. \quad (1.2.13)$$

On putting $\alpha = \beta = -\tfrac{1}{2}$ in (1.2.11) we get

$$P_n^{(-\frac{1}{2}, -\frac{1}{2})}(X) = \frac{(\frac{1}{2})_n}{n!} \; {}_2F_1\left[\begin{matrix} -n, \; n; \\ \tfrac{1}{2} \; ; \end{matrix} \; \frac{1-X}{2}\right] \quad (1.2.14)$$

which reduce to Tchebicheff polynomials of first kind defined as

$$T_n(X) = \frac{n!}{(\frac{1}{2})_n} P_n^{(-\frac{1}{2},\,-\frac{1}{2})}(X) = {}_2F_1\left[\begin{matrix}-n, & n; \\ \frac{1}{2}, & \end{matrix}\ \frac{1-X}{2}\right]. \quad (1.2.15)$$

In 1936 Bateman [5] while constructing inverse Laplace transforms introduced the polynomials

$$Zn(X) = {}_2F_2\left[\begin{matrix}-n, & n+1; \\ 1, & 1;\end{matrix}\ X\right] \quad (1.2.16)$$

which are called as Bateman's polynomials.

Khandekar [20] introduced generalized Rice's polynomials in the form.

$$Hn^{(a,b)}(\xi, \sigma; X) = \frac{(1+a)_n}{n!}\ {}_3F_2\left[\begin{matrix}-n, n+a+b+1, \xi; \\ 1+a, \sigma;\end{matrix}\ X\right] \quad (1.2.17)$$

Which reduce to Rice's polynomials [24] on putting a = b = 0.

Shah [30] defined the generalized Sister Celine's polynomials by the relation

$$f_n^{(a,b)}\left(\begin{matrix}a_2, \dots, a_p; \\ b_2, \dots, b_q;\end{matrix}\ x\right) = \frac{(1+a)_n}{n!}\ {}_{p+1}F_{q+1}\left[\begin{matrix}-n, \ n+a+b+1, \ a_2, \dots, a_p; \\ 1+a, \ \frac{1}{2}, b_2, \dots, b_q;\end{matrix}\ x\right] \quad (1.2.18)$$

Which reduce to Sister Celine's polynomials given by Fasenmyer [16] on putting a = b = 0.

Rainville [22] has studied the generalized Bessel poly--nomials which are of the form

$$y_n(t) = {}_2F_2\left[\begin{matrix}-n, & 2\nu + n; \\ \nu+\frac{1}{2}, & 1+b;\end{matrix}\ t\right] \quad (1.2.19)$$

Bedient [6] in his study of some polynomials associated with Appell's F_2 and F_3 functions introduced Bedient polynomials $\lambda_n(\beta, \gamma;\ X)$ and $Gn(\alpha, \beta;\ X)$ defined by the relations

$$R_n\ (\beta,\gamma;x) = \frac{(\beta)_n\ (2x)^n}{n!}\ {}_3F_2\left[\begin{matrix}-\tfrac{1}{2}, & -\tfrac{1}{2}n+\tfrac{1}{2}, \gamma-\beta; \\ \gamma, & 1-\beta-n\ ;\end{matrix}\ \frac{1}{x^2}\right] \quad (1.2.20)$$

and

$$G_n\ (\alpha,\beta;x) = \frac{(\alpha)_n\ (\beta)_n\ (2x)^n}{n!\ (\alpha+\beta)}\ {}_3F_2\left[\begin{matrix}-\tfrac{1}{2}n, -\tfrac{1}{2}n+\tfrac{1}{2}, & 1-\alpha-\beta-n; \\ 1-\alpha-n, & 1-\beta-n;\end{matrix}\ \frac{1}{x^2}\right].$$

(1.2.21)

The Lommel polynomials defined by Watson [34] are

$$R^{(\frac{1}{x})}_{n,\nu} = (\nu)_n\ (2x)^n\ {}_2F_3\left[\begin{matrix}-\tfrac{1}{2}n, & -\tfrac{1}{2}n+\tfrac{1}{2}; \\ \nu, & -n, & 1-\nu-n;\end{matrix}\ -\frac{1}{x^2}\right]. \quad (1.2.22).$$

Toscano [33] has defined the polynomials as

$$S_n^{(x)} = \frac{(a)_{2n}}{n!\ (a)_n}\ {}_{p+1}F_{q+1}\left[\begin{matrix}-n, & \alpha_1, ---, \alpha_p\ ; \\ a+n, & \beta_1, ---, \ \beta_q\ ;\end{matrix}\ x\right]. \quad (1.2.23)$$

1.3. Here we introduce generalized hypergeometric polynomials in the form

$$F_n(x) = x^{(\delta-1)n}\ {}_{p+\delta}F_{q+\lambda}\left[\begin{matrix}\Delta(\delta, -n), & (a_p)\ ; \\ \Delta(\lambda, \alpha), & (b_q)\ ;\end{matrix}\ ux^{\mu}\right]$$

$$= \sum_{k=0}^{\infty} \frac{\prod_{i=0}^{\delta-1} \left(\frac{-n+i}{\delta}\right)_k \left[(a_p)\right]_k \; u^k \; x^{\{\mu k + (\delta-1)n\}}}{\prod_{j=0}^{\lambda-1} \left(\frac{\alpha+j}{\lambda}\right)_k \left[(b_q)\right]_k \; k!} \quad (1.3.1)$$

where δ, λ, α, n are non negative integers, $\Delta(\delta, b)$ stands for the set of δ parameters

$$\frac{b}{\delta}, \; \frac{b+1}{\delta}, \; \frac{b+2}{\delta}, --- , \frac{b+\delta-1}{\delta};$$

and $(a_p) \equiv a_1, \; a_2, ---, a_p$.

$$\left[(a_p)\right]_n = \prod_{i=1}^{p} (a_i)_n = (a_1)_n \cdot (a_2)_n \cdots (a_p)_n .$$

On specializing the parameters in (1.3.1) the following wellknown polynomials are obtained as particular cases.

(i) By choosing $\delta = p = \lambda = \mu = u = 1$, $q = 0$, $\alpha = 1$, $a_1 = n + 1$ in (1.3.1) we have

$$F_n(x) = {}_2F_1\left[\begin{matrix} -n, \; n+1; \\ 1; \end{matrix} \; x\right] = P_n(1-2x) \quad (1.3.2)$$

which is Legendre polynomial (1.2.3).

ii) Setting $\delta = - \mu = 2$, $u = -1$ and $\lambda = p = q = o$, (1.3.1) reduces to

$$F_n(x) = x^n \; {}_2F_0\left[\begin{matrix} -\tfrac{1}{2}n, & -\tfrac{1}{2}n + \tfrac{1}{2} & ; \\ \text{———} & , & \end{matrix} \; -\frac{1}{x^2}\right] = \frac{H_n(x)}{2^n} \qquad (1.3.3)$$

which are the Hermite polynomials (1.2.3).

(iii) On putting $\delta = \lambda = u = \mu = 1$, $p = q = 0$ and replacing α by $1 + \alpha$ in (1.3.1) , we get

$$F_n(x) = {}_1F_1\left[\begin{matrix} -n; \\ 1+\alpha; \end{matrix} \; x\right] = \frac{n!}{(1+\alpha)_n} L_n^{(\alpha)}(x) \qquad (1.3.4)$$

which are generalized Laguerre polynomials (1.2.8) and on further choosing $\alpha = 0$ (1.3.4) reduces to simple Laguerre polynomials (1.2.10)

$$F_n(x) = {}_1F_1\left[\begin{matrix} -n; \\ 1; \end{matrix} \; x\right] = L_n^{(0)}(x) = L_n(x). \qquad (1.3.5)$$

(iv) On taking $\delta = p = \lambda = u = \mu = 1$, $q = 0$, $a_1 = 1 + \alpha + \beta + n$ and replacing α by $1 + \alpha$ in (1.3.1), we get

$$F_n(x) = {}_2F_1\left[\begin{matrix} -n, & 1+\alpha+\beta+n ; \\ 1+\alpha; & \end{matrix} \; x\right] = \frac{n!}{(1+\alpha)_n} P_n^{(\alpha,\beta)}(1 - 2x)$$

(1.3.6)

which are the Jacobi polynomials (1.2.11). Further on putting $\alpha = \beta = 0$, (1.3.6) reduces to the ~~Gegenbauer~~ Legendre polynomials (1.2.13) and $\alpha = \beta = \nu - \frac{1}{2}$ gives us the Gegenbauer polynomials (1.2.13)

$$F_n(x) = {}_2F_1\left[\begin{matrix} -n, & 2\nu + n; \\ \nu + \frac{1}{2}; & \end{matrix} \; x\right] = \frac{n!}{(2\nu)_n} \; C_n^{\nu}(1 - 2x). \qquad (1.3.7)$$

With $\alpha = \beta = -\frac{1}{2}$ in (1.3.6) we have

$$F_n(x) = {}_2F_1\left[\begin{matrix} -n, & n; \\ \frac{1}{2}; & \end{matrix} \; x\right] = T_n \; (1 - 2x) \qquad (1.3.8)$$

which are the Tchebicheff polynomials of first kind (1.2.15).

(v) On taking $\delta = u = \mu = \lambda = q = 1$, $p = 2$, $a_1 = n + a + b + 1$, $a_2 = \rho$, $\alpha = \sigma$, $b_1 = 1 + a$ in (1.3.1), we obtain

$$F_n(x) = {}_3F_2\left[\begin{matrix} -n, & n+a+b+1, & \rho; \\ 1 + a, & \sigma; & \end{matrix} \; x\right] = \frac{n!\, H_n^{(a,b)}}{(1 + a)_n} \; (\rho, \sigma; x)$$

(1.3.9)

which are generalized Rice's polynomials (1.2.17) and further on setting $a = b = o$, (1.3.9) reduces to Rice's polynomials.

(vi) In (1.3.1) if we take $\delta = \lambda = u = \mu = 1$, $b_1 = \frac{1}{2}$
$a_1 = n + a + b + 1$, $\alpha = 1 + a$, then we have

$$F_n(x) = {}_{p+1}F_{q+1}\left[\begin{matrix} -n, & n+a+b+1, & a_2, \dots, a_p; \\ 1+a, & \frac{1}{2}, & b_2, \dots, b_q; \end{matrix} \; x\right] = \frac{n!}{(1+a)_n} \mathcal{S}_n^{(a,b)}\left(\begin{matrix} a_2, \dots, a_p; \\ b_2, \dots, b_q; \end{matrix} \; x\right)$$

which are generalized Sister Celine's polynomials (1.2.18) which reduce to Sister Celine's polynomials by further putting $a = b = 0$.

(vii) By choosing $\delta = p = q = \lambda = u = \mu = 1, \alpha = \nu + \frac{1}{2}$, $a_1 = 2\nu + n$, $b_1 = 1 + b$ in (1.3.1) we get

$$F_n(x) = {}_2F_2\left[\begin{matrix} -n, & 2\nu + n; \\ \nu + \frac{1}{2}, & 1 + b; \end{matrix}\; x\right] = y_n(x) \tag{1.3.11}$$

which are generalized Bessel polynomials (1.2.19).

(viii) On setting , $\delta = -\mu = 2$, $p = q = \lambda = u = 1$, $a_1 = \gamma - \beta$, $\alpha = \gamma$, $b_1 = 1 - \beta - n$ in (1.3.1) we have

$$F_n(x) = x^n \; {}_3F_2\left[\begin{matrix} -\frac{1}{2}n, & -\frac{1}{2}n + \frac{1}{2}, & \gamma - \beta; \\ \gamma, & 1 - \beta - n ; & \end{matrix}\; \frac{1}{x^2}\right] = \frac{n!\, R_n(\beta, \gamma; x)}{(\beta)_n \; 2^n} \tag{1.3.12}$$

and with $\delta = -\mu = 2$, $p = q = \lambda = u = 1$, $a_1 = 1 - \alpha - \beta - n$, replacing α by $(1 - \alpha - n)$, $b_1 = 1 - \beta - n$, (1.3.1) reduces to the form

$$F_n(x) = x^n \; {}_3F_2\left[\begin{matrix} -\frac{1}{2}n, & -\frac{1}{2}n + \frac{1}{2}, & 1 - \alpha - \beta - n; \\ 1 - \alpha - n, & 1 - \beta - n ; & \end{matrix}\; \frac{1}{x^2}\right]$$

$$= \frac{n!\, (\alpha + \beta)_n}{(\alpha)_n \, (\beta)_n \, 2^n} \, G_n(\alpha, \beta; x) , \tag{1.3.13}$$

where $R_n(\beta,\gamma;x)$ and $G_n(\alpha,\beta;x)$ are Bedient polynomials (1.2.20) and (1.2.21) respectively.

(ix) Putting $\delta = q = \mu = 2$, $\lambda = -u = 1$, $p = o$, $\alpha = \gamma$, $b_1 = -n$, $b_2 = 1-\gamma-n$, (1.3.1) reduces to

$$F_n(x) = x^n \; {}_2F_3\left[\begin{matrix}-\tfrac{1}{2}n, \; -\tfrac{1}{2}n+\tfrac{1}{2}; \\ \gamma, \; -n, \; 1-\gamma-n;\end{matrix}\; -\frac{1}{x^2}\right] = \frac{R_{n,\gamma}\left(\frac{1}{x}\right)}{(\gamma)_n \; 2^n} \tag{1.3.14}$$

where $R_{n,\gamma}\left(\frac{1}{x}\right)$ are Lommel polynomials (1.2.22).

(x) If we choose $\delta = \lambda = u = \mu = 1$; $\alpha = a+n$ in (1.3.1), we get

$$F_n(x) = {}_{p+1}F_{q+1}\left[\begin{matrix}-n, \; a_1, \cdots, a_p; \\ a+n, \; b_1, \cdots, b_q;\end{matrix}\; x\right] = \frac{n!\,(a)_n \; S_n(x)}{(2a)_n} \tag{1.3.15}$$

where $S_n(x)$ are Toscano polynomials (1.2.23).

(xi) On setting $\lambda = o$ in (1.3.1) we have the generalized hypergeometric polynomials defined by Shah [29] in the form

$$F_m(x) = x^{(\delta-1)m} \; {}_{\delta+p}F_q\left[\begin{matrix}\Delta(\delta,-m), \; (a_p); \\ (b_q);\end{matrix}\; ux^{\mu}\right] \tag{1.3.16}$$

1.4. In the present work, an attempt has been made to study some properties of generalized hypergeometric polynomials (1.3.1). In view of the general nature of the polynomials, the results derived by us will not only unify the known results but will also add some new results to the existing field of polynomials.

1.5. In this section we state the known results which we have used in our subsequent work.

(1) Results [15]

(a) $$\int_0^1 x^{\alpha}(1-x)^{-\frac{1}{2}}\, T_n(2x-1)\, dx = \frac{n!\, \pi^{\frac{1}{2}}\, \Gamma(\alpha+1)\, \Gamma(\alpha+\frac{3}{2})}{(\frac{1}{2})_n\, \Gamma(\alpha+n+\frac{3}{2})\, \Gamma(\alpha-n+\frac{3}{2})}$$

$R_e\ (\alpha) > -1$; (1.5.1)

(b) $$\int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}} \left[T_n(2x-1)\right]^2 dx = \frac{\pi}{2},\quad n \neq 0 ; \qquad (1.5.2)$$

and

(c) $$\int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}\, T_n(2x-1)\, T_m(2x-1)\, dx = 0; \qquad (1.5.3)$$

$m \neq n$, where m and n are non-negative numbers.

(2) Results [23]

(i) $(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\ \ldots\ (\alpha+n-1),\ (\alpha)_0 = 1$;

(ii) $$(\alpha)_{n-k} = \frac{(-1)^k\, (\alpha)_n}{(1-\alpha-n)_k} \qquad (1.5.4)$$

and its special case at $\alpha = 1$

$$(-n)_k = \frac{(-1)^K \; n!}{(n-k)!} \; ; \qquad (1.5.5)$$

(iii)

$$(a)_{nk} = k^{nk} \prod_{i=0}^{k-1} \left(\frac{a+i}{k}\right)_n \; ; \qquad (1.5.6)$$

$$\text{(iv)} \quad (a)_{-nk} = \frac{(-1)^{nk}}{k^{nk} \prod_{i=0}^{k-1} \left(\frac{1-a+i}{k}\right)_n} \; ; \qquad (1.5.7)$$

$$\text{(v)} \quad \Gamma(\alpha - nk) = (\alpha)_{-nk} \; \Gamma(\alpha) \qquad (1.5.8)$$

(3) Results [19], Rodrigues formulae of even and Odd Hermite polynomials

$$\text{(i)} \quad H_{2pk}^{(x;k)} = H_{2pk}^{(z)} = \frac{(-1)^p \, (1+p)_p \, z^{-\beta} \, e^{z}}{(1+\beta)_p} \; \frac{d^p}{d_z p}\left[e^{-z} \, z^{p+\beta}\right],$$

(1.5.9)

$$\text{(ii)} \quad H_{2pk+1}^{(x;k)} = H_{2pk+1}^{(z)} = \frac{(-1)^p \; 2^{(2p+1)} \; (\frac{3}{2})_p \; e^{z}}{} \; \frac{d^p}{d_z p}\left[e^{z} \; z^{(p-\beta)}\right],$$

(1.5.10)

where $z = x^{2k}$, $-\beta = \frac{1}{2k}$.

(4) Orthogonality relations [19]

(i) $$\int_{-\infty}^{\infty} x^{2k-2} \exp(-x^{2k}) \left[H_{2pk}^{(x;k)} \right]^2 dx$$

$$= \frac{(-2\beta) \left[(2p)! \right]^2 \Gamma(1+\beta)}{p! \; (1+\beta)_p} \qquad (1.5.11)$$

(ii) $$\int_{-\infty}^{\infty} x^{2k-2} \exp(-x^{2k}) \left[H_{2pk+1}^{(x;k)} \right]^2 dx$$

$$= \frac{(-8\beta) \; (1+p)_{p+1} \; (2p+1)! \; \Gamma(1-\beta)}{(1-\beta)_p} . \qquad (1.5.12)$$

(5) Series manipulations [25]

(i) $$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/\delta]} A(k, n-\delta k), \qquad (1.5.13)$$

(ii) $$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/\delta]} A(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n+\delta k) . \qquad (1.5.14)$$

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