CHAPTER - I

GEOMETRICAL AND DYNAMICAL PRELIMINARIES

1. PRECURSORY ASSUMPTIONS

The preliminary concepts necessary for the development of dissertation work are introduced. We mainly deal with four-dimentional space-time manifold V_4 with the Lorentzian metric of signature (-,-,-,+). The geometrical units are such that C=G=-K=1. The various symbols used are as follows,

- ; : Covarient derivative,
- , : Partial derivative,
- $\overset{*}{X}$: Covarient derivative of X with respect to time-like vector,
- ( ) : Symmetrization bracket,
- [ ] : Antisymmetrization bracket,
- $\underset{c}{L}$: Lie derivative with respect to c.

2. FORMATION OF THE STRESS ENERGY TENSOR FOR INFINITELY CONDUCTING FERROFLUID

Lichnerowicz (1967) has developed a stress energy tensor characterizing perfect fluid with infinite conductivity and constant magnetic permeability. The form of this stress energy tensor which he has used to study the relativistic magnetohydrodynamics is given by

$$\underset{(L)}{T_{ab}} = (r+p+\mu h^2)\, U_a U_b - (p+\tfrac{1}{2}\mu h^2)\, g_{ab} - \mu h_a h_b \,, \qquad (2.1)$$

where p is isotropic pressure and
r is proper matter density .

Here the time-like vector U_a has the unitary character i.e.,

$$U^a U_a = 1 \ . \tag{2.2}$$

This implies that,

$$U^a{}_{;b} U_a = 0. \tag{2.3}$$

The magnetic field vector h_a is normal to the flow vector U_a having its magnitude $(-h^2)$ i.e.,

$$U_a h^a = 0 \text{ and } h_a h^a = -h^2 \ . \tag{2.4}$$

It follows from these results that

$$h_{a;b} h^a = -\frac{1}{2} h^2{}_{;b} \ , \tag{2.5}$$

$$h_{a;b} U^a = -U_{a;b} h^a \ . \tag{2.6}$$

This formalism is used by Ray and Banarji (1980) for relativistic ferrofluid. According to these authors in ferrofluid the magnetic induction vector and polarization vector are linearly related and the magnetic permeability is a variable quantity. They have considered the stress energy tensor characterising ferrofluid with infinite electrical conductivity and variable magnetic permeability as given by

$$T_{ab} = (r+p+\mu h^2)\, U_a U_b - (p+\frac{1}{2}\mu h^2)\, g_{ab} - \mu h_a h_b . \tag{2.7}$$

(B)

Note that though the form (2.7) is just similar to (2.1) it differs only in term of magnetic permeability which is treated as variable in (2.7).

Further in 1972, Maugin has coined a stress energy tensor for polorized, magnetised perfect fluid with infinite electrical conductivity and constant magnetic permeability eshtablished by making use of the Action Principle in the format

$$\underset{(m)}{T}_{ab} = (r+p+\mu h^2)\, U_a U_b - \left(p+\mu\left(1-\frac{\mu}{2}\right)h^2\right) g_{ab} - \mu h_a h_b \cdot (2.8)$$

We consider the infinitely conducting perfect ferrofluid encompassing the effects of polorization and magnetization as considered by Maugin which is described by the stress energy tensor given by (2.8) and by keeping the magnetic permeability as variable. Hence the form of the stress energy tensor which is used throughout the dissertation work is

$$T_{ab} = (r+p+\mu h^2)\, U_a U_b - \left[p+\mu\left(1-\frac{\mu}{2}\right)h^2\right] g_{ab} - \mu h_a h_b \cdot \quad (2.9)$$

This tensor with variable magnetic permeability will describe the propertiesof the infinitely conducting polorized, magnetized perfect fluid which we denote hereafter by the name Infinitely Conducting Ferrofluid.

NOTE :

It is observed that if we keep μ as constant in (2.9) then we have the Maugin's stress energy tensor given by (2.8). Further if we put the constant magnetic permeability as unity then the Maugin's form (2.8) gets reduced to Lichnerowicz form (2.1)

3. CHARACTERISTIC PPROPERTIES OF THE INFINITELY CONDUCTING FERROFLUID STRESS TENSOR

(i) Components of T^{ab} :

For the stress energy tensor of infinitely conducting Ferrofluid given by (2.9). We list the following results derived by contractions.

$$T^{ab}U_a = \left(r+\frac{\mu^2 h^2}{2}\right) U^b \,, \tag{3.1}$$

$$T^{ab}U_aU_b = r+\frac{\mu^2 h^2}{2} \,, \tag{3.2}$$

$$T^{ab}h_a = -\left(p-\frac{\mu^2 h^2}{2}\right) h^b \,, \tag{3.3}$$

$$T^{ab}h_ah_b = \left[p-\frac{\mu^2 h^2}{2}\right]h^2 \,, \tag{3.4}$$

$$T^{ab}g_{ab} = T = r-3p-2\mu h^2+2\mu^2 h^2 \,, \tag{3.5}$$

$$T^{ab}U_ah^b = 0 \,, \tag{3.6}$$

$$T^{ab}T_{ab} = A^2+4B^2-2AB-2B\mu h^2+\mu^2 h^4 \,, \tag{3.7}$$

where

$$A = r+p+\mu h^2 \,,$$

$$B = p+\mu\left(1-\frac{\mu}{2}\right)h^2 \,,$$

DISCUSSION :

It follows from the above relations,

(a) The infinitely conducting ferrofluid has a positive time-like eigen value is given by (3.2).

(b) The infinitely conducting ferrofluid has a space-like eigen value given by (3.4).

(c) The rest mass of infinitely conducting Ferrofluid is given by (3.5).

(ii) Energy Conditions :

The energy conditions followed by stress energy tensor associated with infinitely conducting Ferrofluid space-time are

(a) Dominant Energy Condition ($T^{ab}U_aU_b \geq 0$).

We have

$$T^{ab}U_aU_b = r+\frac{\mu^2 h^2}{2} \ .$$

Therefore this energy condition becomes

$$r+\frac{\mu^2 h^2}{2} \geq 0 \ . \tag{3.8}$$

(b) Strong Energy Condition :

It is given by

$$T_{ab}U^aU^b - \frac{1}{2}\,T \geq 0.$$

For infinitely conducting ferrofluid it provides

3 $$r+6p+2\mu h^2-\mu^2 h^2 \geq 0 \ . \tag{3.9}$$

4. EQUATIONS GOVERNING PHYSICAL FIELDS

(i) Einstein Field Equations :

The well known Einstein Field Equations governing the geometry and dynamics of the space-time are given by

$$G_{ab} = T_{ab} \ , \tag{4.1}$$

where the symmetric Einstein tensor G_{ab} has the usual expression comprising the terms of the Ricci tensor Rab and the curvature scalar R, in the form

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} \,. \tag{4.2}$$

NOTE 1 : The left hand side of (4.1) is forced to be the Einstein tensor under the assumptions,

1. G vanishes when space-time is flat,
2. G is given by (4.2),
3. G satisfies the condition arising out of contracted Bianchi identities i.e. divergence of G is zero.

NOTE 2 : The more general Einstein Field Equations including the cosmoligical term are ~~(Einstein 1917)~~,

$$G_{ab} + \Lambda g_{ab} = T_{ab} \,. \tag{4.3}$$

The inclusion of Λ term is abandoned by Einstein after the conformation of Hubbles law, but still many workers in the field of General Relativity use Einstein Field Equations (4.3). We use the Einstein Field Equations given by the equation (4.1).

NOTE 3 : The right hand side of Einstein Field Equations (4.1) involving a Symmetric Stress Tensor speaks about the dynamical features of the space-time.

(ii) Equations Governing Electromagnetic Field :

The Maxwells equations ~~from~~ for the electromagnetic field with source are given by ,

$$F^{ab}{}_{;b} = 4 \, J^{a} \,, \tag{4.4}$$

$$F_{ab;c} + F_{bc;a} + F_{ca;b} = 0. \quad (4.5)$$

Here the skew-symmetric electromagnetic field tensor F_{ab} is defined through the vector potential A_a in the form,

$$F_{ab} = A_{b;a} - A_{a;b} . \quad (4.6)$$

The term J^a in equation (4.4) gives the four-current vector which is always conservative.
These equations under the constraints that,
(i) infinite conductivity (ii) variable magnetic permeability are ~~related~~ due to (Lichnerowicz, 1967),

$$[\mu(U^a h^b - U^b h^a)] ;b = 0. \quad (4.7)$$

NOTE 1 :
The Maxwell equations in Special Relativistic Space are given by

$$F^{ab}{}_{,b} = 4\pi J^a , \quad (4.8)$$

$$F_{ab,c} + F_{bc,a} + F_{ca,b} = 0. \quad (4.9)$$

Here the components of four-current J are

$J_o = \rho =$ charged density ,

and $(J_1 , J_2 , J_3) =$ components of current density.

NOTE 2 :
In empty space we write the Maxwell Equations in index free notations in the form

$$dF = 0 \,, \tag{4.10}$$

$$d{*}F = 4\pi {*}J \,. \tag{4.11}$$

where d denotes exterior derivative and * denotes the dual.

NOTE 3 :

Source free Maxwell Equations in curved space time are

$$F^{ab}{}_{;b} = 0 \,, \tag{4.12}$$

$$F_{ab;c} + F_{bc;a} + F_{ca;b} = 0 \,. \tag{4.13}$$

We use the set of Maxwell Equations (4.7) throughout the dissertation work for the space-time of infinitely conducting ferrofluid .

(iii) Heat Transfer Equations :

The relativistic laws of thermodynamics available in literature are stated as follows,

(a) First law of Thermodynamics:

The relations interlinking the thermodynamical variables are governed by the law, (Maugin 1970)

$$T_0 dS = d\epsilon + p d\left(\frac{1}{r_0}\right) - \frac{1}{2r_0}\mu(1-\mu) dh^2 \,, \tag{4.14}$$

where the proper material density r_0, the internal energy density ϵ, the specific entropy S, and the rest temperature T_0 are related by ,

$$r = r_0(1+\epsilon) \,, \tag{4.15}$$

$$\text{and } x = 1+\epsilon+p/r_0 \,. \tag{4.16}$$

REMARK : This law is merely the law of conservation of energy in Special Relativity.

(b) Second law of Thermodynamics :

The entropy flux four vector S^a is defined as (Greenberg, 1971)

$$S^a = r_o S U^a + q^a / T_o , \qquad (4.17)$$

where q^a is the four flux vector.

This entropy vector sarisfies Clausinsduhem inequality given by ,

$$S^a{}_{;a} \geqslant 0. \qquad (4.18)$$

REMARK : (i) The second law of thermodynamics plays an effective role whenever there exists heat flux vector in the stress energy tensor for the field under consideration.

(ii) This law also states that the entropy can be generat~~i~~ed but cannot be destroyed.

5. GEOMETRICAL ASPECTS :

(a) Kinematical parameters connected with the time-like vector :

The gradient of the four flow vector U^a is expressible in the form (Greenberg, 1971),

$$U_{a;b} = \sigma_{ab} + W_{ab} + \frac{1}{3}\theta P_{ab} + \dot{U}_a U_b , \qquad (5.1)$$

where the parameters on right hand side are,

$$\theta = U^a{}_{;a} \text{ : (Expansion) ,} \tag{5.2}$$

$$\dot{U}^a = U^a{}_{;b}U^b \text{ : (4-acceleration) ,} \tag{5.3}$$

$$\sigma_{ab} = U_{(a;b)} - \dot{U}_{(a}U_{b)} - \frac{1}{3}\theta P_{ab} \text{ :(Shear tensor),} \tag{5.4}$$

$$W_{ab} = U_{[a;b]} - \dot{U}_{[a}U_{b]} \text{ : rotation tensor,} \tag{5.5}$$

$$P_{ab} = g_{ab} - U_a U_b \text{ : (3-space projection operator) ,} \tag{5.6}$$

It follows from these defining expressions of parameters :

$$\sigma^a_a = W^a_a = 0 \text{ ,} \tag{5.7}$$

$$\sigma_{ab} = \sigma_{ba} \text{ , } W_{ba} = -W_{ab} \text{ ,} \tag{5.8}$$

$$P^a_a = 3 \text{ , } P_{ab}U^b = 0, \ P_{ab} = P_{ba} \text{ ,} \tag{5.9}$$

$$W_{ab}U^b = \sigma_{ab}U^b = 0 \text{ .} \tag{5.10}$$

We define two scalars

$$\sigma_{ab}\sigma^{ab} = 2\sigma^2 \text{ ,} \tag{5.11}$$

$$W_{ab}W^{ab} = 2W^2 \text{ .} \tag{5.12}$$

(b) Lie Derivative : (Carmeli 1982)

Let there be some tensor field $T(x)$ which is defined in our space-time. At the point Q we can evaluate the tensor $T(x)$ in two different ways. First we have the value of T at the points Q , namely, $T(\bar{x})$. Then we have the value of $\bar{T}(\bar{x})$, namely, the transformed tensor $\bar{T}$ using the usual co-ordinates transformations for tensors at the point Q . The difference between these two values of the tensor T evaluated at the point Q with co-ordinates X , leads to the possibility of defining the concept of Lie Derivative of the tensor T.

(1) <u>Definition</u> : Lie derivative of covariant tensor T^{ab} with respect to vector field $\bar{\xi}$ is defined as

$$\underset{\xi}{L} T_{ab} = T_{ab;c} \xi^{c} + T^{c}_{a} \xi_{c;b} + T^{c}_{b} \xi_{c;a} \; . \qquad (5.13)$$

(2) <u>Definition</u> :Lie derivative of contravariant tensor T^{ab} with respect to vector field $\bar{\xi}$ is defined as ,

$$\underset{\xi}{L} T^{ab} = T^{ab}_{;c} \xi^{c} - T^{ac} \xi_{c;b} - T^{bc} \xi_{c;a} \; . \qquad (5.14)$$

(3) <u>Einstein Collineation</u> : The Einstein Collineation with respect to any arbitrary vector field $\bar{\xi}$ is defined by

$$\underset{\xi}{L} G_{ab} = 0 \; ,$$

i.e.,

$$\underset{\xi}{L} (Rab - \frac{1}{2} R \, g_{ab}) = 0 \; ,$$

i.e. ,

$$\underset{\xi}{L} T_{ab} = 0 \text{ (via Einstein field equations)} \qquad (5.15)$$

This by definition (5.13) implies

$$T_{ab;c} \xi^{c} + T_{ac} \xi_{;b} + T_{cb} \xi_{;a} = 0 \; . \qquad (5.16)$$

<u>NOTE</u> : No original results are claimed in this section.