CHAPTER - II

SOME SUBSTITUTION THEOREMS FOR DISTRIBUTIONAL STIELTJES TRANSFORMATION

2.1 Introduction :

In this chapter we shall extend a simple generalization of the Stieltjes transformation to certain class of distributions. We shall also prove some substitution theorems similar to Buschman [1] for this distributional transformation.

The transformation defined by the equation

$$S[j(\tau), \tau \to y] \triangleq J(y) \triangleq \Gamma(\varrho) y^{m\varrho-1} \int_0^\infty \frac{j(\tau)}{(y^m+\tau^m)^\varrho} d\tau, (m, \varrho > 0) \qquad \ldots (2.1.1)$$

(whenever the integral on the righthand side converges for a complex y with $Re.y > c$)
is a simple generalization of the Stieltjes transform of a function $j(\tau) \in L(0,\infty)$, which is defined by

$$F(y) = \int_0^\infty \frac{j(\tau)}{(y+\tau)} d\tau \qquad \ldots (2.1.2)$$

Throughout this chapter $D_y(I)$ will denote the space of all smooth functions with compact supports on I: $(0,\infty)$ and with the parameter y .

We first construct a testing function space $S_{a,b}$ by applying the change of variable $\tau = e^t$ to the definition of $\mathcal{L}_{a,b}$ and setting $\sqrt{\tau}\,\psi(\tau) = \phi(\log \tau)$ in Sect.3.2, Eq.(1) [6]. This yields the following definition :

Given any two real numbers a and b, $S_{a,b}$ is the space of all smooth functions $\psi(\tau)$ on $(0,\infty)$ such that

$$L_k(\psi) = L_{a,b,k}(\psi) = \sup_{0<\tau<\infty} \left| \xi(\log \tau)\left(\tau \frac{d}{d\tau}\right)^k \sqrt{\tau}\,\psi(\tau) \right| < \infty \quad \ldots (2.1.3)$$

for $k = 0, 1, 2, \ldots$

where $$\xi(\log \tau) = \begin{cases} \tau^a & \text{if } 1 \leq \tau < \infty \\ \tau^b & \text{if } 0 < \tau < 1 . \end{cases}$$

The topology of $S_{a,b}$ is that generated by the multinorm $\{L_{a,b,k}\}_{k=0}^{\infty}$. $S_{a,b}$ is sequentially complete, Housdorff, locally convex, first countable, metrizable topological linear space. It is complete and therefore Frechet space [6].

For $a < m\rho - 1/2$ and $b > m\rho - 1/2$, $\psi(\tau) = (y^m + \tau^m)^{-\rho} \in S_{a,b}$.

To prove this it suffices to show that

$$\phi(t) = e^{t/2}(y^m + e^{mt})^{-\rho} \in \mathcal{L}_{a,b} \quad \text{for } a < m\rho - 1/2 \text{ and } b > m\rho - 1/2 .$$

Now we have

$$\gamma_k(\phi) = \sup_{-\infty<t<\infty} \left| K_{a,b}(t)\, D^k[\, e^{t/2} \,(y^m + e^{mt})^{-\rho}\,] \right|$$

where $K_{a,b}(t)$ is defined as in Sect.3.2 [6]

$$= \sup_{-\infty<t<\infty} \left| K_{a,b}(t) \sum_{\nu=0}^{k} \binom{k}{\nu} [\, D^{k-\nu} e^{t/2}][D^{\nu}(y^m + e^{mt})^{-\rho}\,] \right|$$

$$= \sup_{-\infty<t<\infty} \left| K_{a,b}(t). \sum_{\nu=0}^{k} \binom{k}{\nu} [(\tfrac{1}{2})^{k-\nu} e^{t/2}][\sum_{i=1}^{\nu} \beta_i e^{imt} (y^m + e^{mt})^{-\rho-i}] \right|$$

where β_i are suitable constants

$< \infty$ for $a > m\rho - 1/2$ and $b > m\rho - 1/2$.

$\therefore\ \phi(t) = e^{t/2}(y^n + e^{mt})^{-\rho} \in \mathcal{L}_{a,b}$ for $a < m\rho - 1/2$ and $b > m\rho - 1/2$.

As a consequence, $S_{a,b}$ is a complete countably multinormed space, $S'_{a,b}$, the dual of $S_{a,b}$ is also complete [6].

We state the following useful results :

(a) The mapping $\psi(\tau) = \tau^{-1/2}\phi(\log \tau) \to \phi(t)$ is an isomorphism from $S_{a,b}$ onto $\mathcal{L}_{a,b}$ since $L_{a,b,k}[\psi(\tau)] = \gamma_{a,b,k}[\phi(t)]$.

(b) If $j(\tau) \in S'_{a,b}$ for some $a < m\varrho - 1/2$ and $b > m\varrho - 1/2$, then the Stieltjes transform J of j is defined by

$$S[j(\tau), \tau \to y] \triangleq J(y) \triangleq \left\langle j(\tau), \frac{1}{(y^m+\tau^m)^{\varrho}} \right\rangle', \quad o < \tau < \infty \quad \ldots(2.1.4)$$

where for each fixed y the right-hand side has a sense as the application of $j(\tau) \in S'_{a,b}$ to $(y^m + \tau^m)^{-\varrho} \in S_{a,b}$

2.2 Substitution Theorems :

Theorem 2.2.1

If $S[j(\tau), \tau \to y] = J(y)$, $o < y < \infty$ and $\mu(y,u) \in D_y(I)$, then

$$\left\langle K(\tau) j[G(\tau)], (y^m + \tau^m)^{-\varrho} \right\rangle = \int_c^{\infty} J(u)\mu(y,u)du, \quad \ldots (2.2.1)$$

where K, G and $h = G^{-1}$ are single-valued analytic functions, real on (o, ∞) and such that $G(o) = 0$ and $G(\infty) = \infty$ (OR $G(\infty) = 0$ and $G(0) = \infty$) and

$$S[\mu(y,u), u \to p] = \bar{\mu}(y,p)$$
$$= [(y^m + h^m(p))^{-\varrho}] \, K[h(p)] \, |h'(p)|$$

Proof : Let $\psi(\tau)$ be an arbitrary member of $S_{c,d}$, $c < m\varrho - 1/2$ and $d > m\varrho - 1/2$. By Sect. 2.1, result (a), the mapping $\psi(\tau) \to \phi(t)$ is an isomorphism from $S_{c,d}$ onto $\pounds_{c,d}$.

The mapping $\phi(t) \to K(t)\phi(t)$, where K is analytic and real on (o,∞), is an isomorphism from $\mathcal{E}_{c,d}$ onto $\mathcal{E}_{u,v}$ where $u < c$ and $v > d$ [4, Theorem 3.4.1]. Again by Sect.2.1, result (a), the mapping $K(\tau)\psi(\tau) \to K(t)\phi(t)$ is an isomorphism from $S_{u,v}$ onto $\mathcal{E}_{u,v}$. Hence the mapping $\psi(\tau) \to K(\tau)\psi(\tau)$ is an isomorphism from $S_{c,d}$ onto $S_{u,v}$ where $u < c$ and $v > d$.

Furthermore, in accordance with Sect.2.5 [6], it now follows that $j(\tau) \to K(\tau)j(\tau)$ is an isomorphism from $S'_{u,v}$ onto $S'_{c,d}$ and we write

$$\langle K(\tau)j(\tau), \psi(\tau) \rangle = \langle j(\tau), K(\tau)\psi(\tau) \rangle$$

therefore if $S[j(\tau), \tau \to y] = J(y)$, $o < y < \infty$, the equation

$$\langle K(\tau)j(\tau), (y^m + \tau^m)^{-\rho} \rangle = \langle j(\tau), K(\tau)(y^m + \tau^m)^{-\rho} \rangle$$

has sense. Indeed, we have

$j(\tau) \in S'_{u,v}$, $K(\tau)(y^m + \tau^m)^{-\rho} \in S_{u,v}$, $K(\tau)\, j(\tau) \in S'_{c,d}$ and $(y^m + \tau^m)^{-\rho} \in S_{c,d}$.

If $\chi(\tau) = j[G(\tau)] \in S'_{u,v}$, then $\chi(\tau) \to K(\tau)\chi(\tau)$ is an isomorphism from $S'_{u,v}$ onto $S'_{c,d}$ and we can write

$$\langle K(\tau)j[G(\tau)], (y^m + \tau^m)^{-\rho} \rangle = \langle j[G(\tau)], K(\tau)(y^m + \tau^m)^{-\rho} \rangle \qquad \ldots(2.2.2)$$

Here

$$j[G(\tau)] \in S'_{u,v}\ , \quad K(\tau)(y^m + \tau^m)^{-\rho} \in S_{u,v}\ ,$$

$$K(\tau)j[G(\tau)] \in S'_{c,d} \quad \text{and} \quad (y^m + \tau^m)^{-\rho} \in S_{c,d}\ .$$

Let $K(\tau)\psi(\tau) = \eta(\tau)$ be an arbitrary member of $S_{u,v}$. Choose real numbers a and b, $a < u$ and $v < b$ such that

$$\eta\,[h(\tau)]\,|\,h'(\tau)\,| \in S_{a,b}\ .$$

Let $K(t)\phi(t) = \eta(t) \in \mathcal{E}_{u,v}$. By Sect.2.1, result (a), the mapping $\eta(\tau) \to \eta(t)$ is an isomorphism from $S_{u,v}$ onto $\mathcal{E}_{u,v}$. The mapping $\eta(t) \to \eta[h(t)]\,|h'(t)|$ is an isomorphism from $\mathcal{E}_{u,v}$ onto $\mathcal{E}_{a,b}$ [4, Theorem 3.4.1]. Again by Sect.2.1, result (a), the mapping

$\eta[h(\tau)]\,|\,h'(\tau)\,| \longrightarrow \eta[h(t)]\,|\,h'(t)|$ is an isomorphism

from $S_{a,b}$ onto $\mathcal{E}_{a,b}$. Hence the mapping

$\eta(\tau) \to \eta[h(\tau)]\,|\,h'(\tau)\,|$ is an isomorphism from

$S_{u,v}$ onto $S_{a,b}$.

We denote the adjoint of the mapping

$\eta(\tau) \to \eta[h(\tau)]\,|\,h'(\tau)\,|$ by $j(\tau) \to j[G(\tau)]$, since this is what we would have if j were a conventional function, and we write

$$\langle j[G(\tau)],\ \eta(\tau)\rangle = \langle j(\tau),\ \eta[h(\tau)]\,|h'(\tau)|\rangle\ .$$

By Theorem 1.10.2 [6], $j(\tau) \to j[G(\tau)]$ is an isomorphism from $S'_{a,b}$ onto $S'_{u,v}$. Therefore, if

$S[j(\tau), \tau \to y] = J(y)$, $0 < y < \infty$, the equation

$$\langle j[G(\tau)], K(\tau)(y^m + \tau^m)^{-\rho} \rangle = \langle j(\tau), K[h(\tau)](y^m + h^m(\tau))^{-\rho} |h'(\tau)| \rangle \quad \ldots (2.2.3)$$

has sense. Indeed, we have

$j(\tau) \in S'_{a,b}$, $K[h(\tau)](y^m + h^m(\tau))^{-\rho} |h'(\tau)| \in S_{a,b}$,

$j[G(\tau)] \in S'_{u,v}$ and $K(\tau)(y^m + \tau^m)^{-\rho} \in S_{u,v}$.

From equation (2.2.2) and (2.2.3) we conclude that

$j(\tau) \to K(\tau) j[G(\tau)]$ is an isomorphism from $S'_{a,b}$ onto $S'_{c,d}$

where $a < c$ and $d < b$ and we write

$$\langle K(\tau) j[G(\tau)], (y^m + \tau^m)^{-\rho} \rangle = \langle j(\tau), K[h(\tau)](y^m + h^m(\tau))^{-\rho} |h'(\tau)| \rangle \quad \ldots (2.2.4)$$

Indeed, we have

$j(\tau) \in S'_{a,b}$, $K[h(\tau)](y^m + h^m(\tau))^{-\rho} |h'(\tau)| \in S_{a,b}$,

$K(\tau) j[G(\tau)] \in S'_{c,d}$ and $(y^m + \tau^m)^{-\rho} \in S_{c,d}$.

The equation (2.2.4) further can be written as

$$\left\langle K(\tau)J[G(\tau)],\ (y^m + \tau^m)^{-\rho} \right\rangle = \left\langle J(\tau), K[h(\tau)](y^m+h^m(\tau))^{-\rho}|h'(\tau)| \right\rangle$$

$$= \left\langle J(\tau),\ \bar{\mu}(y,\tau) \right\rangle$$

$$= \left\langle J(\tau),\ \int_0^\infty \mu(y,u)(\tau^m+u^m)^{-\rho} du \right\rangle$$

$$= \int_0^\infty \left\langle J(\tau),\ (\tau^m+u^m)^{-\rho} \right\rangle \mu(y,u)du \ldots$$

because of Lemma 3.2.1 [4]

$$= \int_0^\infty J(u)\mu(y,u)du .$$

This completes the proof.

<u>Theorem 2.2.2</u>

Let $S[A(\tau)J(\tau),\ \tau \to y] = J^*(y)$, $0 < y < \infty$ and $\mu^*(y,u) \in D_y(I)$, then

$$\left\langle K(\tau)J[G(\tau)],\ (y^m + \tau^m)^{-\rho} \right\rangle = \int_0^\infty J^*(u)\ \mu^*(y,u)du, \ \ldots \ (2.2.5)$$

where A, K, G and $h = G^{-1}$ are single valued analytic functions, real on $(0, \infty)$ and such that $G(0) = 0$ and $G(\infty) = \infty$ (OR $G(0) = \infty$ and $G(\infty) = 0$) and

$$S[\mu^*(y,u),\ u \to p] = \bar{\mu}^*(y,p)$$

$$= [(y^m + h^m(p))^{-\rho}]K[h(p)]|h'(p)|[A(p)]^{-1}$$

Proof : Let $\psi(\tau)$ be an arbitrary member of $S_{c,d}$, $c < m\rho - 1/2$ and $d > m\rho - 1/2$. By Sect.2.1, result (a), the mapping $\psi(\tau) \to \phi(t)$ is an isomorphism from $S_{c,d}$ onto $\mathcal{L}_{c,d}$. The mapping $\phi(t) \to K(t)\phi(t)$, where K is analytic and real on (o, ∞), is an isomorphism from $\mathcal{L}_{c,d}$ onto $\mathcal{L}_{u,v}$, where $u < c$ and $d < v$ [4, Theorem 3.4.1]. Again by Sect.2.1, result (a), the mapping

$K(\tau)\ \psi(\tau) \to K(t)\ \phi\ (t)$ is an isomorphism from $S_{u,v}$ onto $\mathcal{L}_{u,v}$. Hence the mapping $\psi(\tau) \to K(\tau)\psi(\tau)$ is an isomorphism from $S_{c,d}$ onto $S_{u,v}$, where $u < c$ and $d < v$. Furthermore in accordance with Sect.2.5 [6], it now follows that $j(\tau) \to K(\tau)\ j\ (\tau)$ is an isomorphism from $S'_{u,v}$ onto $S'_{c,d}$ and we write

$$\langle K(\tau)j(\tau),\ \psi(\tau) \rangle = \langle j(\tau),\ K(\tau)\psi(\tau) \rangle$$

Therefore if

$$S[A(\tau)j(\tau),\ \tau \to y\ ] = J^*(y),\ o < y < \infty,$$ the equation

$$\langle K(\tau)j(\tau),\ (y^m + \tau^m)^{-\rho} \rangle = \langle j(\tau), K(\tau)(y^m + \tau^m)^{-\rho} \rangle$$

has sense. Indeed, we have

$$J(\tau) \in S'_{u,v},\quad K(\tau)(y^m + \tau^m)^{-\rho} \in S_{u,v},\quad K(\tau)j(\tau) \in S'_{c,d}$$

and $(y^m + \tau^m)^{-\rho} \in S_{c,d}$.

If $\mathcal{X}(\tau) = j[G(\tau)] \in S'_{u,v}$ then $\mathcal{X}(\tau) = K(\tau)\mathcal{X}(\tau)$ is an isomorphism from $S'_{u,v}$ onto $S'_{c,d}$ and we can write

$$\langle K(\tau)j[G(\tau)],\ (y^m + \tau^m)^{-\rho} \rangle = \langle j[G(\tau)],\ K(\tau)(y^m + \tau^m)^{-\rho} \rangle \qquad \ldots(2.2.6)$$

Here $j[G(\tau)] \in S'_{u,v}$, $K(\tau)(y^m + \tau^m)^{-\rho} \in S_{u,v}$,

$K(\tau)j[G(\tau)] \in S'_{c,d}$ and $(y^m + \tau^m)^{-\rho} \in S_{c,d}$.

Let $K(\tau)\psi(\tau) = \eta(\tau)$ be an arbitrary member of $S_{u,v}$. Choose real numbers a and b, $a < u$ and $v < b$ such that

$$\eta\,[h(\tau)]\,|h'(\tau)| \in S_{a,b}\ .$$

Let $K(t)\,\phi(t) = \eta(t) \in \mathcal{L}_{u,v}$. By Sect.2.1, result (a), the mapping $\eta(\tau) \to \eta(t)$ is an isomorphism from $S_{u,v}$ onto $\mathcal{L}_{u,v}$. The mapping $\eta(t) \to \eta[h(t)]\,|\dot{h}(t)|$ is an isomorphism from $\mathcal{L}_{u,v}$ onto $\mathcal{L}_{a,b}$ [4,Theorem 3.4.1].

Again by Sect.2.1, result (a), the mapping $\eta[h(\tau)]\,|h'(\tau)| \to \eta[h(t)]\,|h'(t)|$ is an isomorphism from $S_{a,b}$ onto $\mathcal{L}_{a,b}$. Hence the mapping $\eta(\tau) \to \eta[h(\tau)]\,|h'(\tau)|$ is an isomorphism from $S_{u,v}$ onto $S_{a,b}$.

We denote the adjoint of the mapping $\eta(\tau) \to \eta[h(\tau)]|h'(\tau)|$ by $j(\tau) \to j[G(\tau)]$, since this is what we would have if $j(\tau)$ were a conventional function, and we write

$$\langle j[G(\tau)],\ \eta(\tau) \rangle = \langle j(\tau),\ \eta[h(\tau)]\ |h'(\tau)| \rangle\ .$$

By Theorem 1.10.2 [6], $j(\tau) \to j[G(\tau)]$ is an isomorphism from $S'_{a,b}$ onto $S'_{u,v}$. Therefore if

$S[A(\tau)j(\tau), \tau \to y] = J^*(y), \quad 0 < y < \infty$, the equation

$$\left\langle j[G(\tau)], K(\tau)(y^m + \tau^m)^{-\rho} A[G(\tau)] [A[G(\tau)]]^{-1} \right\rangle$$

$$= \left\langle j(\tau), K[h(\tau)](y^m + h^m(\tau))^{-\rho} A(\tau)[A(\tau)]^{-1} |h'(\tau)| \right\rangle \quad \ldots (2.2.7)$$

has sense. Indeed, we have

$$j(\tau) \in S'_{a,b}, \quad K[h(\tau)](y^m + h^m(\tau))^{-\rho} A(\tau)[A(\tau)]^{-1} |h'(\tau)| \in S_{a,b}$$

$$j[G(\tau)] \in S'_{u,v} \text{ and } K(\tau)(y^m + \tau^m)^{-\rho} A[G(\tau)][A[G(\tau)]]^{-1} \in S_{u,v}.$$

From equation (2.2.6) and (2.2.7) we conclude that $j(\tau) \to K(\tau)j[G(\tau)]$ is an isomorphism from $S'_{a,b}$ onto $S'_{c,d}$, where $a < c$ and $d < b$ and we write

$$\left\langle K(\tau)j[G(\tau)], (y^m + \tau^m)^{-\rho} A[G(\tau)][A[G(\tau)]]^{-1} \right\rangle$$

$$= \left\langle j(\tau), K[h(\tau)](y^m + h^m(\tau))^{-\rho} A(\tau)[A(\tau)]^{-1} |h'(\tau)| \right\rangle \quad \ldots (2.2.8)$$

Indeed, we have

$$j(\tau) \in S'_{a,b}, \quad K[h(\tau)](y^m + h^m(\tau))^{-\rho} A(\tau)[A(\tau)]^{-1} |h'(\tau)| \in S_{a,b},$$

$$K(\tau)j[G(\tau)] \in S'_{c,d} \text{ and } (y^m + \tau^m)^{-\rho} A[G(\tau)][A[G(\tau)]]^{-1} \in S_{c,d}.$$

The equation (2.2.8) further can be written as

$$\left\langle K(\tau)j[G(\tau)], (y^m+\tau^m)^{-\rho}\right\rangle = \left\langle j(\tau), K[h(\tau)](y^m+h^m(\tau))^{-\rho}|h'(\tau)| A^{-1}(\tau).A(\tau)\right\rangle$$

$$= \left\langle j(\tau), \bar{\mu}^*(y,\tau)\, A(\tau)\right\rangle$$

$$= \left\langle A(\tau)j(\tau), \bar{\mu}^*(y,\tau)\right\rangle$$

$$= \left\langle A(\tau)j(\tau), \int_0^\infty \mu^*(y,u)(u^m+\tau^m)^{-\rho} du\right\rangle$$

$$= \int_0^\infty \left\langle A(\tau)j(\tau), (u^m+\tau^m)^{-\rho}\right\rangle \mu^*(y,u)du$$

$$= \int_0^\infty J^*(u)\ \mu^*(y,u)du .$$

This completes the proof.

<u>Theorem 2.2.3</u> :

Let $S[j(\tau), \tau \to y] = J(y)$, $0 < y < \infty$ and $\theta(\tau, u) \in Dy(I)$, then

$$S^{-1}[K(y)\ J[G(y)]] = \left\langle j(u), \theta(\tau, u)\right\rangle \qquad \ldots (2.2.9)$$

where

$$S[\theta(\tau,u), \tau \to y] = K(y)[(G(y))^m + u^m]^{-\rho}$$

$$= K(y)\ \Phi\ [G(y), u]$$

and K, G are analytic functions.

Proof : Now $J(p) = \left\langle j(u), (p^m + u^m)^{-\rho} \right\rangle$

$$= \left\langle j(u), \Phi(p.u) \right\rangle$$

$$\therefore \quad J[G(y)] = \left\langle j(u), \Phi\, [G(y), u] \right\rangle$$

$$\therefore \quad K(y)J[G(y)] = \left\langle j(u), K(y)\, \Phi\, [G(y), u] \right\rangle$$

$$= \left\langle j(u), \int_0^\infty \Theta(\tau,u)(y^m + \tau^m)^{-\rho}\, d\tau \right\rangle$$

$$= \left\langle \int_0^\infty j(u)\Theta(\tau,u)du, (y^m + \tau^m)^{-\rho} \right\rangle \quad \ldots (2.2.10)$$

$$= S\left[\int_0^\infty j(u)\, \Theta(\tau,u)\, du \right]$$

$$\therefore \quad S^{-1}[K(y)J[G(y)]] = \int_0^\infty j(u)\, \Theta(\tau, u)du$$

$$= \left\langle j(u), \Theta(\tau, u) \right\rangle$$

The right side of (2.2.10) can be justified.

This completes the proof.

<u>Remarks</u> :

1) By substituting $m = \rho = 1$ in the above theorems, we get the same results obtained by Sonavane [4].

2) Our distributional transformation (2.1.1) is not a particular case of any other Stieltjes transformation considered by Ghose [2], Pathak [3] and Tiwari [5].

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