CHAPTER 1

DEFINITIONS AND STATEMENT OF KNOWN RESULTS

A B S T R A C T

In this introductory Chapter 1 we state the several definitions which we are going to use during the course of investigation. The list of relevant references is cited at the end of the chapter.

SOME DEFINITIONS

Definition : Let $U = \{ z : z$ is a complex number and $|z| < 1 \}$

Definition : A complex valued function $f(z)$ is holomorphic in a domain D in the complex plane if it is differentiable at each and every point of the domain D.

Definition : A function $f(z)$ holomorphic in some domain D is said to be univalent in D if $f(z_1) = f(z_2)$ implies $z_1 = z_2$ for all z_1, z_2 in D.

Definition : A domain containing the origin is said to be starlike with respect to the origin if the join of any point in the domain and origin lies completely in that domain. Starlike with respect to the origin will be refered to as simply starlike.

Definition : Let S denote the class of all functions $f(z)$, which are holomorphic and univalent in U and normalized by $f(0) = 0$ and $f'(0) = 1$.

Definition : Let $f(z)$ be holomorphic at $z = 0$ and satisfy $f(0) = 0$ and $f'(0) \neq 0$ there. Then the radius of univalence is defined to be the largest value of r such that $f(z)$ is holomorphic and univalent for $|z| < r$.

Definition : Let $f(z)$ be holomorphic at $z = 0$ and satisfy $f(0) = 0$ and $f'(0) \neq 0$. Let λ be a real number satisfying $0 \leq \lambda < 1$. Then the radius of starlikeness of order λ, denoted by S_{λ} is defined to be the largest value of r such that $f(z)$ is holomorphic and $\mathrm{Re}\{ z f'(z) / f(z) \} > \lambda$; for $|z| < r$.

Definition : Let $f(z)$ be holomorphic at $z = 0$ and satisfy $f(0)=0$ and $f'(0) \neq 0$. Let λ be a real number satisfying $0 \leq \lambda < 1$. The radius of convexity of order λ, denoted by K_λ, is defined to be the largest value of r such that $f(z)$ is holomorphic and $\operatorname{Re}\{ 1 + zf''(z)/f'(z) \} > \lambda$, for $|z| < r$.

Definition :

Let $P(z)=a\prod_{k=1}^{n}(z - z_k)$ be a polynomial of degree n, where n is a positive integer, all of whose zeros lie outside or on the unit circle. We denote the set of all such polynomials by $Q(z)$.

For starlike and convex functions we make use of the following definitions.

Definition : Suppose $f(z)$ is holomorphic in U and $f(0) = 0$ and $f'(0) = 1$, then $f(z) \in S^*$ if and only if $\operatorname{Re}\{ zf'(z)/f(z) \} > 0$, for $z \in U$.

Definition : Suppose $f(z)$ is holomorphic in U and $f(0) = 0$ and $f'(0) = 1$. Then $f(z) \in K$, K denotes the family of convex functions if and only if $\operatorname{Re}\{1 + zf''(z)/f'(z) \} > 0$, for $z \in U$.

Definition : A function $f(z) \in S$ is close-to-convex with respect to the convex function $e^{i\alpha} g(z)$, where $g(z) \in K$, and $0 \leq \alpha < 2\pi$ if $\operatorname{Re} \{ f'(z) / e^{i\alpha} g'(z)\} > 0$, for $z \in U$. We denote by C, the subclass of S of close-to-convex functions.

<u>Definition</u> : Let $f(z)$ be holomorphic at $z = 0$ and satisfy $f(0) = 0$ and $f'(0) \neq 0$. Then the radius of close-to-convexity is defined to be the largest value of r such that $f(z)$ is holomorphic and close-to-convex for $|z| < r$.

We note that the function satisfying the close-to-convex condition, the starlike condition or the convex condition is univalent.

The following is the result due to Kaplan [6] for the close-to-convex function, which is often considered as the definition of close-to-convex function.

<u>Result</u> : Suppose $f(z)$ is holomorphic in U, $f'(z) \neq 0$ $(z \epsilon U)$, $f(0) = 0$ and $f'(0) = 1$. Then $f(z) \epsilon C$ if and only if

$$\int_{\theta_1}^{\theta_2} \mathrm{Re} \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} d\theta > -\pi , \quad (z = re^{i\theta})$$

for any r, θ_1 and θ_2, $0<r<1$, and $0 \leq \theta_1 < \theta_2 \leq \theta_1 + 2\pi$

<u>Note</u>: We observe that the following is the inclusion relationship that holds good for the family of functions, K, S^*, C, S;

$$K \subset S^* \subset C \subset S$$

<u>Definition</u> : A subset `A' of linear space L is convex if and only if for all pairs of points of A, their convex combination is also in A. i.e. for $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, with $\alpha_1 + \alpha_2 = 1$, $\alpha_1 x_1 + \alpha_2 x_2 \epsilon A$.

A point `a' in a convex set `A' is called an extreme point of A if and only if a cannot be expressed as a convex combination of any other two distinct points of A. More explicitly, we shall say that a is on extreme point of convex set A, if and only if

$a \in A$ and $a = tx_1 + (1-t)\, x_2$, with $o < t < 1$, $x_1, x_2 \in A$ implies that $a = x_1 = x_2$.

Ext A will denote the set of extreme points of A.

Definition : Let Aq (q a fixed integer greater than zero) denote the class of functions

$$f(z) = z^q + \sum_{k=q+1}^{\infty} a_k z^k \quad \text{which are holomorphic}$$

in $U = \{z: |z| < 1\}$.

A function $f \in A_q$ is said to be q-valent starlike of order α and type β if the condition

$$\left| \frac{\left(\dfrac{zf'(z)}{f(z)} - q \right)}{2\beta \left(\dfrac{zf'(z)}{f(z)} - \alpha \right) - \left(\dfrac{zf'(z)}{f(z)} - q \right)} \right| < 1$$

is satisfied for some α, β ($0 \le \alpha < q$, $0 < \beta \le 1$) and for all $z \in U$.

We note that for specific values to α, β, q we get the following subclasses studied by the several authors.

(i) $S^*_1(\alpha,\beta) = S^*(\alpha,\beta)$

(ii) $S^*_q(0,1) = S^*_q$ is the class of q-valent starlike functions considered by Goodman [5].

(iii) $S^*_q(\alpha,1) = S^*_q(\alpha)$.

(iv) $S^*_1(0,1) = S^*$ and $S^*_1(\alpha,1) = S^*(\alpha)$ are respectively, the well known class of starlike functions and the class of starlike functions of order α introduced by Robertson [10].

Such type of class was introduced by Kulkarni-Thakare[7] and various interesting properties were investigated for univalent functions. Padmanabhan [9], Eenigenburg[4], Ram Singh [11] McCarty[8] and Wright [12] also studied such type of class with same modification and made an authentic contribution in the study of univalent functions.

REFERENCES :-

1: Aouf, M.K. :

On q-valent starlike functions of order α and type β, Bulletin of the institute of Mathematics Academia Sinica Vol.16(3) September 1988.

2. Barr, A.F. :

The radius of Univalence of certain classes of Analytic functions. Ph. D. dissertation, University of Mississipi. (1971)

3. Dutton, C.E. :

Geometric properties of some classes of univalent starlike functions. Ph.D. dissertation, Temple University, 1978.

4. Eenigenburg, E.G. :

A class of starlike mapping in the unit disc, Compositio Math. (2) 24 (1972); 235-238.

5. Goodman, A.W. :

On the Schwarz-Christoffel transformation and p-valent functions. Trans. Amer. Math. Soci. 68 (1950) 204-223.

6. Kaplan, W. :

Close-to-convex Schlicht function, Michigan Math.J.1(1952), 169-185.

7. Kulkarni, S.R.(Thakare, N.K.) :

Some problems connected with Univalent functions. Ph.D. Thesis, Shivaji university, Kolhapur, (1981).

8. Mc Carty, C.P. :

Starlike functions, Proc. Amer. Math. Soc.(1), 43(1974), 361-366.

9. Padmanabhan, K.S. :

On certain classes of starlike functions in the unit disc, J. Indian Math. Soc. (32) (1968) 89-103.

10. Robertson, M.S. :

On the theory of univalent functions Ann.of Math.37 (1936), 374-408.

11. Singh, R. :

On a class of starlike functions compositio Math. 19 (1967) 78-82.

12. Wright, D.J.

On a class of starlike functions compositio Math. 21(1969) 122-124.

--- xx ---