CHAPTER - II

Approximate Solution of Steady Laminar Flow Past a Horizontal Plate embedded in a Saturated Porous Medium

1) Introduction :

The approximate solution of the velocity boundary layer based on Karman momentum integral equation was first obtained by E. Pohlhausen [3].

The exact solution of the velocity distribution in boundary layer flow past a flat plate was investigated by Blasius (1908) [5] and the corresponding numerical solution for the thermal boundary layer with and without frictional heat was studied by E.Pohlhausen. [ ]

Recently B.C.Chandrasekhara [1] obtained solution for axial and transverse boundary layer equation in the case of steady laminar flow past a horizontal plate embedded in saturated porous medium.

In this problem we assume the governing equation of motion for two dimensional boundary layer following B.C.Chandrasekhara [1], Yekta and B.B.Waghmode [2], where the properties of the fluid and the porous medium such as viscocity permeability are constant and 6th degree velocity profile for the boundary layer flow over a horizontal plate embedded in a saturated porous medium. The velocity distribution is some function η of the ratio y/δ, δ being the boundary layer thickness. ?

The aim of this analysis is to find the velocity distribution graphically and obtain the displacement thickness δ_1 momentum thickness δ_2 and shear stress at the wall τ_o.

2) Mathematical Formulation :

For mathematical analysis we assume the governing equation of motion for two dimensional boundary layer flow following B.C.Chandrasekhara [1] and Yekta and B.B.Waghmode [2] where the properties of the fluid and the porous medium such as viscosity, permeability are constant.

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} - \frac{\nu}{K} u \qquad \ldots (1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \ldots (2)$$

where K : permeability of the porous medium

ν : Kinematic viscosity of the fluid

The boundary conditions for this problem are

$$\left.\begin{array}{ll} y = 0 & v = u = 0 \\ y = \infty & u = U_\infty ,\ v = 0 \\ \frac{du}{dy} = 0 & \text{when } y = \infty \end{array}\right\} \qquad \ldots (3)$$

3) **Analysis of the Problem :**

To solve the above equation we define a stream function

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} \qquad \ldots (4)$$

Substituting these values in (1) we get

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \nu \frac{\partial^3 \psi}{\partial y^3} - \frac{\nu}{K}\frac{\partial \psi}{\partial y}$$

$$\text{i.e.} \quad \frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \nu \frac{\partial^3 \psi}{\partial y^3} - B\frac{\partial \psi}{\partial y} \qquad \ldots (5)$$

where $B = \nu / K$.

Introducing a similarity transformations as

$$\left.\begin{aligned} \eta &= \left(\sqrt{\frac{U_\infty}{\nu x}}\right) y \qquad \therefore\; y = \eta\sqrt{\frac{\nu x}{U_\infty}} \\ \psi(\eta) &= \sqrt{\nu x\, U_\infty}\; f(\eta) \end{aligned}\right) \qquad \ldots (6)$$

$$u = \frac{\partial \psi}{\partial y} = U_\infty f'(\eta)$$

$$u\frac{\partial u}{\partial x} = U_\infty^2 f'(\eta)\frac{\partial f'}{\partial x}$$

$$v = \sqrt{\nu U_\infty x}\,\frac{\partial f}{\partial x} + \sqrt{\frac{\nu U_\infty}{x}}\,\frac{f(\eta)}{2}$$

$$v \frac{\partial u}{\partial y} = U_\infty^2 f''(\eta) \frac{\partial f}{\partial x} + U_\infty^2 \frac{f(\eta)\ f''(\eta)\ x^{-1}}{2}$$

$$\frac{\partial^2 \psi}{\partial y^2} = U_\infty f''(\eta) \sqrt{\frac{U_\infty}{\nu\, x}}$$

$$\therefore \quad \frac{\partial^3 \psi}{\partial y^3} = \frac{U_\infty^2\ f'''(\eta)}{\nu\, x}$$

$$\therefore \quad \nu \frac{\partial^3 \psi}{\partial y^3} = \frac{U_\infty^2\ f'''(\eta)}{x}$$

Putting all the above values in (1) we get

$$U_\infty^2 f' \frac{\partial f'}{\partial x} - U_\infty^2 f''(\eta) \frac{\partial f}{\partial x} - \frac{U_\infty^2 f(\eta) f''(\eta) x^{-1}}{2} =$$

$$U_\infty^2\ f'''(\eta) x^{-1} - B U_\infty f'(\eta) \qquad \ldots (7)$$

where $B = \nu / K$. repeatation

Dividing both side by $U_\infty^2\ x^{-1}$ equation (7) becomes

$$f'''(\eta) + \frac{1}{2} f(\eta)\ f''(\eta) - \left(\frac{Bx}{U_\infty}\right) f'(\eta) = 0$$

i.e. $2f'''(\eta) + f(\eta)\ f''(\eta) - 2Sf'(\eta) = 0$ ... (8)

$$\text{where} \quad S = \frac{Bx}{U_\infty} = \frac{\nu\ x}{KU_\infty} = \frac{\nu\ x^2}{KU_\infty x} = \frac{x^2}{\frac{KU_\infty x}{\nu}} = \frac{x^2}{KRe}$$

where $Re = \dfrac{U_\infty x}{\nu}$

with the Boundary conditions

$$\left.\begin{array}{l} \eta = 0, \quad u = 0, \quad f = 0, \quad f' = 0 \\ \eta = \infty, \quad u = U_\infty, \quad f' = 1 . \end{array}\right\} \qquad \ldots (9)$$

The solution of non-linear equation (8)

$$\frac{u}{U_\infty} = f(\eta) = \sum_{i=0}^{6} a_i \eta^i \qquad \ldots (10)$$

$$\frac{u}{U_\infty} = 1 \quad \text{for } \eta \geqslant 1 \text{ where } \eta = y/\delta$$

We have to determine the coefficient a_0, a_1, a_2, a_3, a_4, a_5 and a_6. We prescribe the following boundary and compatibility conditions

$$\left.\begin{array}{l} y = 0; \quad u = 0, \quad \dfrac{\partial^2 u}{\partial y^2} = 0, \quad \dfrac{\partial^3 u}{\partial y^3} = 0, \\ y = \delta; \quad u = U_\infty, \quad \dfrac{\partial u}{\partial y} = \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial^3 u}{\partial y^3} = 0 \end{array}\right\} \qquad \ldots (11)$$

?

see (?)

By using boundary conditions (11) we get

$$\left.\begin{array}{l} a_0 = 0 \quad a_2 = 0 \quad a_3 = 0 \text{ and} \\ a_1 + a_4 + a_5 + a_6 = 1 \\ a_1 + 4a_4 + 5a_4 + 6a_6 = 0 \\ 12a_4 + 20a_5 + 30a_6 = 0 \\ 24a_4 + 60a_5 + 120a_6 = 0 \end{array}\right\} \qquad \ldots (12)$$

Solving the above equations we get

$$a_1 = 2, \; a_4 = -5 \; , \quad a_6 = -2, \; a_5 = 6 \qquad \ldots (13)$$

Substituting the values of a_0 to a_6 in equation (10) we obtain the velocity in the form

$$\frac{u}{U_\infty} = f(\eta) = 2\eta - 5\eta^4 + 6\eta^5 - 2\eta^6 \qquad \ldots (14)$$

We calculate the displacement thickness δ_1, momentum thickness δ_2 and shear stress at the wall (τ_o)

1) Displacement thickness (δ_1)

$$\delta_1 = \int_o^1 \left(1 - \frac{u}{U_\infty}\right) d\eta$$

$$\delta_1 = 0.2857 \; \delta \qquad \ldots (15)$$

2) Momentum thickness (δ_2)

$$\delta_2 = \int_0^1 \frac{u}{U_\infty} \left(1 - \frac{u}{U_\infty}\right) d\eta$$

$$\delta_2 = 0.1093 \; \delta \qquad \ldots (16)$$

3) Shearing stress at the wall (τ_o)

$$\tau_o = \mu \frac{U_\infty}{\delta} \left(\frac{\partial f}{\partial \eta} \right)_{\eta=0}$$

$$\left(\frac{\partial f}{\partial \eta} \right)_{\eta=0} = 2$$

$$\tau_o = \frac{2\mu\, U_\infty}{\delta} \qquad \ldots (17)$$

The momentum integral equation may be solved easily if δ_2 instead of δ is regarded as known function and for this we write the Karman momentum equation as

$$\frac{U}{\nu}\, \delta_2 \frac{d\delta_2}{dx} + \left(2 + \frac{\delta_1}{\delta_2}\right) \frac{\delta_2^2}{\nu} \frac{dU}{dx} = \frac{\tau_o\, \delta_2}{\mu\, U} \qquad \ldots (18)$$

Substituting the values of δ_1 and δ_2 from (15) and (16) in equation (18) and simplifying we get

$$\delta = 6.0495 \sqrt{\frac{\nu\, x}{U_\infty}} \qquad \ldots (19)$$

From equation (19) equation (15), (16) and (17) will take following forms

$$\delta_1 = 1.97084 \sqrt{\frac{\nu\, x}{U_\infty}}$$

$$\delta_2 = 0.6672 \sqrt{\frac{\nu\, x}{U_\infty}}$$

$$\tau_o = 0.3306 \sqrt{\frac{\nu\, x}{U_\infty}}$$

Table

η	0	0.1	0.2	0.3	0.4
$f(\eta)$	0	0.199558	0.39379	0.572622	0.725248

η	0.5	0.6	0.7	0.8	0.9
$f(\eta)$	0.84375	0.925248	0.972622	0.993792	0.999558

From the table velocity distribution curve has been obtained.

4) Discussion and Conclusions :

1) As η increases from 0 to 1 the velocity increases.

2) Displacement thickness, Momentum thickness and shearing stress at the wall are calculated.

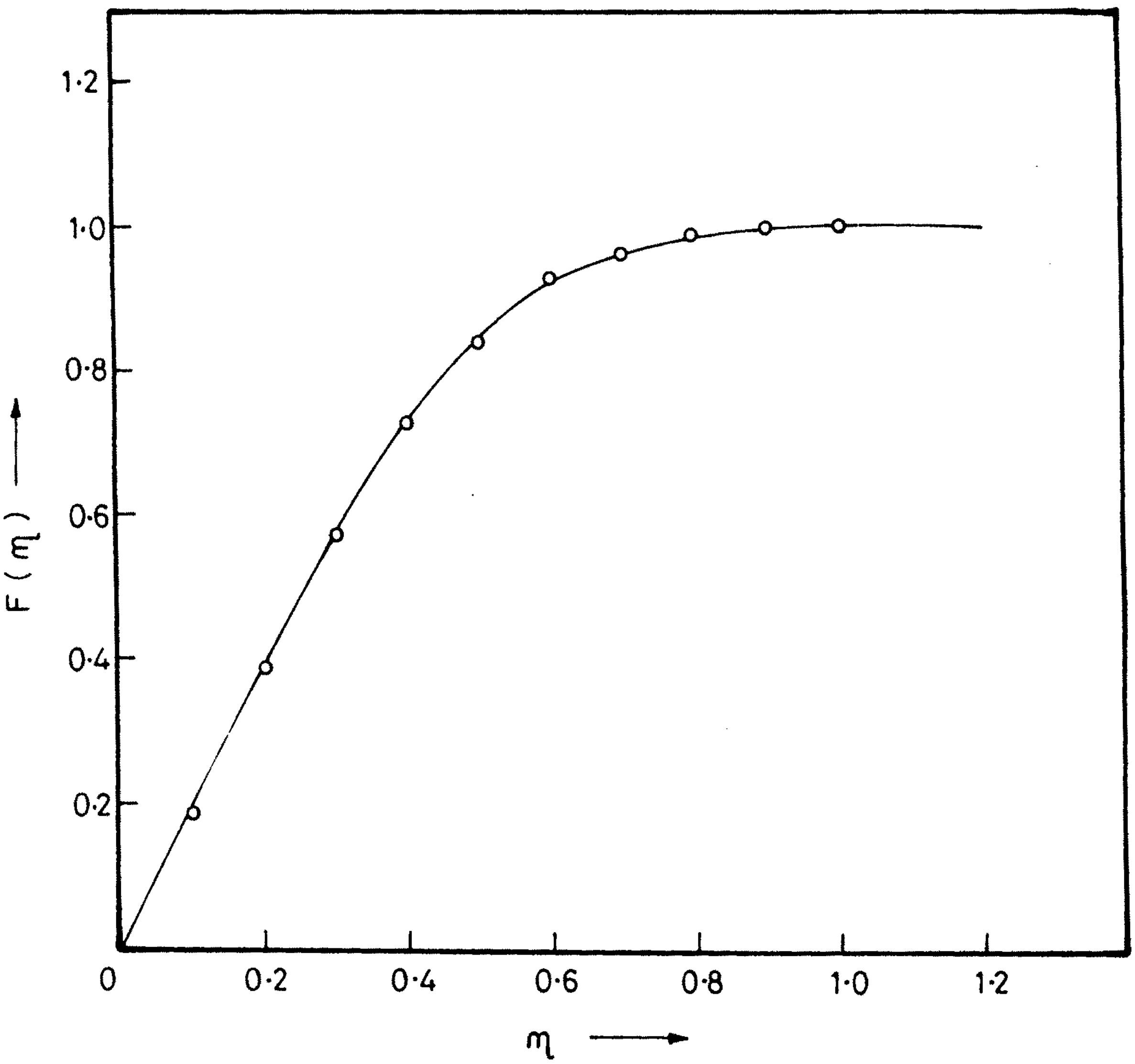

Fig. 1 — VELOCITY DISTRIBUTION .

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