# C H A P T E R – O

## NEWMAN PENROSE FORMALISM

# INTRODUCTION

The prime mathematical tool for studying the strain variation equation in relativistic continuum mechanics in this dissertation is the Newman-Penrose formalism invented in 1962. It has several exquisite advantages over the standard tensorial presentation of Einstein field equations of gravitation in the presence of matter.

At first we observe that Einstein had utilized only four (contracted) Bianchi identities, viz.

$$\left( R^{ab} - \frac{1}{2}Rg^{ab} \right)_{;b} = 0.$$

Leading to the energy-balance equations in Continuum mechanics :

$$T^{ab}_{\;\;;b} = 0.$$

Unfortunately these four equations do not indicate the interaction of free gravitational field Weyltensor ($C_{abcd}$) and the matter field, Tab. One has to look up for the twentyfour Bianchi identities in a 4 - dimensional Riemannian space. The credit of utilizing all the twenty-four identities for studying the interaction of $C_{abcd}$ and Tab lies in the Newman and Penrose formalism (NP formalism for short).

Next advantage in this spin-coefficient formalism is its easy adaptability to other formalisms. From null tetrad formalism we can switch over very easily to a tetrad comprising of one time-like and three space-like vectors. (Vide 1.5, 1.6, 1.7 for explcit example). Eisenhart (1964), Lichnerowicz (1955), Shaha (1974) have used such formalism consisting of one null vector field and three space like vector fields, while studying the Serret-Frenet formulae of a curve representing the history of a massless particle.

Computational ease is an asset for the Np - formalism. The covariant derivative of a vector field is again in terms of the (outer product of) four null vectors. In particular, the derivatives of null vector fields is again in terms of null vectors (vide Appendix vide : Chapt. II). The algebraic computations are facilitated by the idem potents "zero" and "one" appearing in the defining relations (vide : 1.1, 1.2, 1.3). As a matter of fact even the computer time is economized if the Np method is adopted. Campbell and Wainwright (1977) have claimed that the null formalism affords a saving of 60 % of the computer time as compared to the classical method of Einstein.

The efficiency of the Newman Penrose formalism lies in making the tensor equations transparent. This means that the necessary and sufficient conditions for the validity of a TENSOR equation will be expressed in terms of independant SCALAR equations. This is the reason for calling the Newman Penrose formalism as an AMAZINGLY USEFUL formalism, by Flaherty (1976), Carmeli (1982). It is this advantage that is amply exploited in Chapter II and III of this dissertation.

## EXPOSITION

### 0.1 The four null vector fields :

Newman and Penrose (1962) invented a set of our rays:

$$X_i^a = \{ l^a, m^a, \bar{m}^a, n^a \}$$

Where $l^a$, $n^a$ are two real rays and $m^a$, $\bar{m}^a$ (an overhead bar denotes complex conjugation) are complex rays which satisfy the following conditions.

$$l^a l_a = m^a m_a = n^a n_a = \bar{m}^a \bar{m}_a = 0 \tag{1.1}$$

(null relations)

$$l^a m_a = l^a \bar{m}_a = n^a m_a = n^a \bar{m}_a = 0 \tag{1.2}$$

(Orthogonal relations)

$$l^a n_a = -m^a \bar{m}_a = 1 \tag{1.3}$$

(normal relations)

The relation between this tetrad and the geometry of space-time is

$$g_{ab} = l_a n_b + n_a l_b - m_a \bar{m}_b - \bar{m}_a m_b \qquad (1.4)$$

(completeness relation).

Thus at each point of the space-time manifold, a null tetrad of basis vectors is postulated. Accordingly any non zero vector $X^k$ can be expressed as unique combination of the null vectors :

$$X^a = (A) l^a + (C) m^a + (\bar{C}) \bar{m}^a + (B) n^a$$

with

$$A = X^k n_k, \quad B = X^k l_k, \quad C = -X^k \bar{m}_k$$

Thus A, B, C are the complex tetrad components (scalars) of the vector $X^a$.

In continuum mechanics the ubiquitous vector is the unit time-like flow vector. The choice $X^a = u^a$, $u^a u_a = 1$ is consistent with the relation

$$u^a = \frac{1}{\sqrt{2}} l^a + \frac{1}{\sqrt{2}} n^a. \qquad (1.5)$$

A space-like vector $E^a$ can have the form

$$E^a = \frac{1}{\sqrt{2}} l^a - \frac{1}{\sqrt{2}} n^a. \qquad (1.6)$$

The other two space-like vector fields which are orthogonal to these two vectors can be enumerated simply

$$\frac{1}{\sqrt{2}} (m^a + \bar{m}^a), \frac{1}{\sqrt{2}} (m^a - \bar{m}^a) \qquad (1.7)$$

The new basis is, therefore,

$$2^{-1/2}(\underline{l}^a + n^a, \underline{l}^a - n^a, m^a + \bar{m}^a, m^a - \bar{m}^a)$$

with one time-like and three space-like vectors. This establishes the ease with which one can switch over from null tetrad to the classical tetrad.

0.2 The Twelve Spin Coefficient :

The following 12 'Greek' letters have been famous in all the works on gravitational radiation :

$$k = \underline{l}_{a;b}\, m^a l^b,$$

$$\rho = \underline{l}_{a;b} m^a \bar{m}^b,$$

$$\varrho = \underline{l}_{a;b} m^a m^b,$$

$$\tau = \underline{l}_{a;b} m^a n^b,$$

$$\nu = - n_{a;b} \bar{m}^a n^b,$$

$$\mu = - n_{a;b} \bar{m}^a m^b,$$

$$\lambda = - n_{a;b} \bar{m}^a \bar{m}^b,$$

$$\pi = - n_{a;b} \bar{m}^a \underline{l}^b,$$

$$\alpha = 1/2(\underline{l}_{a;b} n^a \bar{m}^b - m_{a;b} \bar{m}^a \bar{m}^b),$$

$$\beta = 1/2(\underline{l}_{a;b} n^a m^b - m_{a;b} \bar{m}^a m^b),$$

$$\gamma = 1/2(\underline{l}_{a;b}n^a n^b - m_{a;b}\bar{m}^a n^b),$$

$$\epsilon = 1/2(\underline{l}_{a;b}n^a \underline{l}^b - m_{a;b}\bar{m}^a \underline{l}^b).$$

0.3 The Eleven Ricci Ray Scalars :

The enumeration of the eleven scalars which are just the tetrad components of the Ricci tensor $R_{ab}$ and R (Ricci scalar) is given by -

$$\phi_{00} = -\frac{1}{2} R_{ab}\underline{l}^a\underline{l}^b ,$$

$$\phi_{01} = -\frac{1}{2}R_{ab}\underline{l}^a m^b ,$$

$$\phi_{02} = -\frac{1}{2}R_{ab}m^a m^b,$$

$$\phi_{10} = -\frac{1}{2}R_{ab}\underline{l}^a \bar{m}^b ,$$

$$\phi_{11} = -\frac{1}{4}R_{ab}(\underline{l}^a n^b + m^a \bar{m}^b) ,$$

$$\phi_{12} = -\frac{1}{2}R_{ab}n^a m^b,$$

$$\phi_{20} = -\frac{1}{2}R_{ab}\bar{m}^a \bar{m}^b ,$$

$$\phi_{21} = -\frac{1}{2}R_{ab}n^a \bar{m}^b,$$

$$\phi_{22} = -\frac{1}{2} R_{ab}n^a n^b,$$

$$\Lambda = \frac{1}{24} R.$$

## 0.4 The Five Complex Weyl Scalars :

The free gravitational part of the curvature tensor $R_{abcd}$ (which is locally not defined by matter tensor $T_{ab}$) is the Weyl tensor $C_{abcd}$. The Weyl tensor $C_{abcd}$ in NP-formalism is expressed by (Compbell and Wainwright., 1977).

$$C_{abcd} = R_e[-2\psi_o U_{ab}U_{cd} + 4\psi_1(U_{ab}M_{cd} + M_{ab}U_{cd}) -$$
$$- 2\psi_2(U_{ab}V_{cd} + 4M_{ab}M_{cd} + V_{ab}U_{cd}) +$$
$$+ 4\psi_3(V_{ab}M_{cd} + M_{ab}V_{cd}) - 2\psi_4 V_{ab}V_{cd}],$$

where

$$U_{ab} = 2\bar{m}_{[a}n_{b]}$$

$$V_{ab} = 2 l_{[a}m_{b]}$$

and

$$M_{ab} = l_{[a}n_{b]} - m_{[a}\bar{m}_{b]}.$$

Then the tetrad components of $C_{abcd}$ are labaled :

$$\psi_o = - C_{abcd}l^a m^b l^c m^d,$$
$$\psi_1 = - C_{abcd}l^a n^b l^c m^d,$$
$$\psi_2 = - C_{abcd}\bar{m}^a n^b l^c m^d,$$
$$\psi_3 = - C_{abcd}\bar{m}^a n^b l^c n^d,$$
$$\psi_4 = - C_{abcd}\bar{m}^a n^b \bar{m}^c n^d.$$

0.4.1 Five types of $C_{abcd}$.

| Petrov type | Propagation vector | Vanishing components | Form of $C_{abcd}$ |
|---|---|---|---|
| I | $n_a$ | $\Psi_1, \Psi_2, \Psi_3, \Psi_4$ | $C_{abcd} = -\Psi_0 U_{ab} U_{cd} - \bar{\Psi}_0 \bar{U}_{ab} \bar{U}_{cd}$ |
| II | $n_a$ | $\Psi_0, \Psi_2, \Psi_3, \Psi_4$ | $C_{abcd} = 2\Psi_1 (U_{ab} M_{cd} + M_{ad} V_{cd}) + C.C.$ |
| III | $l_a$ | $\Psi_0, \Psi_1, \Psi_2, \Psi_4$ | $C_{abcd} = 2\Psi_3 (U_{ab} M_{cd} + M_{ab} V_{cd}) + C.C.$ |
| D | $n_a$ and $l_a$ | $\Psi_0, \Psi_1, \Psi_3, \Psi_4$ | $C_{abcd} = -\Psi_2 (U_{ab} V_{cd} + 4M_{ab} M_{cd} + V_{ab} U_{cd}) + C.C.$ |
| N | $l_a$ | $\Psi_0, \Psi_1, \Psi_2, \Psi_3$ | $C_{abcd} = -\Psi_4 V_{ab} V_{cd} + C.C.$ |

## 0.5 The Four Intrinsic Derivative Operations :

The 4 – Operators D, $\Delta$, $\delta$, $\bar{\delta}$ are defined as follows,

$$D\phi = \phi_{;a}\underline{l}^{a},$$

$$\Delta\phi = \phi_{;a}n^{a},$$

$$\delta\phi = \phi_{;a}m^{a},$$

$$\bar{\delta}\phi = \phi_{;a}\bar{m}^{a}.$$

## 0.6 The Four Commutator Relations :

The commutator relations of successive intrinsic derivatives are :

$$[\Delta, D]\phi = (\gamma + \bar{\gamma})D\phi - (\bar{\tau} + \pi)\delta - (\tau + \bar{\pi})\bar{\delta}\phi + + (\epsilon + \bar{\epsilon})\Delta\phi ,$$

$$[\delta, D]\phi = (\bar{\alpha} + \beta - \bar{\pi})D\phi - (\bar{\rho} + C - \bar{C})\delta\phi - - \sigma\bar{\delta}\phi + K\Delta\phi ,$$

$$[\delta, \Delta]\phi = - \bar{\nu}D\phi + (\mu - \gamma + \bar{\gamma})\delta\phi + \bar{\lambda}\bar{\delta}\phi + + (\tau - \bar{\alpha} - \beta)\Delta\phi,$$

$$[\bar{\delta}, \delta]\phi = (\bar{\mu} - \mu)D\phi - (\bar{\beta} - \alpha)\delta\phi - (\bar{\alpha} - \beta)\bar{\delta}\phi + + (\bar{\rho} - \rho)\Delta\phi,$$

Where

$$[\delta, D] = \delta D - D\delta.$$

If we now substitute $X^a$ for the arbitrary function $\emptyset$ in these equations and use the fact that (Carmeli 1977, p. 332).

$$DX^a = X^a_{;b}\underline{l}^b = \underline{l}^a ,$$

$$\delta X^a = X^a_{;b}m^b = m^a ,$$

$$\bar{\delta} X^a = X^a_{;b}\bar{m}^b = \bar{m}^a ,$$

$$\Delta X^a = X^a_{;b}n^b = n^a .$$

0.7 <u>We obtain the four metric Equations</u> :

$$\Delta\underline{l}^a - Dn^a = (\gamma + \gamma)\underline{l}^a - (\bar{\tau} + \pi)m^a - (\tau + \bar{\pi})\bar{m}^a + + (\epsilon + \bar{\epsilon})n^a ,$$

$$\delta\underline{l}^a - Dm^a = (\bar{\alpha} + \beta - \bar{\pi})\underline{l}^a - (\bar{\rho} + \epsilon - \bar{\epsilon})m^a - - \sigma\bar{m}^a + kn^a ,$$

$$\delta n^a - \Delta m^a = - \bar{\nu}\underline{l}^a + (\mu - \gamma + \bar{\gamma})m^a + \lambda\bar{m}^a + + (\tau - \bar{\alpha} - \beta)n^a ,$$

$$\bar{\delta} m^a - \delta\bar{m}^a = (\bar{\mu} - \mu)\underline{l}^a - (\bar{\beta} - \alpha)m^a - (\bar{\alpha} - \beta)\bar{m}^a + +(\bar{\rho} - \rho)n^a .$$

## 0.8 The Eighteen Newman Penrose Equations :

NP1 : $D\rho - \bar{\delta}k = (\rho^2 + \sigma\bar{\sigma}) + (\epsilon + \bar{\epsilon})\rho - \bar{k}\tau - k(3\alpha + \bar{\beta} - \pi) + \Phi_{00}$,

NP2 : $D\sigma - \delta k = (\rho + \bar{\rho})\sigma + (3\epsilon - \bar{\epsilon})\sigma - (\tau - \bar{\pi} + \bar{\alpha} + 3\beta)k + \Psi_0$,

NP3 : $D\tau - \Delta k = (\bar{\tau} + \bar{\pi})\rho + (\bar{\tau} + \pi)\sigma + (\epsilon - \bar{\epsilon})\tau - (3\gamma + \bar{\gamma}) + \Psi_1 + \Phi_{01}$,

NP4 : $D\alpha - \bar{\delta}\epsilon = (\rho + \bar{\epsilon} - 2\epsilon)\alpha + \beta\bar{\sigma} - \bar{\beta}\epsilon - k\lambda - \bar{k}\gamma + (\epsilon + \rho)\pi + \Phi_{10}$,

NP5 : $D\beta - \delta\epsilon = (\alpha + \pi)\sigma + (\bar{\rho} - \bar{\epsilon})\beta - (\mu + \gamma)k - (\bar{\alpha} - \bar{\pi})\epsilon + \Psi_1$

NP6 : $D\gamma - \Delta\epsilon = (\tau + \bar{\pi})\alpha + (\bar{\tau} + \pi)\beta - (\epsilon + \bar{\epsilon})\gamma - (\gamma + \bar{\gamma})\epsilon + (\ \pi - \nu k + \Psi_2 - \Lambda + \Phi_{11}$,

NP7 : $D\lambda - \bar{\delta}\pi = (\rho\lambda + \bar{\sigma}\mu) + \pi^2 + (\alpha - \bar{\beta})\pi - \nu\bar{k} - (3\epsilon - \bar{\epsilon})\lambda + \Phi_{20}$,

NP8 : $D\mu - \delta\pi = (\bar{\rho}\mu + \sigma\lambda) + \pi\bar{\pi} - (\epsilon + \bar{\epsilon})\mu - \pi(\bar{\alpha} - \beta) - \nu k + \Psi_2 + 2\Lambda$,

NP9 : $D\nu - \Delta\pi = (\pi + \bar{\tau})\mu + (\bar{\pi} + \tau)\lambda + (\gamma - \bar{\gamma})\pi - (3\epsilon + \bar{\epsilon})\nu + \Psi_3 + \Phi_{21}$,

NP10 : $\Delta\lambda - \bar{\delta}\nu = -(\mu + \bar{\mu})\lambda - (3\gamma - \bar{\gamma})\lambda + (3\alpha + \bar{\beta} + \pi - \bar{\tau})\nu - \Psi_4$

NP11 : $\delta\rho - \bar{\delta}\sigma = \rho(\bar{\alpha}+\beta) - \sigma(3\alpha - \bar{\beta}) + (\rho - \bar{\rho})\tau + (\mu - \bar{\mu})\kappa - \Psi_1 + \Phi_{01},$

NP12 : $\delta\alpha - \bar{\delta}\beta = (\mu\rho - \lambda\sigma) + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta + \gamma(\rho - \bar{\rho}) + \varepsilon(\mu - \bar{\mu}) - \Psi_2 + \Lambda + \Phi_{11},$

NP13 : $\delta\lambda - \bar{\delta}\mu = (\rho - \bar{\rho})\nu + (\mu - \bar{\mu})\pi + \mu(\bar{\alpha} + \beta) + \lambda(\bar{\alpha} - \beta) - \Psi_3 + \Phi_{21},$

NP14 : $\delta\nu - \Delta\mu = (\mu^2 + \lambda\bar{\lambda}) + (\gamma + \bar{\gamma})\mu - \bar{\nu}\pi + (\tau - 3\beta - \bar{\alpha})\nu + \Phi_{22},$

NP15 : $\delta\gamma - \Delta\beta = (\tau - \bar{\alpha} - \beta)\gamma + \mu\tau - \sigma\nu - \varepsilon\bar{\nu} - \beta(\gamma - \bar{\gamma} - \mu) + \alpha\bar{\lambda} + \Phi_{12},$

NP16 : $\delta\tau - \Delta\sigma = (\mu\sigma + \bar{\lambda}\rho) + (\tau + \beta - \bar{\alpha})\tau - (3\gamma - \bar{\gamma})\sigma - \kappa\bar{\nu} + \Phi_{02},$

NP17 : $\Delta\rho - \bar{\delta}\tau = -(\rho\bar{\mu} + \sigma\lambda) + (\bar{\beta} - \alpha - \bar{\tau})\tau + (\gamma + \bar{\gamma})\rho + \nu\kappa - \Psi_2 - 2\Lambda,$

NP18 : $\Delta\alpha - \bar{\delta}\gamma = (\rho + \varepsilon)\nu - (\tau + \beta)\lambda + (\bar{\gamma} - \bar{\mu})\alpha + (\bar{\beta} - \bar{\tau})\gamma - \Psi_3.$

0.9 The eleven Bianchi Identities :

B1 : $\delta\Psi_0 - D\Psi_1 + D\phi_{01} - \delta\phi_{00} = (4\alpha - \pi)\Psi_0 - 2(2\rho + \epsilon)\Psi_1 +$

$+ 3k\Psi_2 + (\bar{\pi} - 2\bar{\alpha} - 2\beta)\phi_{00} + 2(\epsilon + \bar{\rho})\phi_{01} + 2\sigma\phi_{10} -$

$+ 2k\phi_{11} - \bar{k}\phi_{02}$ ,

B2 : $\Delta\Psi_0 - \delta\Psi_1 + D\phi_{02} - \delta\phi_{01} = (4\gamma - \mu)\Psi_0 - 2(2\tau + \beta)\Psi_1 +$

$+ 2\sigma\Psi_2 - \bar{\lambda}\phi_{00} + 2(\bar{\pi} - \beta)\phi_{01} + 2\sigma\phi_{11} +$

$+ (2\epsilon - 2\bar{\epsilon} + \bar{\rho})\phi_{02} - 2k\phi_{12}$ ,

B3 : $3(\bar{\delta}\Psi_1 - D\Psi_2) + 2(D\phi_{11} - \delta\phi_{10}) + \bar{\delta}\phi_{01} - \Delta\phi_{00} =$

$= 3\lambda\Psi_0 - 9\rho\Psi_2 + 6(\alpha - \pi)\Psi_1 + 6k\Psi_3 +$

$+ (\bar{\mu} - 2\mu - 2\gamma - 2\bar{\gamma})\phi_{00} + (2\alpha + 2\pi + 2\bar{\tau})\phi_{01} +$

$+ 2(\tau - 2\bar{\alpha} + \bar{\pi})\phi_{10} + 2(2\bar{\rho} - \rho)\phi_{11} +$

$+ 2\sigma\phi_{20} - \bar{\sigma}\phi_{02} - 2\bar{k}\phi_{12} - 2k\phi_{21}$ ,

B4 : $3(\Delta\Psi_1 - \delta\Psi_2) + 2(D\phi_{12} - \delta\phi_{11}) + (\bar{\delta}\phi_{02} - \Delta\phi_{01}) =$

$= 3\nu\Psi_0 + 6(\gamma - \mu)\Psi_1 - 9\tau\Psi_2 = 6\sigma\Psi_3 - \bar{\nu}\phi_{00} +$

$+ 2(\bar{\mu} - \mu - \gamma)\phi_{01} - 2\bar{\lambda}\phi_{10} + 2(\tau + 2\bar{\pi})\phi_{11} +$

$+ (2\alpha + 2\pi + \bar{\tau} - 2\bar{\beta})\phi_{02} + (2\bar{\rho} - 2\rho - 4\epsilon)\phi_{12} +$

$+ 2\sigma\phi_{21} - 2k\phi_{22}$ ,

B5 :
$$3(\bar{\delta}\Psi_2 - D\Psi_3) + D\Phi_{21} - \delta\Phi_{20} + 2(\bar{\delta}\Phi_{11} - \Delta\Phi_{10}) =$$
$$6\lambda\Psi_1 - 9\pi\Psi_2 + 6(\epsilon - \rho)\Psi_3 + 3\kappa\Psi_4 - 2\nu\Phi_{00} +$$
$$+ 2\lambda\Phi_{01} + 2(\bar{\mu} - \mu - 2\tau)\Phi_{10} + (2\pi + 4\bar{\tau})\Phi_{11} +$$
$$+ (2\beta + 2\tau + \bar{\pi} - 2\bar{\alpha})\Phi_{20} - 2\bar{\sigma}\Phi_{12} +$$
$$+ 2(\bar{\rho} - \rho - \epsilon)\Phi_{21} - \bar{\kappa}\Phi_{22} ,$$

B6 :
$$3(\Delta\Psi_2 - \delta\Psi_3) + D\Phi_{22} - \delta\Phi_{21} + 2(\bar{\delta}\Phi_{12} - \Delta\Phi_{11}) =$$
$$6\nu\Psi_1 - 9\mu\Psi_2 + 6(\beta - \tau)\Psi_3 + 3\sigma\Psi_4 - 2\nu\Phi_{01} -$$
$$- 2\bar{\nu}\Phi_{10} + 2(2\bar{\mu} - \mu)\Phi_{11} + 2\bar{\lambda}\Phi_{02} - \bar{\lambda}\Phi_{20} +$$
$$+ 2(\pi + \bar{\tau} - 2\bar{\beta})\Phi_{12} + 2(\beta + \tau + \bar{\pi})\Phi_{21} + (\bar{\rho} - 2\epsilon -$$
$$- 2\bar{\epsilon} - 2\rho)\Phi_{22},$$

B7 :
$$\bar{\delta}\Psi_3 - D\Psi_4 + \bar{\delta}\Phi_{21} - \Delta\Phi_{20} = 3\lambda\Psi_2 - 2(\alpha + 2\pi)\Psi_3 +$$
$$+ (4\epsilon - \rho)\Psi_4 - 2\nu\Phi_{10} + 2\lambda\Phi_{11} + (2\gamma - 2\bar{\gamma} + \mu)\Phi_{20} +$$
$$+ 2(\bar{\tau} - \alpha)\Phi_{21} - \bar{\sigma}\Phi_{22} ,$$

B8 :
$$\Delta\Psi_3 - \delta\Psi_4 + \bar{\delta}\Phi_{22} - \Delta\Phi_{21} = 3\nu\Psi_2 - 2(\gamma + 2\mu)\Psi_3 +$$
$$+ (4\beta - \tau)\Psi_4 - 2\nu\Phi_{11} - \bar{\nu}\Phi_{20} + 2\lambda\Phi_{12} +$$
$$+ 2(\gamma + \bar{\mu})\Phi_4 + (\tau - 2\bar{\beta} - 2\alpha)\Phi_{22} .$$

B9 : $D\phi_{11} - \delta\phi_{10} + \bar{\delta}\phi_{01} + \Delta\phi_{00} + 3\,D\Lambda = (2\gamma - \mu + 2\,\bar{\gamma} -$

$- \bar{\mu})\phi_{00} + (\pi - 2\alpha - 2\,\bar{\tau})\phi_{01} + (\bar{\pi} - 2\bar{\alpha} - 2\tau)\phi_{10} +$

$+ 2(\rho + \bar{\rho})\phi_{11} + \bar{\sigma}\,\phi_{02} + \sigma\,\phi_{20} - \bar{k}\phi_{12} - k\phi_{21}$ ,

B10 : $D\phi_{12} - \delta\phi_{11} - \bar{\delta}\phi_{02} + \Delta\phi_{01} + 3\,\delta\Lambda = (2\gamma - \mu -$

$- 2\,\bar{\mu})\phi_{01} + \bar{\nu}\phi_{00} - \bar{\lambda}\phi_{10} + 2(\bar{\pi} - \tau)\phi_{11} + (\pi + 2\,\bar{\beta} -$

$- 2\alpha - \bar{\tau})\phi_{02} + (2\,\rho + \bar{\rho} - 2\,\bar{\varepsilon})\phi_{12} + \sigma\phi_{21} - k\phi_{22}$,

B11 : $D\phi_{22} - \delta\phi_{21} - \bar{\delta}\phi_{12} + \Lambda\phi_{11} + 3\Delta\Lambda = \nu\phi_{01} +$

$+ \bar{\nu}\phi_{10} - 2\,(\mu + \bar{\mu})\,\phi_{11} - \lambda\phi_{02} - \bar{\lambda}\phi_{20} + (\,2\pi - \bar{\tau} +$

$+ 2\,\beta)\,\phi_{12} + (2\,\beta - \tau + 2\,\pi)\,\phi_{21} + (\rho + \bar{\rho} - 2\,\varepsilon -$

$- 2\,\bar{\varepsilon}\,)\,\phi_{22}$ .

## 0.10 Spin coefficient for Asymptotically flat spacetimes:

| Definition | Expression for Asymptotically Flat Spacetimes |
|---|---|
| 1) $\rho = l_{a;b} m^a \bar{m}^b$ | $\rho = -r^{-1} - \sigma^0 \bar{\sigma}^0 r^{-3} + o(r^{-5})$ |
| 2) $\sigma = l_{a;b} m^a m^b$ | $\sigma = \sigma^0 r^{-2} + (\bar{\sigma}^0 \sigma^{02} - \frac{1}{2}\Psi^0_0) r^{-4} + o(r^{-5})$ |
| 3) $\tau = l_{a;b} m^a n^b$ | $\tau = -(\frac{1}{2r^3}) \Psi^0_1 + (\frac{1}{r^4})(2\bar{A}^{ok} \Psi^0_{0,k} - 8\alpha^0 \Psi^0_0 + \sigma^0 \bar{\Psi}^0_1) + O(r^{-5})$ |
| 4) $\lambda = n_{a;b} \bar{m}^a \bar{m}^b$ | $\lambda = \lambda^0 r^{-1} - \bar{\sigma}^0 \mu^0 r^{-2} - r^{-3}(\sigma^0 \bar{\sigma}^0 \lambda^0 + \frac{1}{2} \bar{\sigma}^0 \Psi^0_2) + o(r^{-4})$ |
| 5) $\mu = -n_{a;b} \bar{m}^a m^b$ | $\mu = \mu^0 r^{-1} - (\sigma^0 \lambda^0 + \Psi^0_2) r^{-2} + (\sigma^0 \bar{\sigma}^0 \mu^0 - \alpha^0 \Psi^0_1 + \frac{1}{2}\bar{A}^{ok} \Psi^0_{1,k}) r^{-3} + o(r^{-4})$ |
| 6) $\nu = n_{a;b} \bar{m}^a n^b$ | $\nu = \nu^0 - r^{-1} \Psi^0_3 + \frac{1}{2} r^{-2} \bar{A}^{ok} \Psi_{o,k} + o(r^{-3})$ |
| 7) $\alpha = \frac{1}{2}(l_{a;b} n^a \bar{m}^b - m_{a;b} \bar{m}^a \bar{m}^b)$ | $\alpha = \beta^0 r^{-1} + \bar{\sigma} \bar{\beta}^0 r^{-2} + \sigma^0 \bar{\sigma}^0 \beta^0 r^{-3} + o(r^{-4})$ |

| Definition | Expression for Asymptotically Flat Spacetimes |
|---|---|
| 8) $\beta = \frac{1}{2}( \quad_{a;b} n^a \bar{m}^b - m_{a;b} \bar{m}^a m^b)$ | $\beta = - \bar{\alpha}^o r^{-1} - \sigma^o \alpha^o r^{-2} - (\sigma^o \bar{\sigma}^o \bar{\alpha}^o + \frac{1}{2} \psi_1^o) r^{-3} + O(r^{-4})$ |
| 9) $\gamma = \frac{1}{2}(l_{a;b} n^a n^b - m_{a;b} \bar{m}^a n^b)$ | $\gamma = \gamma^o - \frac{1}{2r^2} \psi_2^o + \frac{1}{r^3} (\frac{1}{3} \bar{A}^{ok} \psi^o_{1,k} - \frac{1}{6} \bar{\alpha} \bar{\psi}_1^o - \frac{1}{2} \alpha^o \psi_1^o) + o(r^{-4})$ |

0.11 Weyl Scalars for Asymptotically Flat space times :

| Definition | Expression for Asymptotically Flat space-times. |
|---|---|
| $\psi_0 = - C_{abcd} l^a m^b l^c m^d$ | $\psi_0 = \psi_0^o r^{-5} + O(r^{-6})$ |
| $\psi_1 = - C_{abcd} l^a n^b l^c m^d$ | $\psi_1 = \psi_1^o r^{-4} + (4\alpha^o \psi_0^o - \bar{A}^{ok} \psi^o_{0,k}) r^{-5} + O(r^{-6})$ |
| $\psi_2 = - \frac{1}{2} C_{abcd} (l^a n^b l^c n^d + l^a n^b m^c \bar{m}^d)$ | $\psi_2 = \psi_2^o r^{-3} + (2\alpha^o \psi_1^o - \bar{A}^{ok} \psi^o_{1,k}) r^{-4} + O(r^{-5})$ |

| Definition | Expression for Asymptotically flat space-times. |
| --- | --- |
| $\psi_3 = C_{abcd} \bar{l}^a n^b n^c \bar{m}^d$ | $\psi_3 = \psi^{3}_{\circ} r^{-2} - \bar{A}^{ok} \psi^{\circ}_{2} r^{-3} + O(r^{-4})$ |
| $\psi_4 = - C_{abcd} n^a \bar{m}^b n^c \bar{m}^d$ | $\psi_4 = \psi^{\circ}_{4} r^{-1} - (2\, \alpha^{o}\, \psi^{\circ}_{3} + A^{ok} \psi^{\circ}_{3,k}) r^{-2} + O(r^{-3})$ |