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CHAPTER-III

1-DISTRIBUTIVE LATTICES

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3.1 INTRODUCTION

The concept of 1-distributivity follows dually that of 0-distributivity. So far nobody has paid attention to 1-distributive lattices. On the same line of 0-distributivity we have studied 1-distributivity delightfully.

The definition of a 1-distributive lattice alongwith a five element lattice N_5 as an illustration is given in Article 2. How 1-distributivity generalizes the distributivity and quasicomplementedness is also demonstrated.

Article 3 deals with the important result which contains characterizations of a 1-distributive lattice.

It is easy to observe that the set of all dual dense elements of L is an ideal in L. Also a sufficient condition for an ideal topology on a 1-distributive lattice to be Hausdorff can be proved as

Result : Let L be a 1-distributive lattice. If L is dual atomic and dually weakly complemented, then the ideal topology on L is Hausdorff.

3.2 DEFINITION AND EXAMPLES

3.2.1 Definition : A lattice L with 1 is said to be 1-distributive if a V b = 1 and a V c = 1 imply that a V (b $\wedge$ c)=1 for all a,b,c ε L.

3.2.2 Example : The lattice N_5 sketched in the following diagram is a 1-distributive lattice.

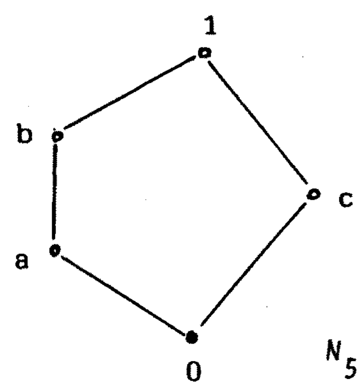

Fig.5

3.2.3 Example : Every lattice need not be 1-distributive.

The lattice M_5 sketched in the following diagram is not 1-distributive.

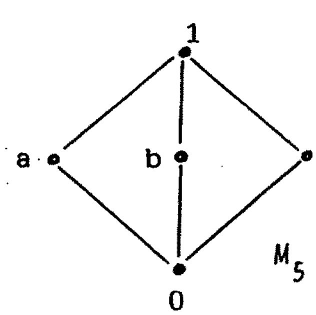

Fig.6

From the figure it follows that a V b = 1 and a V c = 1 but a V(b $\wedge$ c) = a V o = a$\neq$ 1 and hence M_5 is not 1-distributive.

As M_5 is a modular lattice it follows that

3.2.4 Remark : A modular lattice with 1 need not be 1-distributive.

In the following results we prove that the class of all 1-distributive lattices contains the class of all distributive lattices with 1 and the class of all quasicomplemented lattices (see Def.1.1.11).

3.2.5 Result : Every distributive lattice with 1 is 1-distributive.

Proof : Let L be a distributive lattice with 1.

For x,y,a in L let $x \vee a = 1$ and $y \vee a = 1$.

consider $(x \wedge y) \vee a$

$= (x \vee a) \wedge (y \vee a)$ (since L is distributive)

$= 1 \wedge 1$

$= 1$

Thus $x \vee a = 1$ and $y \vee a = 1$ imply that $(x \wedge y) \vee a = 1$.

Hence L is 1-distributive. ■

3.2.6 Remark : Converse of Result 3.2.5 need not be true.

The lattice sketched in the following diagram is a 1-distributive lattice which is not distributive.

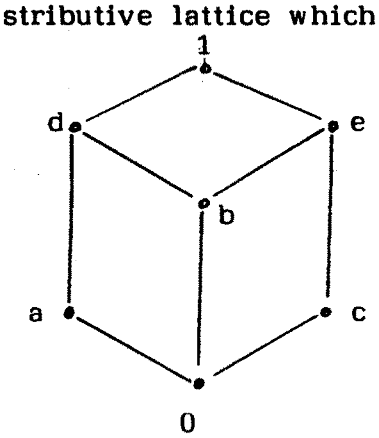

Fig.7

From the figure it is observed that

$b \wedge (a \vee c) \neq (b \wedge a) \vee (b \wedge c)$

and hence the lattice of Fig.7 is not distributive.

3.2.7 Result : Any quasicomplemented lattice is 1-distributive .

Proof : Let L be a quasicomplemented lattice. Then $a^{\perp}$ exists for each element a in L (see Def.1.1.11).

Let $a,b,c \in L$ such that $a \vee b = 1$ and $a \vee c = 1$.

Then $b \geq a^{\perp}$ and $c \geq a^{\perp}$.

This imply that $b \wedge c \geq a^{\perp}$.

i.e; $a \vee (b \wedge c) = 1$.

Thus $a \vee b = 1$ and $a \vee c = 1$ imply that $a \vee (b \wedge c) = 1$ for all $a,b,c \in L$.

Hence L is 1-distributive. ■

3.2.8 Remark : Converse of Result 3.2.7 need not be true.

The lattice $\tilde{N}$ = dual lattice of $\langle N, \wedge, \vee \rangle$ (see Remark 2.2.8) is a 1-distributive lattice, which is not quasicomplemented.

Since a lattice L is quasicomplemented if and only if $\{a\}^{\perp} = \{x \in L / x \vee a = 1\}$ is a principal filter for each $a \in L$, a sufficient condition for a 1-distributive lattice to be quasicomplemented çan be stated as

3.2.9 Result : A 1-distributive lattice L is quasicomplemented if $\{a\}^{\perp}$ is a principal filter for every a in L.

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3.3 CHARACTERIZATIONS

3.3.1 Definition : Let L be a lattice with 1 and A be any nonempty subset of L. Then the set $A^{\perp}$ is defined as $A^{\perp} = \{x \in L / x \vee a = 1, \forall a \in A\}$.

Dualizing the results in Article 2.3 we get the following characterizations for a 1-distributive lattice.

3.3.2 Result : Let L be a lattice with 1. Then following are equivalent.

1) L is 1-distributive.
2) $\{a\}^{\perp}$ is a filter for every a in L.
3) $A^{\perp}$ is a filter for every nonempty subset A of L.
4) The set of all filters D(L) of L is pseudocomplemented.
5) D(L) is 0-distributive.
6) Every maximal ideal in L is prime.
7) $\cap \{F / F \in \mathcal{F}(L)\} = \{1\}$, where $\mathcal{F}(L)$ is the set of all prime filters in L.
8) For $a \neq 1$ there exists a prime filter in L not containing a.
9) For any ideal I disjoint with $\{a\}^{\perp}$, $(a \in L)$, there exists a prime ideal in L containing I and disjoint with $\{a\}^{\perp}$.

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