# ARTICLE THREE

## Some Self-reciprocal Generalized Functions

In this section we shall extend the transform defined as follows,

$$h_{\nu,\mu,\lambda}(f) = \int_{o}^{\infty} (xy)^{\lambda(\nu/2-\mu/2+1)-1/2} J_{\frac{\nu+\mu}{2}}(x^{\lambda}y^{\lambda})f(y)dy \quad (3)$$

to a class of distributions and study different properties of this transformation.

* 3.1 : The Testing Function space $H_{\mu,\lambda}$ And Its Dual $H'_{\mu,\lambda}$:

Let $\mu$ be any real number and $\lambda > o$. $H_{\mu,\lambda}$ is the space of all complex-valued smooth functions $\phi(x)$ defined on the open interval $I \equiv (o, \infty)$, such that for each pair of non-negative integers m and k.

$$\gamma^{\mu,\lambda}_{m,k}(\phi(x)) = \sup_{o<x<\infty} \left| x^{m\lambda}(x^{1-2\lambda}D_x)^k x^{-\lambda\mu-\lambda+1/2}\phi(x) \right| < \infty$$

If we assign to $H_{\mu,\lambda}$, the topology generated by the countable multinorm $\left\{ \gamma^{\mu,\lambda}_{m,k} \right\}_{m,k=o}^{\infty}$, then $H_{\mu,\lambda}$ becomes a testing function space. Putting $\lambda = 1$, we have $H_{\mu,\lambda} = H_{\mu}$ [30, p.129]. The space consisting of continuous linear functionals defined on $H_{\mu,\lambda}$ is called the dual space of $H_{\mu,\lambda}$, and it is denoted by $H'_{\mu,\lambda}$. We assign to $H'_{\mu,\lambda}$, the weak topology generated by multinorm $\{ \xi_{\phi}(f) \}$, where $\xi_{\phi}(f) = |\langle f, \phi \rangle|$, $\phi \in H_{\mu,\lambda}$. Thus, the dual space $H'_{\mu,\lambda}$ is also complete.

★ 3.2 : Properties of $H_{\mu,\lambda}$ And $H'_{\mu,\lambda}$. :

The following properties were developed by Ghosh[9,VIII] in a similar way to Zemanian [30, pp. 130-133].

(1) $\phi(x)$ is a member of $H_{\mu,\lambda}$ if and only if it satisfies the following conditions.

(i) $\phi(x)$ is a smooth complex-valued function on open set $0 < x < \infty$. By a smooth function, we mean a function that possesses derivatives of all orders which are continuous at all points of its domain.

(ii) For each nonnegative integer k, and any real numbers $\mu, \lambda > 0$.

$$\phi(x) = x^{\lambda\mu+\lambda-1/2}[a_0 + a_2 x^{2\lambda} + \ldots + a_{2k} x^{2k\lambda} + R_{2k}(x)]$$

where $a_i$ 's are constants given by

$$a_{2k} = \frac{1}{k!(2\pi)^k} \operatorname{Lim}_{x \to 0+} (x^{1-2\lambda} D_x)^k \, x^{-\lambda\mu-\lambda+1/2} \phi(x)$$

and the reminder term $R_{2k}(x)$ satisfies

$$(x^{1-2\lambda} D_x)^k \, R_{2k}(x) = O(1), \quad x \to 0+.$$

(iii) For each nonnegative integer k, $D^k \phi(x)$ is of rapid descent as $x \to \infty$ (i.e. $D^k \phi(x)$ tends to zero faster than any power of $(1/x)$ as $x \to \infty$).

(2) Let D(I) be the space of all complex valued smooth functions defined on I. The space D(I) is a sub-space of $H_{\mu,\lambda}$ for every choice of $\mu$ and $\lambda > o$, and the covergence in D(I) implies convergence in $H_{\mu,\lambda}$. Consequently, the restriction of any $f \in H'_{\mu,\lambda}$ to D(I) is a member of D'(I).

(3) If q is an even positive integer, then $H_{\mu+q,\lambda}$ is a sub-space of $H_{\mu,\lambda}$. The topology of $H_{\mu+q,\lambda}$ is stronger than that induced on it by $H_{\mu,\lambda}$, and hence restriction of $f \in H'_{\mu,\lambda}$ to $H_{\mu+q,\lambda}$ is a member of $H'_{\mu+q,\lambda}$. However, $H_{\mu+q,\lambda}$ is not dense in $H_{\mu,\lambda}$.

(4) For each $\mu$ and $\lambda > o$, $H_{\mu,\lambda}$ is a dense sub-space of E(I), where E(I) is the space of all complex valued smooth functions on $o < x < \infty$, Zemanian [30, p.36]. Moreover D(I) is a dense in E(I). The topology of $H_{\mu,\lambda}$ is stronger than that induced on it by E(I). Hence E'(I), the dual of E(I), is a subspace of $H'_{\mu,\lambda}$ whatever be the choice of $\mu$.

(5) For each choice of nonnegative integer r, set

$$\rho_r^{\mu,\lambda}(\phi) = \max \gamma_{m,k}^{\mu,\lambda}(\phi), \qquad \begin{matrix} o \leq m \leq r \\ o \leq k \leq r \end{matrix}$$

then for each $f \in H'_{\mu,\lambda}$, there exists a positive constant C and a nonegative integer r such that

$$\left| \langle f, \phi \rangle \right| = C \rho_r^{\mu,\lambda}(\phi), \text{ for every } \phi \in H_{\mu,\lambda}.$$

Here, C and r depends on f but not on $\phi$.

(6) If f(x) be a locally integrable function on $o < x < \infty$ such that f(x) is of slow growth as $x \to \infty$, and $x^{\lambda\mu+1/2} f(x)$ is absolutely integrable on $o < x < 1$. Then, f(x) generates a regular generalized function f in $H'_{\mu,\lambda}$ by the definition

$$\langle f, \phi \rangle = \int_o^\infty f(x)\, \phi(x)\, dx, \quad \phi \in H_{\mu,\lambda}.$$

## * 3.3 : Some Operations on $H_{\mu,\lambda}$ and $H'_{\mu,\lambda}$ :

The following results were developed by Ghosh[9,VIII] in a similar way to [30, p. 134].

(i) Multipliers in $H_{\mu,\lambda}$ :- A multiplier is a continuous linear mapping from a testing function space into itself. Let $\Theta_\lambda$ be the linear space of all smooth functions $\Theta(x)$ defined on $o < x < \infty$ such that, for each nonnegative integer $\nu$, there exist an integer $n_\nu$ for which $(x^{1-2\lambda} D_x)^\nu \Theta(x) / (1+x^{\lambda n_\nu})$ is bounded on $o < x < \infty$. Clearly the product of any two members of $\Theta_\lambda$ is also in $\Theta_\lambda$. Now $\Theta \in \Theta_\lambda$ is a multiplier for $H_{\mu,\lambda}$, for every $\mu$ and $\lambda > 0$. Infact $\Theta(x) \in \Theta_\lambda$ implies that $\Theta(x^{1/\lambda}) \in \Theta_\lambda$, it follows from [ 30, p. 134] for $\lambda = 1$. consequently, the linear mapping $\phi \rightarrow \Theta(x^{1/\lambda})\, \phi$ is a continuous mapping of $H_\mu$ on to itself. It is clear that the mapping $\phi \rightarrow x^{1/2\lambda - 1/2}\, \phi_1(x^{1/\lambda})$ is a continuous linear mapping from $H_{\mu,\lambda}$ onto $H_\mu$, the inverse mapping being $\phi_1 \rightarrow x^{\lambda/2-1/2}\, \phi_1(x^\lambda)$.

Now combining the facts that, $\phi_1 \rightarrow x^{1/2\lambda-1/2}\, \phi_1(x^{1/\lambda})$ $\phi_1(x^{1/\lambda})$ is a continuous linear mapping from $H_{\mu,\lambda}$ onto $H_\mu$; $\phi(x) \rightarrow \Theta(x^{1/\lambda})\, \phi(x)$ is a continuous linear mapping of $H_\mu$ onto itself and $\phi(x) \rightarrow x^{\lambda/2-1/2}\, \phi(x^\lambda)$ is a continuous linear mapping of $H_\mu$ onto $H_{\mu,\lambda}$. This implies that $\phi_1 \rightarrow \Theta(x)\, \phi_1$ is a continuous linear mapping from $H_{\mu,\lambda}$ into itself and hence, $\Theta \in \Theta_\lambda$ is a multiplier for $H_{\mu,\lambda}$. We can define the adjoint operator $f \rightarrow \Theta f$ on $H'_{\mu,\lambda}$ by the relation.

$$\langle \Theta f, \phi \rangle = \langle f, \Theta\phi \rangle, \quad f \in H'_{\mu,\lambda}, \ \phi \in H_{\mu,\lambda};\ \Theta \in \Theta_\lambda.$$

Clearly, $f \rightarrow \Theta f$ is a continuous linear mapping from $H'_{\mu,\lambda}$ into itself. We emphasize that the linear space $\Theta_\lambda$ does not depends upon $\mu$, it is a space of multipliers for $H_{\mu,\lambda}$ no matter what real value $\mu$ assumes.

(ii) The mapping $\phi(x) \longrightarrow x^{n\lambda} \phi(x)$ is an isomorphism from $H_{\mu,\lambda}$ onto $H_{\mu+n,\lambda}$. Consequently, the mapping $f(x) \longrightarrow x^{n\lambda} f(x)$, which is defined by

$$\langle x^{n\lambda} f(x), \phi(x) \rangle = \langle f(x), x^{n\lambda} \phi(x) \rangle$$ is an isomorphism from $H'_{\mu+n,\lambda}$ onto $H'_{\mu,\lambda}$.

Infact,

$$\gamma_{m,k}^{\mu+n,\lambda} (x^{n\lambda}\phi) = \gamma_{m,k}^{\mu,\lambda} (\phi).$$

Some Differential And Integral Operators :

We define two linear differential operations $N_{\mu,\lambda}$ and $M_{\mu,\lambda}$ and a integral operator $N^{-1}_{\mu,\lambda}$ by

$$N_{\mu,\lambda}\, \phi(x) = x^{\lambda\mu+1/2}\, D_x\, x^{-\lambda\mu-\lambda+1/2}\, \phi(x) \qquad (3.3.1)$$

$$M_{\mu,\lambda}\, \phi(x) = x^{-\lambda\mu-\lambda+1/2}\, D_x\, x^{\lambda\mu+1/2}\, \phi(x) \qquad (3.3.2)$$

and $$N^{-1}_{\mu,\lambda}\, \phi(x) = x^{\lambda\mu+\lambda-1/2} \int_{\infty}^{x} t^{-\lambda\mu-1/2}\, \phi(t)\, dt.. \qquad (3.3.3)$$

The operator $N^{-1}_{\mu,\lambda}$ is certainly defined on every locally integrable function of rapid descent and therefore on every $\phi \in H_{\mu+1,\lambda}$. Moreover, $N_{\mu,\lambda}$ and $N^{-1}_{\mu,\lambda}$ are inverses of each other whenever $\phi$ and its derivatives are continuous on

$0 < x < \infty$, and of rapid descent as $x \to \infty$.

(iii) The differential operator $N_{\mu,\lambda}$ defined in (3.3.1) is a continuous transformation of $H_{\mu,\lambda}$ into $H_{\mu+1,\lambda}$ i.e. the mapping $\phi \to N_{\mu,\lambda}\,\phi$ is a continuous linear mapping of $H_{\mu,\lambda}$ into $H_{\mu+1,\lambda}$.

(iv) The operator $\phi \longrightarrow M_{\mu,\lambda}\,\phi$ is a continuous linear mapping of M $H_{\mu+1,\lambda}$ into $H_{\mu,\lambda}$.

(v) The generalized differential operator $N_{\mu,\lambda}$, defined on $H'_{\mu,\lambda}$ by

$$\langle N_{\mu,\lambda}(f), \phi \rangle = \langle f, -M_{\mu,\lambda}(\phi) \rangle, \quad f \in H'_{\mu,\lambda}, \; \phi \in H_{\mu+1,\lambda}.$$

Consequently, $f \longrightarrow N_{\mu,\lambda}(f)$ is a continuous linear mapping of $H'_{\mu,\lambda}$ into $H'_{\mu+1,\lambda}$.

(vi) The generalized differential operator $M_{\mu,\lambda}$, defined on $H'_{\mu+1,\lambda}$ by the relation

$$\langle M_{\mu,\lambda}(f), \phi \rangle = \langle f, -N_{\mu,\lambda}(\phi) \rangle; \quad f \in H'_{\mu+1,\lambda}, \; \phi \in H_{\mu,\lambda}.$$

Therefore, $f \longrightarrow M_{\mu,\lambda}(f)$ is an isomorphism from $H'_{\mu+1,\lambda}$ onto $H_{\mu,\lambda}$. It follows by generalization of the Lemma [30, p.137].

(vii) We can define the generalized differential operator $M_{\mu,\lambda}\ N_{\mu,\lambda}$ for $f \in H'_{\mu,\lambda}$, $\phi \in H_{\mu,\lambda}$ by

$$\langle M_{\mu,\lambda}\ N_{\mu,\lambda}\ (f),\ \phi \rangle = \langle f,\ M_{\mu,\lambda}\ N_{\mu,\lambda}\ (\phi) \rangle$$

Consequently, $f \longrightarrow M_{\mu,\lambda}\ N_{\mu,\lambda}\ (f)$ is a continuous linear mapping of $H'_{\mu,\lambda}$ into itself.

Lemma : 3.3 : $\phi \dashrightarrow N^{-1}_{\mu,\lambda}\ (\phi)$ is a continuous linear mapping of $H_{\mu+1,\lambda}$ into $H_{\mu,\lambda}$.

Proof : Assume that, $\phi \in H_{\mu+1,\lambda}$ and k be a fixed positive integer. Then, we have

$$\gamma^{\mu+1,\lambda}_{m,k}\ (N_{\mu,\lambda}\ \phi(x))$$

$$= \sup_{0<x<\infty} \left| x^{m\lambda} (x^{1-2\lambda} D_x)^k\ x^{-\lambda\mu-\lambda+1/2} (x^{\lambda\mu+\lambda-1/2} \int_{\infty}^{x} t^{-\lambda\mu-1/2} \phi(t)dt) \right|$$

$$= \sup_{0<x<\infty} \left| x^{m\lambda} (x^{1-2\lambda} D_x)^k\ (x^{1-2\lambda} D_x) \int_{\infty}^{x} t^{-\lambda\mu-1/2}\ \phi(t)dt \right|$$

$$= \sup_{0<x<\infty} \left| x^{m\lambda} (x^{1-2\lambda} D_x)^{k-1}\ (x^{1-2\lambda}\ x^{-\lambda\mu-1/2}\ \phi(x)\ ) \right|$$

$$= \sup_{0<x<\infty} \left| x^{m\lambda}\ (x^{1-2\lambda} D_x)^{k-1}\ x^{-\lambda(\mu+1)-1/2}\ \phi(x) \right|$$

$$= \gamma^{\mu+1,\lambda}_{m,k-1} \quad ; \quad k = 1, 2, 3, \ldots \qquad (a)$$
$$m = 0, 1, 2, \ldots$$

Similar result for the case k = 0, can be obtained as follows :

$$\left| x^{m\lambda} x^{-\lambda\mu-\lambda+1/2} N^{-1}_{\mu,\lambda} (\phi) \right|$$

$$= \left| x^{m\lambda} x^{-\lambda\mu-\lambda+1/2} (x^{\lambda\mu+\lambda-1/2} \int_x t^{-\lambda\mu-1/2} \phi(t)\, dt \right|$$

$$\leq x^{m\lambda} \int_x^{\infty} \left| t^{-\lambda\mu-1/2} \phi(t) \right| dt$$

$$\leq \int_x^{\infty} \left| t^{m\lambda-\lambda\mu-1/2} \phi(t) \right| dt$$

$$\leq \int_0^{\infty} \left| t^{m\lambda-\lambda\mu-1/2} \phi(f) \right| dt$$

$$\leq \frac{1}{\lambda} \int_0^{\infty} \frac{dt}{1+t^{2\lambda}} (\lambda t^{\lambda-1}) \sup_{o\ t\ \infty} \left\{ \left| (t^{(m+1)\lambda} + t^{(m+3)\lambda}) t^{-\lambda\mu-2\lambda+1/2} \phi\,(t) \right| \right\} dt$$

Therefore,

$$\gamma^{\mu,\lambda}_{m,o} (N^{-1}_{\mu,\lambda}\, \phi) \leq (\tfrac{1}{\lambda})\, \frac{\pi}{2} \left[ \gamma^{\mu+1,\lambda}_{m+1,o} (\phi) + \gamma^{\mu+1,\lambda}_{m+3,o} (\phi) \right] \qquad (b)$$

$$m = 0, 1, 2, \ldots\ldots$$

From (a) and (b) the result follows.

## * 3.4 : Operation Transform Formulae :

We have the following results from Ghosh [9,VIII].

(1) For $\mu \geqslant -1/2$ and $\lambda > 0$, the mapping $\phi \longrightarrow h_{\mu,\lambda}(\phi)$ is an automorphism from $H_{\mu,\lambda}$ onto itself.

(2) For $\mu \geqslant -1/2$, $\lambda > 0$. If $\phi \in H_{\mu,\lambda}$, then

$$\lambda\, h_{\mu+1,\lambda}(-x^{\lambda}\phi) = N_{\mu,\lambda}\, h_{\mu,\lambda}\, (\phi) \qquad (3.4.1)$$

$$h_{\mu+1,\lambda}(N_{\mu,\lambda}\phi) = -\lambda\, y^{\lambda}\, h_{\mu,\lambda}\,(\phi) \qquad (3.4.2)$$

$$\lambda^2\, h_{\mu,\lambda}\,(-x^{2\lambda}\,\phi) = M_{\mu,\lambda} N_{\mu,\lambda}\, h_{\mu,\lambda}\,(\phi) \qquad (3.4.3)$$

$$h_{\mu,\lambda}\,(M_{\mu,\lambda}\, N_{\mu,\lambda}\phi) = -\lambda^2\, y^{2\lambda}\, h_{\mu,\lambda}(\phi) \qquad (3.4.4)$$

If $\phi \in H_{\mu+1,\lambda}$, then

$$\lambda\, h_{\mu,\lambda}\,(x^{\lambda}\,\phi) = M_{\mu,\lambda}\, h_{\mu+1,\lambda}\,(\phi) \qquad (3.4.5)$$

$$h_{\mu,\lambda}\,(M_{\mu,\lambda}\phi) = \lambda\, y^{\lambda}\, h_{\mu+1,\lambda}\,(\phi) \qquad (3.4.6)$$

## 3.5 An Isomorphism From $H_{\mu,\lambda}$ Onto $H_{\nu,\lambda}$ :

Theorem : 3.5.1 : For $\nu + \mu \geqslant -1$ and $\lambda > 0$ the mapping $\phi \rightarrow h_{\nu,\mu,\lambda}(\phi)$ is an isomorphism from $H_{\mu,\lambda}$ onto $H_{\nu,\lambda}$, the inverse mapping being $\phi \rightarrow h_{\mu,\nu,\lambda}(\phi)$.

Proof For $\phi \in H_{\mu,\lambda}$, we have

$$h_{\nu,\mu,\lambda}(\phi) = \int_0^\infty (xy)^{\lambda(\nu/2-\mu/2+1)-1/2} J_{\frac{\nu+\mu}{2}}(x^\lambda y^\lambda)\, \phi(y)\, dy$$

$$= x^{\lambda(\nu/2-\mu/2)} \int_0^\infty (xy)^{\lambda-1/2} J_{\frac{\nu+\mu}{2}}(x^\lambda y^\lambda) (y^{\lambda(\nu/2-\mu/2)} \phi(y))\, dy$$

$$= x^{\lambda(\nu/2-\mu/2)} h_{\frac{\nu+\mu}{2},\lambda} (y^{\lambda(\nu/2-\mu/2)} \phi(y))$$

or $$x^{\lambda(\mu/2-\nu/2)} h_{\nu,\mu,\lambda}(\phi) = h_{\frac{\nu+\mu}{2},\lambda} (y^{\lambda(\nu/2-\mu/2)} \phi(y)) \quad (3.5.1)$$

Now, the mapping $\phi(y) \longrightarrow y^{\lambda(\nu/2-\mu/2)} \phi(y)$ is a continuous linear mapping of $H_{\mu,\lambda}$ onto $H_{\frac{\nu+\mu}{2},\lambda}$.

For,

$$\gamma_{m,\lambda}^{\frac{\nu+\mu}{2},\lambda} (y^{\lambda(\nu/2-\mu/2)} \phi(y))$$

$$= \sup_{0<y<\infty} \left| y^{m\lambda} (y^{1-2\lambda} D_y)^k y^{-\lambda(\nu/2+\mu/2+1)+1/2} (y^{\lambda(\nu/2-\mu/2)} \phi(y)) \right|$$

$$= \sup_{0<y<\infty} \left| y^{m\lambda} (y^{1-2\lambda} D_y)^k y^{-\lambda\mu-\lambda+1/2} \phi(y) \right|$$

$$= \gamma_{m,k}^{\mu,\lambda} (\phi(y)).$$

Again, it can be easily shown that the mapping $\phi \to y^{\lambda(\mu/2-\nu/2)}\phi$ is the inverse of the above mapping $\phi \to y^{\lambda(\nu/2-\mu/2)}\phi$; and is a continuous linear mapping from $H_{\frac{\nu+\mu}{2},\lambda}$ onto $H_{\mu,\lambda}$.

Infact,

$$\gamma_{m,k}^{\mu,\lambda}\left(y^{\lambda(\mu/2-\nu/2)}\phi(y)\right).$$

$$= \sup_{0<y<\infty}\left|y^{m\lambda}(y^{1-2\lambda}D_y)^k\, y^{-\lambda\mu-\lambda+1/2}\left(y^{\lambda(\mu/2-\nu/2)}\phi(y)\right)\right|$$

$$= \sup_{0<y<\infty}\left|y^{m\lambda}(y^{1-2\lambda}D_y)^k\, y^{-\lambda(\nu/2+\mu/2)-\lambda+1/2}\phi(y)\right|$$

$$= \gamma_{m,k}^{\frac{\nu+\mu}{2},\lambda}(\phi(y)) \qquad (3.5.2)$$

In view of, the known result (3.4.(1)), we have $h_{\frac{\nu+\mu}{2},\lambda}$ is an automorphism on $H_{\frac{\nu+\mu}{2},\lambda}$.

Again from (3.5.2) it follows that $\phi \to y^{\lambda(\nu/2-\mu/2)}\phi(y)$ is a continuous linear mapping of $H_{\frac{\nu+\mu}{2},\lambda}$ onto $H_{\nu,\lambda}$. Combining the above results we have

$\phi(y) \longrightarrow x^{\lambda(\nu/2-\mu/2)}.\ h_{\frac{\nu+\mu}{2},\lambda}(y^{\lambda(\nu/2-\mu/2)}\phi(y))$ is a continous linear mapping from $H_{\mu,\lambda}$ onto $H_{\nu,\lambda}$. It follows, from (3.5.1) that $\phi \to h_{\nu,\mu,\lambda}(\phi)$ is a continuous linear mapping from $H_{\mu,\lambda}$ onto $H_{\nu,\lambda}$. It is also clear from (3.5.1) that $h_{\mu,\nu,\lambda}$

is the inverse of $h_{\nu,\mu,\lambda}$ and so is a continuous linear mapping from $H_{\nu,\lambda}$ onto $H_{\mu,\lambda}$. Hence, the result.

Theorem : 3.5.2 : For $\nu+\mu \geq -1$ and $\lambda > o$. If $\phi$ $H_{\mu,\lambda}$, then

$$\lambda h_{\nu+1,\mu+1,\lambda}(-x^{\lambda}\phi) = N_{\nu,\lambda}\ h_{\nu,\mu,\lambda}\ (\phi) \qquad (3.5.3)$$

$$h_{\nu+1,\mu+1,\lambda}(N_{\mu,\lambda}\phi) = -\lambda y^{\lambda} h_{\nu,\mu,\lambda}\ (\phi) \qquad (3.5.4)$$

$$h_{\nu,\mu,\lambda}\ (M_{\nu,\lambda}\ N_{\mu,\lambda}\ \phi) = -\lambda^2\ y^{2\lambda}\ h_{\nu,\mu,\lambda}\ (\phi) \qquad (3.5.5)$$

$$\lambda^2\ h_{\nu,\mu,\lambda}\ (-x^{2\lambda}\ \phi) = M_{\mu,\lambda}\ N_{\nu,\lambda}\ h_{\nu,\mu,\lambda}\ (\phi) \qquad (3.5.6)$$

If $\phi \in H_{\mu+1,\lambda}$, then

$$\lambda h_{\nu,\mu,\lambda}\ (x^{\lambda}\ \phi) = M_{\mu,\lambda}\ h_{\nu+1,\mu+1,\lambda}\ (\phi) \qquad (3.5.7)$$

$$h_{\nu,\mu,\lambda}\ (M_{\mu,\lambda}\ \phi) = \lambda y^{\lambda}\ h_{\nu+1,\ \mu+1,\lambda}\ (\phi) \qquad (3.5.8)$$

Proof : For $\nu+\mu \geq -1$, $\lambda > o$ and if $\phi \in H_{\mu,\lambda}$ then by theorem (3.5.1), we obtain $x^{\lambda(\nu/2-\mu/2)}\ \phi \in H_{\frac{\nu+\mu}{2},\lambda}$

and $y^{\lambda(\mu/2-\nu/2)}\ h_{\nu,\mu,\lambda}\ (\phi) = h_{\frac{\nu+\mu}{2},\lambda}\ (x^{\lambda(\nu/2-\mu/2)}\phi)$ by (3.5-1). (3.5.9).

Therefore,

$$\lambda y^{\lambda(\mu/2\ -\ \nu/2)}\ h_{\nu+1,\mu+1,\lambda}\ (-x^{\lambda}\phi) = \lambda h_{\frac{\nu+\mu}{2}+1,\lambda}(-x^{\lambda(\nu/2-\mu/2+1)}\phi(x))$$

$$= N_{\frac{\nu+\mu}{2},\lambda}\ h_{\frac{\nu+\mu}{2},\lambda}(x^{\lambda(\nu/2-\mu/2)}\phi) \text{ by}(3.4.1)$$

or $\quad \lambda\, y^{\lambda(\mu/2-\nu/2)}\, h_{\nu+1,\mu+1,\lambda}\,(-x^{\lambda}\phi)$

$$= N_{\frac{\nu+\mu}{2},\lambda}\,(y^{\lambda(\mu/2-\nu/2)}\, h_{\nu,\mu,\lambda}\,(\phi)) \qquad \text{by} \qquad (3.5.9)$$

i.e. $\lambda\, h_{\nu+1,\mu+1,\lambda}(-x^{\lambda}\phi) = y^{\lambda(\nu/2-\mu/2)}\, N_{\frac{\nu+\mu}{2},\lambda}(y^{\lambda(\mu/2-\nu/2)}\, h_{\nu,\mu,\lambda}(\phi))$

$$= N_{\nu,\lambda}\, h_{\nu,\mu,\lambda}\,(\phi)$$

Hence (3.5.3) follows.

Since $\phi \in H_{\mu,\lambda}$ and by theorem (3.5.1), it follows that $x^{\lambda(\nu/2-\mu/2)}\phi \in H_{\frac{\nu+\mu}{2},\lambda}$, and in view of (3.4.2), we obtain

$$h_{\frac{\nu+\mu}{2},\lambda}\,(N_{\frac{\nu+\mu}{2},\lambda}\,(x^{\lambda(\nu/2-\mu/2)}\,\phi)) = -\lambda\, y^{\lambda}\, h_{\frac{\nu+\mu}{2},\lambda}(x^{\lambda(\nu/2-\mu/2)}\phi)$$

It is clear that

$$N_{\frac{\nu+\mu}{2},\lambda}\,(x^{\lambda(\nu/2-\mu/2)}\phi) = x^{\lambda(\nu/2-\mu/2)}\, N_{\mu,\lambda}\,(\phi).$$

Therefore,

$$h_{\frac{\nu+\mu}{2}+1,\lambda}(x^{\lambda(\nu/2-\mu/2)}\, N_{\mu,\lambda}(\phi)) = -\lambda\, y^{\lambda}\, h_{\frac{\nu+\mu}{2},\lambda}(x^{\lambda(\nu/2-\mu/2)}\phi)$$

or $\quad y^{\lambda(\mu/2-\nu/2)}\, h_{\nu+1,\mu+1,\lambda}(N_{\mu,\lambda}\phi) = -\lambda\, y^{\lambda}\, y^{\lambda(\mu/2-\nu/2)}\, h_{\nu,\mu,\lambda}(\phi).$

i.e. $\quad h_{\nu+1,\mu+1,\lambda}\,(N_{\mu,\lambda}\,\phi) = -\lambda\, y^{\lambda}\, h_{\nu,\mu,\lambda}\,(\phi).$

Hence (3.5.4) follows.

If $\nu + \mu \geq -1$, $\lambda > 0$ and $\phi \in H_{\frac{\nu+\mu}{2},\lambda}$, then in view of (3.4.3)

we get

$$\lambda^2 \; h_{\frac{\nu+\mu}{2},\lambda} \; (-x^{2\lambda}\phi) = M_{\frac{\nu+\mu}{2},\lambda} \; N_{\frac{\nu+\mu}{2},\lambda} \; h_{\frac{\nu+\mu}{2},\lambda} \; (\phi)$$

Since $\phi \in H_{\mu,\lambda}$ implies that $x^{\lambda(\nu/2-\mu/2)} \phi \in H_{\frac{\nu+\mu}{2},\lambda}$

Therefore,

$$\lambda^2 \; h_{\frac{\nu+\mu}{2},\lambda}(-x^{\lambda(\nu/2-\mu/2+2)} \phi)$$

$$= M_{\frac{\nu+\mu}{2},\lambda} \; N_{\frac{\nu+\mu}{2},\lambda} \; h_{\frac{\nu+\mu}{2},\lambda} \; (x^{\lambda(\nu/2-\mu/2)}\phi)$$

or $\lambda^2 \; y^{\lambda(\mu/2-\nu/2)} \; h_{\nu,\mu,\lambda} \; (-x^{2\lambda} \phi)$

$$= y^{-\lambda(\nu/2+\mu/2-1)+1/2} D_y \; y^{\lambda(\nu/2+\mu/2)+1/2}$$

$$\Big(y^{\lambda(\nu/2+\mu/2)+1/2} D_y \; y^{-\lambda(\nu/2+\mu/2)+1/2}$$

$$h_{\frac{\nu+\mu}{2},\lambda} \; (x^{\lambda(\nu/2-\mu/2)}\phi)\Big)$$

i.e. $\lambda^2 \; h_{\nu,\mu,\lambda}(-x^{2\lambda}\phi) = y^{-\lambda\mu-\lambda+1/2} D_y (y^{\lambda(\mu+\nu)+1} D_y \; y^{-\lambda\nu-\lambda+1/2} h_{\nu,\mu,\lambda}$

i.e. $\lambda^2 \; h_{\nu,\mu,\lambda}(-x^{2\lambda}\phi) = M_{\mu,\lambda} \; N_{\mu,\lambda} \; h_{\nu,\mu,\lambda} \; (\phi)$

Thus, the result (3.5.6) follows.

Again in view of (3.4.6), we know that for $\nu+\mu \geq -1$, $\lambda > 0$

and if $\phi \in H_{\frac{\nu+\mu}{2}+1,\lambda}$ then $x^{\lambda(\nu/2-\mu/2)}\phi \in H_{\frac{\nu+\mu}{2}+1,\lambda}$

Therefore,

$$h_{\frac{\nu+\mu}{2},\lambda}\left\{M_{\frac{\nu+\mu}{2},\lambda}\,(x^{\lambda(\nu/2-\mu/2)}\phi)\right\}$$

$$= \lambda y^{\lambda}\, h_{\frac{\nu+\mu}{2}+1,\lambda}\,(x^{\lambda(\nu/2-\mu/2)}\phi).$$

i.e. $h_{\frac{\nu+\mu}{2},\lambda}\,(x^{\lambda(\nu/2-\mu/2)}M_{\nu,\lambda}\,\phi) = \lambda y^{\lambda}\,y^{\lambda(\mu/2-\nu/2)}h_{\nu+1,\mu+1,\lambda}(\phi)$

i.e. $h_{\nu,\mu,\lambda}\,(M_{\nu,\lambda}\,\phi) = \lambda y^{\lambda}\,h_{\nu+1,\mu+1,\lambda}\,(\phi).$

Hence, (3.5.8) follows.

The results (3.5.5) and (3.5.7) can be obtained by similar argument from the results of (3.4(2)).

## * 3.6 : Definition of Self-Reciprocal Generalized Function $R'_{\mu,\lambda}$ :

We shall now give a definition of self-reciprocal generalized function $R'_{\mu,\lambda}$.

Definition : For $\mu \geq -1/2$ and $\lambda > 0$, a generalized function f in $H'_{\mu,\lambda}$ is said to be a self-reciprocal generalized function

$R_{\mu,\lambda}$ if $h'_{\mu,\lambda}$ (f) = f in $H'_{\mu,\lambda}$.

i.e. if $\langle h'_{\mu,\lambda}$ (f), $\phi \rangle = \langle f, \phi \rangle$, for all $\phi \in H_{\mu,\lambda}$. Thus, a generalized function f in $H'_{\mu,\lambda}$ is a self-reciprocal generalized function $R'_{\mu,\lambda}$ if $\langle h'_{\nu,\mu,\lambda}$ (f), $\phi \rangle = \langle f, \phi \rangle$, for $\nu+\mu \geq -1$ and for all $\phi \in H_{\mu,\lambda}$. It is clear that $h_{\nu,\mu,\lambda} = h_{\mu,\lambda}$ for $\mu \geq -1/2$. Also, when $\lambda = 1$ and $\mu \geq -1/2$, it reduce to $h'_{\mu}(f) = f$ i.e. if $\langle h'_{\mu}(f), \phi \rangle = \langle f, \phi \rangle$ for all $\phi \in H_{\mu}$, $f \in H'_{\mu}$. Which implies that f is a self-reciprocal generalized function $R'_{\mu}$.

## * 3.7 : The Distributional Hankel Transformation $h'_{\nu,\mu,\lambda}$ on $H'_{\mu,\lambda}$:

We shall give a definition of $h'_{\nu,\mu,\lambda}(f)$ for $f \in H'_{\mu,\lambda}$ as follows :

Definition : For $\nu + \mu \geq -1$, $\lambda > 0$ and $\phi \in H_{\mu,\lambda}$, we define a distributional Hankel transformation $h'_{\nu,\mu,\lambda}$ on $H'_{\mu,\lambda}$ as the adjoint of $h_{\nu,\mu,\lambda}$ defined on $H_{\mu,\lambda}$ by the relation.

$$\langle h'_{\nu,\mu,\lambda}(f), \phi \rangle = \langle f, h_{\nu,\mu,\lambda}(\phi) \rangle , \quad f \in H'_{\mu,\lambda} \qquad \ldots(3.7.1)$$

From the consequence of known result of Zemanian [30, p. 29] and Theorem 3.5-1, we have the following

Theorem : 3.7 : The operator $f \longrightarrow h'_{\nu,\mu,\lambda}(f)$ is an isomorphism from $H'_{\nu,\lambda}$ onto $H'_{\mu,\lambda}$. Moreover, the inverse operator being $f \longrightarrow h'_{\mu,\nu,\lambda}(f)$.

Proof : Since $\phi \longrightarrow h_{\nu,\mu,\lambda}(\phi)$ is an isomorphism from $H_{\mu,\lambda}$ onto $H_{\nu,\lambda}$, and $h'_{\nu,\mu,\lambda}$ is the adjoint of $h_{\nu,\mu,\lambda}$ defined on $H'_{\nu,\lambda}$, $h'_{\nu,\mu,\lambda}$ is a continuous linear mapping of $H'_{\nu,\lambda}$ into $H'_{\mu,\lambda}$. Now, let $\psi = h_{\nu,\mu,\lambda}(\phi) \in H_{\nu,\lambda}$ for $\phi \in H_{\mu,\lambda}$. For any $f \in H'_{\nu,\lambda}$ we have $(h'^{-1}_{\nu,\mu,\lambda})\ (h'_{\nu,\mu,\lambda} f) = f$

because, $\langle f, \psi \rangle = \langle f, (h_{\nu,\mu,\lambda})\ (h^{-1}_{\nu,\mu,\lambda}\psi) \rangle$

$$= \langle (h'^{-1}_{\nu,\mu,\lambda})\ h'_{\nu,\mu,\lambda}(f), \psi \rangle$$

Similarly, for any $g \in H'_{\mu,\lambda}$ we have $(h'_{\nu,\mu,\lambda})(h'^{-1}_{\nu,\mu,\lambda})g = g$

because for $\phi \in H_{\mu,\lambda}$,

$$\langle g, \phi \rangle = \langle g, (h^{-1}_{\nu,\mu,\lambda})(h_{\nu,\mu,\lambda})\ \phi \rangle$$

$$= \langle (h'_{\nu,\mu,\lambda})\ h'^{-1}_{\nu,\mu,\lambda}(g), \phi \rangle$$

It follows that $h'_{\nu,\mu,\lambda}$ is a one-to-one mapping from $H'_{\nu,\lambda}$ on to $H'_{\mu,\lambda}$. Hence, the theorem.

•

***3.7.1 : Operation Transform Formula :**

In this section, we shall establish a few results in the form of theorem with the help of Theorem 3.5.2.

Theorem : 3.7.1 : For $\nu + \mu \geqslant -1$, $\lambda > 0$ and if $f \in H'_{\nu,\lambda}$ then

$$\lambda\, h'_{\nu,\mu,\lambda}(-x^{\lambda} f) = N_{\nu,\lambda}\, h'_{\nu,\mu,\lambda}(f) \tag{3.7.1}$$

$$h'_{\nu+1,\mu+1,\lambda}(N_{\mu,\lambda}(f)) = -\lambda\, y^{\lambda}\, h'_{\nu,\mu,\lambda}(f) \tag{3.7.2}$$

$$\lambda^{2}\, h'_{\nu,\mu,\lambda}(x^{2\lambda} f) = -M_{\mu,\lambda} N_{\nu,\lambda}\, h'_{\nu,\mu,\lambda}(f) \tag{3.7.3}$$

$$h'_{\nu,\mu,\lambda}(M_{\nu,\lambda} N_{\mu,\lambda}(f)) = -\lambda^{2}\, y^{2\lambda}\, h'_{\nu,\mu,\lambda}(f) \tag{3.7.4}$$

If $f \in H'_{\nu+1,\lambda}$, then

$$\lambda\, h'_{\nu,\mu,\lambda}(x^{\lambda} f) = M_{\mu,\lambda}\, h'_{\nu+1,\mu+1,\lambda}(f) \tag{3.7.5}$$

$$h'_{\nu,\mu,\lambda}(M_{\nu,\lambda}(f)) = \lambda y^{\lambda}\, h'_{\nu+1,\mu+1,\lambda}(f) \tag{3.7.6}$$

Proof : Firstly, we shall establish the result (3.7.2).

Let $\phi(y) \in H_{\mu+1,\lambda}$.

We have

$$\left\langle h'_{\nu+1,\mu+1,\lambda}(N_{\mu,\lambda}(f)),\ \phi(y)\right\rangle$$

$$= \left\langle N_{\mu,\lambda}(f),\ h_{\nu+1,\mu+1,\lambda}(\phi(y))\right\rangle$$

i.e. $\langle h'_{\nu+1,\ \mu+1,\lambda}(N_{\mu,\lambda}(f)),\quad \phi(y)\rangle$

$$= \langle f,\quad -M_{\mu,\lambda}(h_{\nu+1,\ \mu+1,\lambda}(\phi(y)))\rangle \text{ by (3.3(v))}$$

$$= \langle f,\quad -\lambda\, h_{\nu,\mu,\lambda}(y^{\lambda}\phi(y))\rangle \qquad \text{by (3.5.7)}$$

$$= \langle(-\lambda)\ h'_{\nu,\mu,\lambda}(f),\quad y^{\lambda}\ \phi(Y)\rangle$$

$$= \langle(-\lambda)\ y^{\lambda} h'_{\nu,\mu,\lambda}(f),\quad \phi(y)\rangle$$

Hence

$$h'_{\nu+1,\ \mu+1,\lambda}(N_{\mu,\lambda}f) = -\lambda\ y^{\lambda}\ h'_{\nu,\mu,\lambda}(f).$$

The equality has a sense in $H'_{\mu+1,\lambda}$ because $(-\lambda)y^{\lambda}\ h'_{\nu,\mu,\lambda}(f)$ belongs to $H'_{\mu-1,\lambda}$. So its restriction to $H_{\mu+1,\lambda}$ is in $H'_{\mu+1,\lambda}$.

No we prove the result (3.7.1).

For $f \in H'_{\mu,\lambda}$. Let $F = h'_{\mu,\nu,\lambda}(f)$ then i.e.

$f = h'_{\nu,\mu,\lambda}(F)$ and $F \in H'_{\nu,\lambda}$.

Now,

$$N_{\nu,\lambda}(h'_{\nu,\mu,\lambda}(F)) = N_{\nu,\lambda}(f)$$
$$= (h'_{\nu+1,\mu+1,\lambda})(h'_{\nu+1,\mu+1,\lambda})N_{\nu,\lambda}(f)$$
$$= h'_{\nu+1,\mu+1,\lambda}\left\{ -\lambda y^{\lambda} h'_{\nu,\mu,\lambda}(f) \right\}$$
$$= h'_{\nu+1,\mu+1,\lambda}(-\lambda\ y^{\lambda}F)$$

Therefore,

$$N_{\nu,\lambda}\ h'_{\nu,\mu,\lambda}(f) = -\lambda h'_{\nu+1,\mu+1,\lambda}(y^{\lambda}F)$$

This is the same case as (3.7.1), by replacing y by x and F by f respectively. This equality has a sense in $H'_{\mu+1,\lambda}$.

Again the result (3.7.6) established as follows

Let $\phi \in H_{\mu,\lambda}$, we can write

$$\left\langle h'_{\nu,\mu,\lambda}(M_{\nu,\lambda}f),\ \phi(y)\right\rangle = \left\langle M_{\nu,\lambda}(f), h_{\nu,\mu,\lambda}(\phi(y))\right\rangle$$
$$= \left\langle f,\ -N_{\nu,\lambda}\ h_{\nu,\mu,\lambda}\phi(y)\right\rangle \text{ by } (3.3(vi))$$
$$= \left\langle f,\ \lambda h_{\nu+1,\mu+1,\lambda}(y^{\lambda}\phi)\right\rangle \text{ by } (3.5.3)$$
$$= \left\langle \lambda y^{\lambda} h'_{\nu+1,\mu+1,\lambda}(f),\ \phi(y)\right\rangle$$

Therefore,

$$h'_{\nu,\mu,\lambda}(M_{\nu,\lambda}f) = \lambda\ y^{\lambda}\ h'_{\nu+1,\mu+1,\lambda}\ (f).$$

It is clear that the equality is in $H'_{\mu,\lambda}$. Lastly, we shall prove (3.7.5) as follows :

Let $\emptyset \in H_{\mu,\lambda}$, then we have

$$\langle M_{\mu,\lambda} h'_{\nu+1,\mu+1,\lambda}(f), \emptyset(y)\rangle = \langle h'_{\nu+1,\mu+1,\lambda}(f), -N_{\mu,\lambda}\emptyset(y)\rangle \quad \text{by (3.3.(vi))}$$

$$= \langle f , -h_{\nu+1,\mu+1,\lambda}(N_{\mu,\lambda}\emptyset(y))\rangle$$

$$= \langle f , \lambda x^{\lambda} h_{\nu,\mu,\lambda}(\emptyset(y))\rangle \quad \text{by (3.5.4)}$$

$$= \langle \lambda\, h'_{\nu,\mu,\lambda}(x^{\lambda} f), \emptyset(y)\rangle$$

Thus,

$$M_{\mu,\lambda} h'_{\nu+1,\mu+1,\lambda}(f) = \lambda h'_{\nu,\mu,\lambda}(x^{\lambda} f).$$

The results (3.7.3) and (3.7.4) can be obtained by similar argument.

*3.8 : <u>The Generalized Hankel Transform of Aribtrary Order</u> :

The definition of the transform $h_{\nu,\mu,\lambda}$ on $H_{\mu,\lambda}$, where $\nu$ and $\mu$ are any pair of real numbers, can be obtained as a generalization of the transform $h_{\nu,\mu,\lambda}$ defined on $H_{\mu,\lambda}$ by the equation (3) for $\nu+\mu \geq -1$. Let k be any positive integer such that $\nu+\mu+2k \geq -1$. We define the transformation $h_{\nu,\mu,k,\lambda}$ on $H_{\mu,\lambda}$ as follows. For $\nu+\mu+2k \geq -1$, $\lambda > 0$ and if $\emptyset \in H_{\mu,\lambda}$ and $\Phi(y) = h_{\nu,\mu,\lambda}(\emptyset)$ then

$$\bar{\Phi}(y) = h_{\nu,\mu,k,\lambda}(\phi)$$

$$= (-1)^k \lambda^{-k} y^{-\lambda k} h_{\nu+k,\mu+k,\lambda}(N_{\mu+k-1,\lambda} \cdots N_{\mu,\lambda}\phi) \quad ..(3.8.1)$$

and

$$\phi(x) = h^{-1}_{\nu,\mu,k,\lambda}(\bar{\Phi}(y))$$

$$= (-1)^k \lambda^k (N^{-1}_{\mu,\lambda} \cdots N^{-1}_{\mu+k-1,\lambda}) h_{\mu+k,\nu+k,\lambda}(y^{\lambda k}\bar{\Phi}(y)). \quad (3.8.2)$$

Where $h_{\mu+k,\nu+k,\lambda}$ is the inverse of $h_{\nu+k,\mu+k,\lambda}$.

When $\nu + \mu \geqslant -1$, it is clear that

$h_{\nu,\mu,k,\lambda} = h_{\nu,\mu,\lambda}$ and $h^{-1}_{\nu,\mu,k,\lambda} = h^{-1}_{\nu,\mu,\lambda}$ and so we can consider $h^{-1}_{\nu,\mu,k,\lambda}$ is the inverse of $h_{\nu,\mu,k,\lambda}$. Clearly $h_{\nu,\mu,k,\lambda}$ is an isomorphism from $H_{\mu,\lambda}$ onto $H_{\nu,\lambda}$. It can be shown that the transformations defined as in (3.8.1) and (3.8.2) are unique in the sanse that

$h_{\nu,\mu,k,\lambda} = h_{\nu,\mu,p}$ where $\nu+\mu+2k \geqslant -1$ and $\nu+\mu+2p \geqslant -1$, p is a positive integer.

Similarly

$h^{-1}_{\nu,\mu,k,\lambda} = h^{-1}_{\nu,\mu,p,\lambda}$ where $\nu+\mu+2k \geqslant -1$ and $\nu+\mu+2p \geqslant -1$.

The generalization of $\bar{\Phi}(y) = h_{\nu,\mu,\lambda}(\phi)$ and its inverse for $\phi \in H_{\mu,\lambda}$ can also be given by

$$\bar{\Phi}(y) = \bar{h}_{\nu,\mu,k,\lambda}(\phi)$$

$$= (-1)^k \lambda^k (N^{-1}_{\nu,\lambda} \cdots N^{-1}_{\nu+k-1,\lambda}) h_{\nu+k,\mu+k,\lambda}(x^{\lambda k}\phi). \quad ..(3.8.3)$$

Similarly, $\overline{h}^{-1}_{\nu,\mu,k,\lambda}$ is defined on $H_{\mu,\lambda}$ by

$$\phi(x) = \overline{h}^{-1}_{\nu,\mu,k,\lambda}(\Phi\,(y))$$

$$= (-1)^{k}\lambda^{-k}x^{-\lambda k}\,h_{\mu+k,\nu+k,\lambda}(N_{\nu+k-1,\lambda}\cdots N_{\nu,\lambda}\Phi(y)). \qquad ..(3.8.4)$$

It is clear that

$h_{\nu,\mu,\lambda} = \overline{h}_{\nu,\mu,\lambda}$ and $h^{-1}_{\nu,\mu,\lambda} = h^{-1}_{\nu,\mu,\lambda}$ for $\nu+\mu \geqslant -1$.

This extension is also unique. We shall show that the two generalizations discussed above are actually the same.

From the result Jahnke, Emde, and Losch [30, pp. 139-140], we have

$$D_x(x^{\lambda(\mu+1)}J_{\mu+1}(x^{\lambda}y^{\lambda})) = \lambda y^{\lambda(\mu+2)-1}J_{\mu}(x^{\lambda}y^{\lambda}) \qquad ..(3.8.5)$$

and

$$D_x(x^{-\lambda\mu}J_{\mu}(x^{\lambda}y^{\lambda})) = -\lambda y^{\lambda}x^{-\lambda(\mu-1)-1}J_{\mu+1}(x^{\lambda}y^{\lambda}) \qquad ..(3.8.6)$$

Theorem : 3.8. : For $\nu + \mu + 2k \geqslant -1$, $\lambda > 0$ and if $\phi \in H_{\mu,\lambda}$, then

$$h_{\nu,\mu,k,\lambda}\,(\phi) = \overline{h}_{\nu,\mu,k,\lambda}(\phi) \qquad ..(3.8.7)$$

and

$$h^{-1}_{\nu,\mu,k,\lambda}(\phi) = \overline{h}^{-1}_{\nu,\mu,k,\lambda}\,(\phi) \qquad ..(3.8.8)$$

Proof : To establish the equation (3.8.7). It is sufficient to show that for $\emptyset$ belongs to $H_{\mu,\lambda}$

$$h_{\nu,\mu,k,\lambda}(\emptyset)=(-1)^{k}\lambda^{k}(N^{-1}_{\nu,\lambda}\cdots N^{-1}_{\nu+k-1,\lambda})h_{\nu+k,\mu+k,\lambda}(x^{\lambda k}\emptyset(x))$$

i.e. to show that

$$(N_{\nu+k-1,\lambda}\cdots N_{\nu,\lambda})\ \lambda^{k} h_{\nu,\mu,k,\lambda}(\emptyset) = (-1)^{k}\lambda^{2k} h_{\nu+k,\mu+k,\lambda}(x^{\lambda k}\emptyset(x)).$$

Now, we consider only one term of the L.H.S. of the above equation.

$$N_{\nu,\lambda}(\lambda^{k} h_{\nu,\mu,k,\lambda}(\emptyset(x))$$

$$= N_{\nu,\lambda}\ (-1)^{k} y^{-\lambda k} h_{\nu+k,\mu+k,\lambda}(N_{\mu+k-1,\lambda}\cdots N_{\mu,\lambda}\emptyset)$$

$$= (-1)^{k} y^{\lambda\nu+1/2} D_y y^{-\lambda\nu-\lambda+1/2}\Big(y^{-\lambda k}\int_0^\infty (xy)^{\lambda(\nu/2-\mu/2+1)-1/2} J_{\frac{\nu+\mu}{2}+k,\lambda}(x^{\lambda}y^{\lambda})\psi(x)dx\Big)$$

where $\psi(x) = N_{\mu+k-1,\lambda}\cdots N_{\mu,\lambda}\emptyset(x)$.

i.e. $N_{\nu,\lambda}\{\lambda^{k} h_{\nu,\mu,k,\lambda}(\emptyset(x))\}$

$$= (-1)^{k} y^{\lambda\nu+1/2}\int_0^\infty D_y\Big\{y^{-\lambda(\frac{\nu+\mu}{2}+k)} J_{\frac{\nu+\mu}{2}+k}(x^{\lambda}y^{\lambda})\Big\} x^{\lambda(\nu/2-\mu/2+1)-1/2}\psi(x)\,dx$$

$$= -\lambda(-1)^k y^{-\lambda k} \int_0^\infty \left\{ (xy)^{\lambda(\nu/2-\mu/2+1)-1/2} \right.$$

$$\left. J_{\frac{\nu+\mu}{2}+k+1}(x^\lambda y^\lambda)(\psi(x)x^\lambda)\right\} dx \text{ by } (3.8.6)$$

$$= -\lambda(-1)^k y^{-\lambda k} h_{\nu+k+1,\mu+k+1,\lambda}(x^\lambda \psi(x))$$

Differentiation within the sign of integration may be justified.

Thus, we have

$$N_{\nu,\lambda}(\lambda^k h_{\nu,\mu,k,\lambda}(\phi) = -\lambda(-1)^k y^{-\lambda k} h_{\nu+k+1,\mu+k+1,\lambda}(x^\lambda \psi(x)).$$

By similar procedure, we obtain

$$(N_{\nu+k-1,\lambda} \cdots N_{\nu+1,\lambda} N_{\nu,\lambda})(-1)^k y^{-\lambda k} h_{\nu+k,\mu+k,\lambda}(\psi(x))$$

$$= \lambda^k y^{-\lambda k} h_{\nu+2k,\mu+2k,\lambda}(x^{\lambda k}\psi(x)) \qquad ..(3.8.9)$$

i.e. $(N_{\nu+k-1,\lambda} \cdots N_{\nu,\lambda})\lambda^k h_{\nu,\mu,k,\lambda}(\phi)$

$$= \lambda^k y^{-\lambda k} h_{\nu+2k,\mu+2k,\lambda}(x^{\lambda k}\psi(x))$$

$$= \lambda^k y^{-\lambda k} h_{\nu+2k,\mu+2k,\lambda}\left(x^{\lambda k} N_{\mu+k-1,\lambda} \cdots\cdots N_{\mu,\lambda}(\phi)\right)$$

$$= \lambda^k y^{-\lambda k} (-1)^k \lambda^k y^{\lambda k} h_{\nu+k,\mu+k,\lambda}(x^{\lambda k}\phi(x))$$

$$= (-1)^k \lambda^{2k} h_{\nu+k,\mu+k,\lambda}(x^{\lambda k}\phi(x))$$

by repeated application of (3.7.2).

Hence, the result follows. Similarly the result (3.8.8.) can be established.

Corollary : For $\nu+k \geqslant -1/2$ and $\lambda > 0$. If $\phi \in H_{\mu,\lambda}$, then

$$h_{\nu,k,\lambda}(\phi) = \bar{h}_{\nu,k,\lambda}(\phi).$$

The proof of this follows by putting $\nu = \mu$ in above theorem.

* 3.8.1 : By Theorem 3.8.1, we can say that either (3.8.1) or (3.8.3) can be considered as the extension of the Theorem 3.5.1 for any pair of real numbers $\nu$ and $\mu$ such that $\nu + \mu + 2k \geqslant -1$ for any positive integer k Inverse transform of this extension is given by either (3.8.2) or (3.8.4).

Now, we can define the transform $h_{\nu,\mu,\lambda}$ for $f \in H'_{\nu,\lambda}$ by the relation

$$\langle h'_{\nu,\mu,\lambda}(f), \phi \rangle = \langle f, h_{\nu,\mu,k,\lambda}(\phi) \rangle, \qquad ..(3.8.10)$$

for $\phi \in H_{\mu,\lambda}$, $f \in H'_{\nu,\lambda}$ and $\nu + \mu + 2k \geqslant -1$, $\lambda > 0$.

Since $\phi \longrightarrow h_{\nu,\mu,k,\lambda}(\phi)$ is an isomorphism from $H_{\mu,\lambda}$ onto $H_{\nu,\lambda}$. In view of the known result [30, p.29] it follows that the transformation $h_{\nu,\mu,\lambda}$ defined by (3.8.10), is an isomorphism from $H'_{\nu,\lambda}$ on to $H_{\mu,\lambda}$. Where as the corresponding inverse mapping $h^{-1}_{\nu,\mu,\lambda}$ is given by

$$\left\langle {h'}^{-1}_{\nu,\mu,\lambda}(f), \phi \right\rangle = \left\langle f, h^{-1}_{\nu,\mu,k,\lambda}(\phi) \right\rangle$$

$$= \left\langle f, h_{\mu,\nu,k,\lambda}(\phi) \right\rangle$$

$$= \left\langle h'_{\mu,\nu,\lambda}(f), \phi \right\rangle, \ \phi \in H_{\nu,\lambda}$$

Therefore,

$${h'}^{-1}_{\nu,\mu,\lambda}(f) = h'_{\mu,\nu,\lambda}(f).$$

★ 3.9 : <u>On Self-Reciprocal Distribution</u> :

In this section we shall prove a theorem on self-reciprocal distribution which is a generalization of theorem(2.2).

Theorem : 3.9 : If f is a self-reciprocal generalized function $R'_{\nu,\lambda}$, then $h_{\nu,\mu,\lambda}(f)$ is a self-reciprocal generalized function $R'_{\mu,\lambda}$ for $\mu-2 > \nu \geqslant -1/2$.

Proof : To prove the theorem, we have to show that

$h_{\nu,\mu,\lambda}(f) \in H'_{\mu,\lambda}$ and $h_{\mu,\lambda}(h_{\nu,\mu,\lambda}(f) = h_{\nu,\mu,\lambda}(f)$ in $H'_{\mu,\lambda}$ since the mapping $\phi \longrightarrow h_{\nu,\mu,\lambda}(\phi)$ is an isomorphism from $H_{\mu,\lambda}$ onto $H_{\nu,\lambda}$ for $\nu + \mu \geqslant -1$ and by the consequence of [30, p-29], we have for $f \in H'_{\nu,\lambda}$ implies that $h_{\nu,\mu,\lambda}(f) \in H'_{\mu,\lambda}$.

Therefore, it remains to show that

$$\left\langle h'_{\mu,\lambda}\,(h'_{\nu,\mu,\lambda}(f),\ \phi \right\rangle = \left\langle h'_{\nu,\mu,\lambda}(f),\phi\right\rangle,\ \phi \in H_{\mu,\lambda} \qquad ..(3.9.1)$$

The L.H.S. of (3.9.1) can be written as follows

$$\text{L.H.S.} = \left\langle h'_{\nu,\mu,\lambda}(f),\ h_{\mu,k,\lambda}(\phi)\right\rangle \text{ for } \mu + k \geqslant -1/2.$$

$$= \left\langle f,\ h_{\nu,\mu,k,\lambda}(h_{\mu,k,\lambda}(\phi))\right\rangle,\ \nu + \mu + 2k \geqslant -1. \qquad ..(3.9.2)$$

Now, the equation (3.9.1) can be written as follows -

$$\left\langle f,\ h_{\nu,\mu,k,\lambda}(h_{\mu,k,\lambda}(\phi))\right\rangle = \left\langle h'_{\nu,\mu,\lambda}(f),\ \phi\right\rangle \text{ by } (3.9.2) ..(3.9.3)$$

Again, in order to prove (3.9.3), it is sufficient to show that

$$\left\langle f,\ h_{\nu,\mu,k,\lambda}(h_{\mu,k,\lambda}(\phi))\right\rangle = \left\langle f,\ h_{\nu,k,\lambda}(h_{\nu,\mu,k,\lambda}(\phi))\right\rangle \qquad ..(3.9.4)$$

for $\nu+\mu+2k \geqslant -1,\ \phi \in H_{\mu,\lambda}$.

For,

$$\left\langle f,\ h_{\nu,k,\lambda}(h_{\nu,\mu,k,\lambda}(\phi))\right\rangle = \left\langle h'_{\nu,\lambda}(f),\ h_{\nu,\mu,k,\lambda}(\phi)\right\rangle$$

$$= \left\langle f,\ h_{\nu,\mu,\lambda}(\phi)\right\rangle$$

$$= \left\langle h'_{\nu,\mu,\lambda}(f),\ \phi\right\rangle$$

$$= \text{R.H.S. of } (3.9.3).$$

We shall prove (3.9.4), by showing that

$$h_{\nu,\mu,k,\lambda}(h_{\mu,k,\lambda}(\phi)) = h_{\nu,k,\lambda}(h_{\pi,\mu,k,\lambda}(\phi)) \qquad ..(3.9.5)$$

For $\phi \in H_{\mu,\lambda}$ and considering both sides of (3.9.5) to be a function of variable z.

Now, R.H.S. of (3.9.5)

$$= (-1)^k \lambda^{-k} z^{-\lambda k} h_{\nu+k,\lambda} N_{\nu+k-1,\lambda} \cdots N_{\nu,\lambda} \; h_{\nu,\mu,k,\lambda}(\emptyset(x))$$

$$= z^{-\lambda k} h_{\nu+k,\lambda}(h_{\nu+k,\mu+k,\lambda}(x^{\lambda k}\emptyset)) \quad \text{by (3.8.1) and (3.8.9)}$$

Similarly, L. H. S. of (3.9.5)

$$= (-1)^k \lambda^{-k} z^{-\lambda k} h_{\nu+k,\mu+k,\lambda} \; N_{\mu+k-1,\lambda} \cdots N_{\mu,\lambda}(h_{\mu,k,\lambda}(\emptyset(x)))$$

by (3.8.1)

$$= (-1)^k \lambda^{-k} z^{-\lambda k} h_{\nu+k,\mu+k,\lambda} N_{\mu+k-1,\lambda} \cdots N_{\mu,\lambda} \left\{ (-1)^k \lambda^k y^{-\lambda k} h_{\mu+k,\lambda} \; N_{\mu+k-1,\lambda} \cdots N_{\mu,\lambda} \emptyset(x) \right\}$$

$$= z^{-\lambda k} h_{\nu+k,\mu+k,\lambda} \; (h_{\mu+k,\lambda}(x^{\lambda k}\emptyset(x))) \quad \text{by} \quad (3.8.9)$$

Again the result (3.9.5) follows, if we can show that

$h_{\nu+k,\mu+k,\lambda}(h_{\mu+k,\lambda}(x^{\lambda k}\emptyset)) = h_{\nu+k,\lambda} \; (h_{\nu+k,\mu+k,\lambda}(x^{\lambda k}\emptyset))$ and for this, we shall show that

$$h_{\nu,\mu,\lambda} \; (h_{\mu,\lambda} \; (x^{\lambda k}\emptyset)) = h_{\nu,\lambda} \; (h_{\nu,\mu,\lambda}(x^{\lambda k}\emptyset)), \qquad ..(3.9.6)$$

where $\mu-2 > \nu \geq 1/2$.

Now by the known result of Erdelyi (6, p.48), for any variable $z > 0$ and $\lambda > 0$, we have

$$\int_0^\infty (zy)^{\lambda(\nu/2-\mu/2+1)-1/2} J_{\frac{\nu+\mu}{2}}(z^\lambda y^\lambda)\ ((xy)^{\lambda-1/2} J_\mu(x^\lambda y^\lambda))\ dy$$

$$= \int_0^\infty (zy)^{\lambda-1/2} J_\nu(z^\lambda y^\lambda)(xy)^{\lambda(\nu/2-\mu/2+1)-1/2} J_{\frac{\nu+\mu}{2}}(x^\lambda y^\lambda)\ dy \qquad ..(3.9.7)$$

For $\mu > \nu \geqslant -1/2$ and in view of (3.9.7), we have

$$\int_0^\infty \emptyset(x)dx \int_0^\infty (zy)^{\lambda(\nu/2-\mu/2+1)-1/2} J_{\frac{\nu+\mu}{2}}(z^\lambda y^\lambda)(xy)^{\lambda-1/2} J_\mu(x^\lambda y^\lambda)\ dy$$

$$= \int_0^\infty \emptyset(x)\ dx \int_0^\infty (zy)^{\lambda-1/2} J_\nu(z^\lambda y^\lambda)\ (xy)^{\lambda(\nu/2-\mu/2+1)-1/2} J_{\frac{\nu+\mu}{2}}(x^\lambda y^\lambda)\ dy \qquad ..(3.9.8)$$

Changing the order of integration in (3.9.8) by Fubin's theorem, we obtain

....

$$\int_0^\infty (zy)^{\lambda(\nu/2-\mu/2+1)-1/2} J_{\frac{\nu+\mu}{2}}(z^\lambda y^\lambda)\, dy$$

$$\int_0^\infty (xy)^{\lambda-1/2} J_\mu(x^\lambda y^\lambda)\, \phi(x)\, dx$$

$$= \int_0^\infty (zy)^{\lambda-1/2} J_\nu(z^\lambda y^\lambda)\, dy \int_0^\infty (xy)^{\lambda(\nu/2-\mu/2+1)-1/2}$$

$$J_{\frac{\nu+\mu}{2}}(x^\lambda y^\lambda)\, \phi(x)\, dx.$$

Therefore ,

$$h_{\nu,\mu,\lambda}(h_{\mu,\lambda}(\phi)) = h_{\nu,\lambda}(h_{\nu,\mu,\lambda}(\phi)).$$

Hence, the theorem.

...