A R T I C L E - O N E

I n t r o d u c t i o n

The theory of integral transform is a classical object in Mathematics whose literature can be traced back through at least 150 years, but the theory of generalized functions is comparatively of recent origin. The idea of specifying a function not by its value but by its behaviour as a functional on some space of testing functions is a new concept. S. L. Sobolev (1936) was perhaps the first mathematician who used the new concept to develop the theory of generalized functions as functionals defined on a certain function space. In 1950-1951 appeared L. Schwartz's monograph 'Theorie des Distributions' [18], in which he has systematized the theory of distributions based on the theory of functional analysis. He unified all the earlier approaches, his results were deep and of far reaching significance.

Recently, Zemanian [26] extended the various types of transforms to a certain class of generalized functions. The work of Koh and Zemanian [14], Koh [12], Lee [15], and Pandey [17] on some transformations of generalized functions are worth mentioning. For each integral transform, one can define the corresponding generalized transformation by constructing the appropriate testing function space suited to the needs of kernal function occuring in the transformation. The theory of distributions gained acceptance very rapidly as an extremely significant mathematical innovation.

*1.1 Generalized Functions (Distributions) :

The term generalized function (distribution) refer to the method of bringing symbolic function, such as the Dirac [4] delta function $\delta(x)$ within the scope of a rigorous mathematical treatment. The intuitive definition of $\delta(x)$, as a function which is zero for $x \neq 0$ and infinite at $x = 0$ in such a way that $\int_{0}^{\infty} \delta(x)dx = 1$, is certainly not sufficient for this purpose; the integral of a function which vanishes except at one point must always be zero. In some of the problems encountered in applied mathematics, the transform methods are applied to analyse physical situations in which impulsive forces or point sources are involved. Formal calculations are much simplified by the introduction of the Dirac 'delta function' but the rules for its manipulations do not follow, naturally from the methods of Classical Analysis. This led to introduce the concept of a generalized function.

Generalized functions were first introduced in science as a result of Dirac's research in quantum mechanics. At present the theory of ~~a~~ generalized functions is for advanced and has numerous applications in Physics, Mathematics and Engineering. The advantage of generalized functions and distributions is that by widening the class of functions, many theorems and operations are free from tedious restrictions. The extension of integral transformations from classical functions to generalized functions has attracted a lot of attention in recent years. Lions [16] was perhaps the first to extend Hankel transformation to a generalized functions in such a way that an inversion formula could be stated for it. His results were obtained as a perticular case in a study of general types of operators, called transmutation operators acting on certain testing function spaces of Schwartz. Fenyo [7] gave a theory for the case where the order of the Hankel transformation is a non-negative integer. Denoting the linear space of Hankel transforms of functions belonging to D_+ (the linear space of infinitely smooth functions with compact support contained in R_+) by H_n, he defined the Hankel transform $F_n(k)$ for the distribution K belong to H the dual of H_n, by means of the relation $F_n(k)\phi = K(P\, F_n(\phi(x)/x))$, $\phi \in D_+$ and showed that every distribution H_n' in D_+' is the Hankel transform of a distribution belonging to H_n'. He also obtained some elementary properties for transforms of distributions and their derivatives.

Zemanian [30] gave an alternative theory designed specifically for the Hankel transformation. He has defined [30, P.129], H_μ as a testing function space. A generalized function on the open interval I is any continuous linear functional on any testing function space on I. The collection of all continuous linear functional on the testing function space H_μ, is called the dual of H_μ, and it is denoted by H_μ'.

members of H_μ' are generalized functions (distributions) on which the Hankel transformation is defined, and these act like distribution of slow growth as x ∞. The generalized Hankel transformation h_μ' is defined, as the adjoint of h_μ, on H_μ' by the relation

$$\langle h_\mu'(f), \phi \rangle = \langle f, h_\mu(\phi) \rangle \quad \text{where } \mu \geqslant -1/2, \quad \phi \in H_\mu;\ f \in H_\mu'.$$

If the function f(x) be a locally integrable on $o < x < \infty$ such that f(x) is of slow growth as $x \to \infty$ and $x^{\mu + 1/2} f(x)$ is absolutely integrable on $o < x < 1$, then f(x) generates a regular generalized function f in H_μ' by

$$\langle f, \phi \rangle = \int_o^\infty f(x)\, \phi(x) dx, \quad \text{for } \mu \geqslant -1/2$$

Zemanian [30] has also extended the Hankel transformation to a certain class of generalized functions having no restriction on their growth as x ∞. The method adopted is simpler to that used by Gelfand and Shilov [8] to extend Fourier transformation to all distribution. He [26] also considered distribution of slow growth and transform of order $\mu \geqslant -1/2$. Koh [13] extended the theory of transforms of arbitrary order for distributions of rapid growth. He has also studied the n-dimensional distributional Hankel transform. Chaudhary [3] has extended the n - dimensional Hankel transformation to a class of generalized functions of arbitrary order. Ghosh [9] has given an inversion formula for the generalized transform, and also studied other properties of this transform. Another generalization of Hankel transform given by

$$h_{\nu,\mu}(f) = \int_0^\infty (xy)^{\nu/2-\mu/2+1/2} J_{\frac{\nu+\mu}{2}}(xy)\, f(y)\, dy$$

has also been extended by him to a generalized functions. A self reciprocal relation for this transform in the generalized sense for $\nu + \mu \geqslant -1$, and several other results have also been derived by him. The notations and terminology of this work follow that of Zemanian [30], Ghosh [9] and Chaudhary [3].

The simple generalization of Hankel transforms are defined as follows -

$$h_{\mu,\lambda}(f) = \lambda \int_0^\infty f(x)\, (xy)^{\lambda-1/2} J_\mu(x^\lambda y^\lambda)\, dx, \quad \lambda > 0. \tag{1.2}$$

and

$$h_{\nu,\mu,\lambda}(f) = \lambda \int_0^\infty f(x)\, (xy)^{\lambda(\nu/2-\mu/2+1)-1/2} J_{\frac{\nu+\mu}{2}}(x^\lambda y^\lambda)\, dx, \lambda > 0. \tag{1.3}$$

where $J_\mu(z)$ is the Bessel function of first kind of order μ. If $h_{\mu,\lambda}(f) = f$, we say that $f(x)$ is a self-reciprocal function w.r.t. $h_{\mu,\lambda}$. If f is a self-reciprocal w.r.t. $h_{\mu,\lambda}$, then we say that f is a self-reciprocal function $R_{\mu,\lambda}$. We define the n - dimensional generalized Hankel transformation of a function $f(x_1, x_2, \ldots\ldots, x_n)$ by

$$(h_{\mu,\lambda}f)\ (y_1,y_2,\ldots,y_n) = \lambda \int_0^\infty \cdots\cdots \int_0^\infty \prod_{i=1}^{n} (x_i\ y_i)^{\lambda-1/2} J_\mu(x_i^\lambda\ y_i^\lambda)\ f(x_1,\ldots,x_n)\ dx_1\ldots dx_n. \quad (1.4)$$

In the present work we have extended the transforms (1.3) and (1.4) to a class of distributions for arbitrary order. We have also shown that if f is a self-reciprocal function $R_{\mu,\lambda}$, then $h_{\nu,\mu,\lambda}(f)$ is a self-reciprocal function $R_{\nu,\lambda}$. We have also discussed the distributional version of this result.

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