CHAPTER-I

STATEMENTS AND DEFINITIONS OF KNOWN RESULTS

ABSTRACT

This chapter contains some definitions and statements of the results which we need in the course of investigation. The relevant references are given at the end of this chapter.

STATEMENT AND DEFINITION OF KNOWN RESULTS

Definition : Let $E = \{ Z : Z$ is a complex number and $|Z| < 1 \}$

Definition : A complex valued function $f(z)$ is holomorphic in a domain D in the complex plane if it has uniquely determined derivative at each point of D.

Definition : A function $F(z)$ holomorphic in some domain D is said to be univalent in D if $f(Z_1) = f(Z_2)$ implies $Z_1 = Z_2$ for all Z_1, Z_2 in D. In other words, a single valued function f is said to be univalent or schlicht in a domain $D \subset C$ (C-a complex plane) if it never takes the same value twice,

i.e. $f(Z_1) \neq f(Z_2)$ for all points Z_1 and Z_2 in D with $Z_1 \neq Z_2$.

Remark :- A holomorphic univalent function is a conformal mapping because of its angle preserving property.

Definition :- Let S be the class of all functions $F(Z)$ holomorphic and univalent in E and normalised by $F(0)=0$ and $F'(0)=1$.

Remark :- Note that each F in S has a Taylor-series expansion of the form

$$F(Z) = Z + a_2 Z^2 + a_3 Z^3 + \text{--------}, |Z| < 1 .$$

Definition :- A domain containing the origin is said to be starlike with respect to the origin if it is intersected by any straight line through the origin in a linear segment. Starlike with respect to the origin will be referred to as simply starlike.

Definition :- Let $\overset{*}{S}$ be the subclass of S whose members map every disc $|Z| < \xi$, $(0 < \xi < 1)$ on to a starlike domain.

Definition :- Let F be holomorphic in D with $F(0)=0$ and $F'(0)=1$,

then F is in S^* if and only if $Z \dfrac{F'(Z)}{F(Z)}$ is in P, P denoting the set of all functions with positive real part.

Definition :- Let $F(Z)$ be holomorphic at $Z=0$ and satisfy $F(0)=0$, $F'(0) \neq 0$, Then the radius of univalence Uo is defined to be the largest value of r such that $F(Z)$ is holomorphic and univalent for $|Z| < r$.

Definition :- Let $F(Z)$ be holomorphic at $Z=0$ and satisfy $F(0)=0$ and $F'(0) \neq 0$ there. Let α be a real number satisfying $0 \leq \alpha < 1$ the radius of starlikeness of order α denoted by $S^*(\alpha)$ is defined to be the largest value of r such that $F(Z)$ is holomorphic and

$$\operatorname{Re}\left\{ Z \frac{F'(Z)}{F(Z)} \right\} > \alpha \qquad \text{for } |z| < r$$

Definition :- The set F, subset of C is said to be convex if it is starlike with respect to each of its point, that is, if the linear segment joining any two points of F lies entirely in F.

Definition :- A convex function is one which maps the unit disc conformally onto a convex domain.

Definition :- Let K be the subclass of S, whose members map every disc $|Z| < \xi$; $0 < \xi < 1$ onto a convex domain.

Definition :- Let $F(Z)$ be holomorphic at $Z=0$ and satisfy $F(0)=0$; $F'(0) \neq 0$. Let α be a real number satisfying $0 \leq \alpha < 1$. The radius of convexity of order α denoted by K_α is defined to be the largest value of r, such that $F(Z)$ is holomorphic and

$$\operatorname{Re}\left\{ 1+Z \frac{F''(Z)}{F'(Z)} \right\} > \alpha \quad ; \text{ for } \quad |Z| < r$$

In Other Words :

Let F be holomorphic in domain D with $F(0)=0$ and $F'(0)=1$. Then F belongs to K if and only if

$$\left\{ 1+Z \frac{F''(Z)}{F'(Z)} \right\} \text{ is in } P \ ;$$

where P is the set of all functions with positive real part.

Definition :- A function F holomorphic in the unit disc is said to be close-to-convex if there is a convex function g such that

$$\left\{ \operatorname{Re} \frac{F'(Z)}{g'(Z)} \right\} > 0 \quad ; \text{ for all } Z \text{ in } E.$$

We shall denote by 'C' the class of close-to-convex function F normalised by usual conditions $F(0)=0$ and $F'(0)=1$.

(*) Note that every convex function is obviously close-to-convex. More generally,every starlike function is close-to-convex.

These remarks are summarised by the following chain of proper inclusions.

$$K \subset S^* \subset C \subset S.$$

<u>Definition</u> :- Let $F(Z)$ be holomorphic at $Z=0$ and satisfy $F(0)=0$ and $F'(0) \neq 0$ there. Then the radius of close-to-convexity is defined to be the largest value of r such that $F(Z)$ is holomorphic and close-to-convex for $|Z| < r$.

<u>Remark</u> :- A function satisfying either the close-to-convex condition, the starlike condition or the convex condition is univalent .

(*) A logarithmic spiral is a curve in the complex plane of the form.

$$W = W_0 \, e^{-\lambda t} \qquad -\infty < t < \infty$$

Where W_0 and λ are complex constants with $W_0 \neq 0$; $\mathrm{Re}\{\lambda\} \neq 0$.

without loss of generality assume that $\lambda = e^{i\alpha}$ with $-\pi/2 < \alpha < \pi/2$. The curve is then called as α - spiral.

<u>Definition</u> :- A domain D containing the origin is said to be α - spiral like if for each point $W \neq 0$ in D the arc of the α - spiral from W_0 to the origin lies entirely in D.

<u>Definition</u> :- A function F holomorphic and Univalent in the unit disc with $F(0)=0$ is said to be α - spiral like function if its range is α - spiral like.

We note that zero spiral like function is simply the starlike function.

<u>Definition</u> :- Let F be holomorphic in D with $F(0)=0$, $F'(0) \neq 0$ and $F(Z) \neq 0$ for $0 < |Z| < 1$ then F is α - spiral like if and only if $-\pi/2 < \alpha < \pi/2$

$$\mathrm{Re}\left\{ e^{-i\alpha} \, Z \, \frac{F'(Z)}{F(Z)} \right\} > 0 \qquad ; \qquad |Z| < 1$$

Definition :- Principle of Subordination

Let the function F(z) and g(z) be holomorphic in E and let g(z) be univalent there. Let further D′ and D denote the domains on to which E is mapped under W=F(z) and W=g(z) respectively. If f(0)=g(0) and $D' \subset D$ then

f(z) = g{m(z)} ; where m(z) is holomorphic in E and

$$|m(z)| < |z|$$

The sign of equality is possible only if D=D′. If f(z) and g(z) are related by the above, then f(z) is said to be subordinate to g(z); This is written as $f(z) \prec\prec g(z)$.

Definition :- p-Valent Starlike Function

Let $S_p(\alpha)$ denoted the class of functions of the form

$$F(z) = Z^p + \sum_{k=1}^{\infty} a_{p+k} Z^{p+k}$$

p is a fixed integer greater than zero

∴ F(z) is holomorphic in unit disc $E = \{z; |z| < 1\}$ then a function F(z) is said to be p-valent starlike function of order α if it satisfies the condition.

$$\text{Re}\left\{ Z \frac{F'(z)}{F(z)} \right\} > \alpha$$

For $Z \in E$; $0 \le \alpha < P$

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