ARTICLE FOUR

n - Dimensional Generalized Hankel Transform of Arbitrary Order

For $\lambda > 0$, the n-dimensional generalized Hankel transform of a function $\phi(x_1, x_2, \ldots\ldots x_n)$ defined by

$$(h_{\mu,\lambda}\phi)(y_1, y_2 \ldots y_n) = \lambda \int_0^\infty \cdots \int_0^\infty \phi(x_1, x_3, \ldots, x_n) \left(\prod_{i=1}^{n} (x_i y_i)^{\lambda - 1/2} J_\mu (x_i^\lambda y_i^\lambda)\right) dx_1 \ldots dx_n \qquad (4)$$

Where $J_\mu(z)$ is the Bessel function of first kind of order μ. In this section we extend this transform to a class of of generalized function when μ is any real number.

I denotes the open set $x \in R^n : o < x_i < \infty$ $i = 1, 2, \ldots, n$. A function on a sub set of R^n shall be denoted by $f(x) = f(x_1, x_2, \ldots, x_n)$. If $x = (x_1, x_2, \ldots, x_n) \in R^n$, then $[x]$ means the product $x_1\ x_2 \ldots x_n$. Thus $[x^m] = x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$ where

$m = (m_1, m_2, \ldots m_n) \subset R^n$. A nonnegative integer in R^n means the element in R^n whose components are all nonnegative integers.

We shall use the following operators :

$$(1) \qquad (x^{1-2\lambda} D_x)^k = \prod_{i=1}^{n} \left(x_i^{1-2\lambda} \frac{\partial}{\partial x_i}\right)^{k_i}$$

where $k = (k_1, k_2, \ldots, k_n)$ is a nonegative integer in R^n.

$$(2) \qquad N_{\mu,\lambda} = [x]^{\lambda\mu + 1/2} \frac{\partial^n}{\partial x_1 \partial x_2 \ldots \partial x_n} [x]^{-\lambda\mu - \lambda + 1/2}$$

(3) $M_{\mu,} = [x]^{-\lambda\mu-\lambda+1/2} \dfrac{\partial^n}{\partial x_1, \partial x_2 \ldots \partial x_n} [x]^{\lambda\mu+1/2}$

Thus $M_{\mu,\lambda} N_{\mu,\lambda} = \prod_{i=1}^{n} \left\{ x_i^{2(1-\lambda)} \dfrac{\partial^2}{\partial x_i^2} + 2(1-\lambda)\, x_i^{1-\lambda} \dfrac{\partial}{\partial x_i} (\lambda\mu+\lambda+1/2)(\lambda\mu-\lambda-1/2) x_i^{-2\lambda} \right\}.$

(4) $N^{-1}_{\mu,\lambda} \phi = [x]^{\lambda\mu+\mu-1/2} \int_{\infty}^{x_1} \cdots \int_{\infty}^{x_n} [t]^{-\lambda\mu-1/2} \phi(t)\, dt_n \ldots dt_1$

The $N^{-1}_{\mu,\lambda}$ is the inverse of $N_{\mu,\lambda}$. By a smooth function, we mean a function that possesses partial derivatives of at all points of its domain.

★ 4.1 : The Generalized n-Dimensional Hankel Transform of Order $\mu \geqslant -1/2$

The results in this section were developed by Ghosh[9], and Choudhary [3]. Let μ be any real number and $\lambda > 0$. $H_{\mu,\lambda}$ is the space of complex valued smooth function $\phi(x)$ defined on I such that for each pair of nonegative integers m and k in R^n

$$\gamma^{\mu,\lambda}_{m,k}(\phi) = \sup_{x \in I} \left| [x^m]^{\lambda} (x^{1-2\lambda} D_x)^k [x]^{-\lambda\mu-\lambda+1/2} \phi(x) \right| < \infty \quad \ldots (4.1.$$

We shall list a few properties related to these spaces $H_{\mu,\lambda}$ [9].

(i) $H_{\mu,\lambda}$ is complete countably multinormed space.

$H'_{\mu,\lambda}$ the dual of $H_{\mu,\lambda}$ is also a complete.

(ii) For any integer p, and for any μ, $\emptyset \to [x]^{\lambda p}\emptyset$ is an isomorphism from $H_{\mu,\lambda}$ onto $H_{\mu+p,\lambda}$. Thus the operator $f(x) \to [x]^{\lambda p} f(x)$ defined by

$$\langle [x]^{\lambda p} f(x), \emptyset(x) \rangle = \langle f(x), [x]^{\lambda p} \emptyset(x) \rangle \qquad (4.1.2)$$

is an isomorphism from $H'_{\mu+p,\lambda}$ onto $H'_{\mu,\lambda}$

(iii) For any μ, $\emptyset \to N_{\mu,\lambda}\emptyset$ is an isomorphism from $H_{\mu,\lambda}$ onto $H_{\mu+1,\lambda}$ the inverse mapping being

$$\emptyset \longrightarrow N^{-1}_{\mu,\lambda}(\emptyset)..$$

(iv) For any μ, $\emptyset \to M_{\mu,\lambda}\emptyset$ is continuous linear mapping from $H_{\mu+1,\lambda}$ into $H_{\mu,\lambda}$. Thus, $M_{\mu,\lambda}N_{\mu,\lambda}$ is a continuous linear mapping of $H_{\mu,\lambda}$ into itself.

(v) The weak differential operator $N_{\mu,\lambda}$ defined by

$$\langle N_{\mu,\lambda} f, \emptyset \rangle = \langle f, (-1)^n M_{\mu,\lambda}\emptyset \rangle, \; f \in H'_{\mu,\lambda}, \emptyset \in H_{\mu+1,\lambda}. \qquad (4.1.3)$$

is a continuous linear mapping from $H'_{\mu,\lambda}$ into $H'_{\mu+1,\lambda}$

(vi) The weak differential operator $M_{\mu,\lambda}$ defined by

$$\langle M_{\mu,\lambda} f, \emptyset \rangle = \langle f, (-1)^n N_{\mu,\lambda}\emptyset \rangle, f \in H'_{\mu+1,\lambda}, \emptyset \in H_{\mu,\lambda} \qquad (4.1.4)$$

is an isomorphism from $H'_{\mu+1,\lambda}$ onto $H'_{\mu,\lambda}$. Thus, the weak differenti[al] operator $M_{\mu,\lambda} N_{\mu,\lambda}$ is a continuous linear mapping from $H'_{\mu,\lambda}$ into itself.

(vii) The Hankel transformation $h_{\mu,\lambda}$ defined by (4) is an automorphism on $H_{\mu,\lambda}$, when $\mu \geq -1/2$. When $\mu \geq -1/2$, the n-dimensional, distributional Hankel transformation $h'_{\mu,\lambda}$ on $H'_{\mu,\lambda}$ if defined as follows : For $\phi \in H_{\mu,\lambda}$ and $f \in H'_{\mu,\lambda}$, the Hankel transform $F = h'_{\mu,\lambda}$ is defined by

$$\langle h'_{\mu,\lambda} f, \phi \rangle = \langle f, h_{\mu,\lambda} \phi \rangle \tag{4.1.5}$$

(viii) If $\mu \geq -1/2$, the distributional Hankel transformation $h'_{\mu,\lambda}$ is an automorphism on $H'_{\mu,\lambda}$.

*4.2 : The Generalized n-Dimensional Hankel transformation of arbitrary order :

Let μ be any real number, $\lambda > 0$ and p any positive integer such that $\mu + p \geq -1/2$. We define the transformation $h_{\mu,p,\lambda}$ and $h^{-1}_{\mu,p,\lambda}$ on $H_{\mu,\lambda}$ as follows

$$h_{\mu,p,\lambda}(\phi(y)) = (-1)^{-np} [x]^{-\lambda p} h_{\mu+p,\lambda} N_{\mu+p-1,\lambda} \cdots N_{\mu,\lambda} \phi(y), \phi \in H_{\mu,\lambda} \tag{4.2.1}$$

$$h^{-1}_{\mu,p,\lambda}(\phi(x)) = (-1)^{np} \{ N^{-1}_{\mu,\lambda} N^{-1}_{\mu+1,\lambda} \cdots N^{-1}_{\mu+p-1,\lambda} N^{-1}_{\mu+p,\lambda} h_{\mu+p,\lambda} ([x]^{\lambda p} \phi(x)) \} \quad \phi \in H_{\mu,\lambda} \tag{4.2.2}$$

Lemma 4 : Let μ be any real number, $\lambda > 0$ and p any positive integer a such that $\mu + p \geqslant -1/2$. Then

(a) $h_{\mu,p,\lambda}$ defined by (4.2.1) is an automorphism on $H_{\mu,\lambda}$

(b) $h^{-1}_{\mu,p,\lambda}$ defined by (4.2.2) is the inverse of $h_{\mu,p,\lambda}$, and

(c) When $\mu \geqslant -1/2$, $h_{\mu,p,\lambda}$ coincides with $h_{\mu,\lambda}$ as defined by (4).

Proof :- In view of property (iii), sec. 4.1, the mapping

$\phi \longrightarrow N_{\mu+p-1,\lambda} \ldots N_{\mu,\lambda}\, \phi$ is an isomorphism from $H_{\mu,\lambda}$ onto $H_{\mu+p,\lambda}$. By virtue of properties (vii) and (ii), Sec. 4.1,

$\phi \longrightarrow h_{\mu+p,\lambda}\phi$ is an automorphism on $H_{\mu+p,\lambda}$, and

$\phi \longrightarrow [x]^{-\lambda p}\, \phi$ is an isomorphism from $H_{\mu+p,\lambda}$ onto $H_{\mu,\lambda}$,

and hence (a) follows. When $\mu + p \geqslant -1/2$, $h^{-1}_{\mu+p,\lambda} = h_{\mu+p,\lambda}$ is clear from [9]. Hence, (b) follows from the properties (ii) and (iii) again. To rprove (c), let $\mu \geqslant -1/2$ and $\phi \in H_{\mu,\lambda}$. First suppose p = 1, then

$$h_{\mu,1,\lambda}\phi = (-1)^n\,[x]^{-\lambda}\, h_{\mu+1,\lambda}\, N_{\mu,\lambda}\phi(y)$$

$$= (-1)^n [x]^{-\lambda} \int_0^\infty \cdots \int_0^\infty [y]^{\lambda\mu+1/2} \frac{\partial^n}{\partial y_1 \ldots \partial y_n} [y]^{-\lambda\mu-\lambda+1/2}\phi(y) \prod_{i=1}^{n} (x_i y_i)^{\lambda-1/2} J_{\mu+1}(x_i^\lambda y_i^\lambda)\, dy_1 \ldots dy_n$$

$$= \int_0^\infty \cdots \int_0^\infty \phi(y) \left(\prod_{i=1}^{n} (x_i y_i)^{\lambda-1/2} J_\mu(x_i^\lambda y_i^\lambda)\right) dy_1, \ldots dy_n \qquad (4.2.3)$$

Equation (4.2.3) is obtained by an integration by parts through each variable $y_1, y_2, \ldots, y_n$ and using the identities

$$\frac{\partial}{\partial y_i} y_i^{\lambda(\mu+1)} J_{\mu+1}(x_i^\lambda y_i^\lambda) = \lambda\, x_i^\lambda y_i^{\lambda(\mu+2)-1} J_\mu(x_i^\lambda y_i^\lambda)$$

The limit terms vanish since $D_y^k \phi(y)$ is of rapid descent for each non-negative integer k in R^n, by [9] and $\left\{(x_i y_i)^{\lambda-1/2} J_{\mu+1}(x_i^\lambda y_i^\lambda)\right\}$ remains bounded as $y_i \to \infty$ while $\left\{(x_i y_i)^{\lambda-1/2} J_{\mu+1}(x_i^\lambda y_i^\lambda)\right\} = O(y_i)$, $\phi(y) = O(1)$ as $y_i \to 0$, for each i. Thus, when μ -1/2, $h_{\mu,1,\lambda} = h_{\mu,\lambda}$. The general statement for larger values of p follows by induction from this result.

Consequences of the Lemma : (i) $h_{\mu,p,\lambda}$ is independent of the choice of a positive integer p, so long as $\mu + p \geq -1/2$. That is $h_{\mu,p,\lambda} = h_{\mu,q,\lambda}$ if p and q are positive integers such that $\mu + p \geq -1/2$ and $\mu + q \geq -1/2$. (ii) $h^{-1}_{\mu,p,\lambda} = h_{\mu,\lambda}$ if $\mu \geq -1/2$, and (iii) $h^{-1}_{\mu,p,\lambda}$ is independent of the choice of p so long as $\mu + p \geq -1/2$. In view of these consequences, it is reasonable to define the generalized Hankel transformation $h_{\mu,\lambda}$ for $\mu \geq -1/2$ on $\phi \subset H_{\mu,\lambda}$ by $h_{\mu,\lambda}\phi = h_{\mu,p,\lambda}\phi$ where p is a positive integer no less than $-\mu - 1/2$. The inverse of Hankel transformation $h^{-1}_{\mu,\lambda}$ is defined by $h^{-1}_{\mu,\lambda}\phi = h^{-1}_{\mu,p,\lambda}\phi$, $\phi \in H_{\mu,\lambda}$. When $\mu \geq -1/2$, $h^{-1}_{\mu,\lambda} = h_{\mu,\lambda}$ but this is not true when $\mu < -1/2$.

We now define the distributional Hankel transformation $h'_{\mu,\lambda}$ of any real order μ.

<u>Definition</u> : Let μ be any real number, $\lambda > 0$ and $f \in H'_{\mu,\lambda}$. Let p be any positive integer such that $\mu + p \geqslant -1/2$. Then, the distributional Hankel transformation $h'_{\mu,\lambda}$ is defined as the adjoint of $h_{\mu,\lambda} = h_{\mu,p,\lambda}$ of $H_{\mu,\lambda}$. That is for $\Phi \in H_{\mu,\lambda}$ and $\phi = h_{\mu,\lambda}\Phi = h_{\mu,p,\lambda}\,\Phi$, the Hankel transformer $F = h'_{\mu,\lambda} f$ of $f \in H'_{\mu,\lambda}$ is defined by $\langle h'_{\mu,\lambda} f,\ \Phi\rangle = \langle f, h_{\mu,p,\lambda}\,\Phi\rangle$ (4.2.4)

Theorem - 4.2 : The generalized Hankel transformation $h'_{\mu,\lambda}$ is an automorphism on $H'_{\mu,\lambda}$, whatever be the real number μ. and λ $\lambda > 0$. The equation(4.2.4) also defines the inverse of $h'_{\mu,\lambda}$ as the adjoint of $h^{-1}_{\mu,p,\lambda}$: $\langle F, h^{-1}_{\mu,p,\lambda}\phi\rangle = \langle (h'_{\mu,\lambda})^{-1}F,\ \phi\rangle$ (4.2.5)
where $F = h'_{\mu,\lambda} f$ and $\phi = h_{\mu,p,\lambda}\phi$. When $\mu \geqslant -1/2$, the definition (4.2.4) of $h'_{\mu,\lambda}$ coincides with (4.1.5).

4.3 : <u>An Operation-Transform Foumula</u> :

In view of property (vi) Sec. 4.1, the weak differential operator $M_{\mu,\lambda}N_{\mu,\lambda}$ is a continuous linear mapping from $H_{\mu,\lambda}$ into itself. If $\mu \geqslant -1/2$, for $f \in H'_{\mu,\lambda}$,

$$h'_{\mu,\lambda}(M_{\mu,\lambda}N_{\mu,\lambda}) = (-1)^{n}\lambda^{2}[y]^{2\lambda}h_{\mu,\lambda}f \qquad (4.3.1)$$

The same thing is also true for any real number μ if extended function of $h'_{\mu,\lambda}$ is used. To establish this, by using the integration by parts through each variable, differentiation under the integral sign and the same technique used in the case of one dimensional [30], we can obtain

<u>Lemma 4.1</u>: Let $\lambda > 0$ and μ be any fixed real number and p a positive integer $\geq -\mu - 1/2$. Then, for every $\phi \in H_{\mu,\lambda}$,

$$M_{\mu,\lambda} N_{\mu,\lambda} h_{\mu,p,} \quad \Phi = h_{\mu,p,\lambda}((-1)^{n} [y]^{2\lambda} \Phi) \qquad (4.3.2)$$

<u>Theorem 4.3</u> : For arbitrary real μ, $\lambda > 0$ and $f \in H'_{\mu,\lambda}$, then

$$h'_{\mu,\lambda}(M_{\mu,\lambda} N_{\mu,\lambda} f) = (-1)^{n} [y]^{2\lambda} \, h'_{\mu,\lambda} \, f \qquad (4.3.3)$$

Proof : Let $\Phi \in H_{\mu,\lambda}$ and p any positive integer $\geq -\mu - 1/2$. By definition of $M_{\mu,\lambda} N_{\mu,\lambda}$ and Lemma 4.1 we have

$$\begin{aligned}
\langle h'_{\mu,\lambda} M_{\mu,\lambda} N_{\mu,\lambda} f, \Phi \rangle &= \langle M_{\mu,\lambda} N_{\mu,\lambda} f, h_{\mu,p,\lambda} \phi \rangle \\
&= \langle f, M_{\mu,\lambda} N_{\mu,\lambda} h_{\mu,p,\lambda} \Phi \rangle \\
&= \langle f, (-1)^{n\lambda^2} [y]^{2\lambda} \phi \rangle \\
&= \langle h'_{\mu,\lambda} f, (-1)^{n} \lambda^{2} [y]^{2\lambda} \Phi \rangle \\
&= \langle (-1)^{n} \lambda^{2} [y]^{2\lambda} h'_{\mu,\lambda} f, \phi \rangle
\end{aligned}$$

which implies (4.3.3.)

<u>Remarks</u> : (1) When n = 1, the results in this work reduce to the one-dimensional case [9].

(2) When $\lambda = 1$, the results in this work reduce to [3]

(3) When n = 1 and $\lambda = 1$, the results in this work reduce to [30].