CHAPTER - 3

ON OPERATIONS ON FUZZY SETS

ABSTRACT : Two operations, m - union $\underset{m}{\cup}$, and m - intersection $\underset{m}{\cap}$ on the set of all fuzzy subsets of U (where m belongs to valuation set V) are definited in the last chapter. Now, to continue, a notion of m-membership of an element x in a fuzzy subset A is defined for every m ∈ V. Complement $\bar{A}$ of the fuzzy subset A : U ⟶ V is also defined. To see that these notations are appropriate, some propositions are proved which show how usual theoryk of (ordinary) sets can be replicated for each mm V. Further we see that these definitions proposed by Zengo are weaker that classical one (proposed by Zaden).

A definition of sub multiple Boolean algebra is given.

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INTRODUCTION :

According to classical definition due to Zadeh a fuzzy set in a universe U (or fuzzy subset of U) is a map $A : U \longrightarrow [0,1]$. Insted of $[0,1]$, Zenzo choses a finite valuation set.

$$V = \left\{ 0, \frac{1}{p-1}, \frac{2}{p-1}, \ldots, \frac{p-2}{p-1}, 1 \right\}$$

Zadeh defines, subset relation, union and intersection as follows : for any two fuzzy subsets , A and B of U.

$$A \subseteq B \quad \text{iff } A(x) \leq B(x) \quad \text{for all } x \in U.$$

$$(A \cup B)x = \max (A(x), B(x)) \quad \text{for all } x \in U$$

$$(A \cap B)x = \min (A(x), B(x)) \quad \text{for all } x \in U .$$

Complement $\bar{A}$ is defined by

$$\bar{A}(x) = 1 - A(x) \qquad \text{for all } x \in U .$$

By taking the finite valuation set V, Zenzo defines for every m V, the relations : m - memberships $\in_m$, $\overline{\in}_m$ m - inclusion $\subseteq_m$ and m - equality ' $=_m$ ' , additional to the already defined $\cup_m$ and $\cap_m$. In a sense, this is a reverse process, that is we try to build a model of ordinary set theory for every truth value m. The results proved show that the above defined relations, really ' behave ' like the corresponding operations in the ordinary set theory, for a fixed m.

Moreover, for every m ∈ V, Zadeh's inclusion implies m - inclusion and Zadeh's equality implies m- equaltiy. Like sub- Boolean algebra, sub - multiple Boolean algebra can be defined. But here also we see that we have to assume some relation between absorbing elements and identity elements to achieve the required accuracy.

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1. Relations on Fuzzy sets :

Let F be the set of all the fuzzy sets with unive se U and truth set V where

$$V = \left\{ 0, \quad \frac{1}{p-1} \quad \frac{2}{p-1}, \ldots, \frac{p-2}{p-1}, 0 \right\}$$

Definition 3.1. : for any $m \in V$, $x \in U$, $A \in F$, we write $x \in_m A$ iff $A(x) \geq m$.

Definition 3.2 : for any $m \in V$, $x \in U$, $A \in F$ we write $x \bar{\in}_m a$ iff $A(x) \leq 1-m$

We shall study the properties of the new operations.

Theorem 3.1 : $x \bar{\in}_m A$ iff $x \in_m \bar{A}$

Proof : Let $x \bar{\in}_m A$

$\Rightarrow \quad A(x) \leq 1-m$

$\Rightarrow \quad 1 - A(x) \geq m$

$\Rightarrow \quad \bar{A}(x) \geq m \qquad (\because \bar{A}(x) = 1-A(x))$

$\Rightarrow \quad x \in_m \bar{A}$

Similarly ' if' part can be proved.

Theorem 3.2 : $x \in_m A \cup_m B$ iff $x \in_m A$ or $x \in_m B$

Proof : Let $x \in_m A$

$\Rightarrow \quad A(x) \geq m.$

Now either $B(x) \geqslant m$ or $B(x) < m$

If $B(x) \geqslant m$ then.

$(A \underset{m}{\cup} B)\, x = \min(A(x), B(X))$ as $A(x)$, $B(x) \geqslant m$

i.e. $(A \underset{m}{\cup} B)\, x \geqslant m$.

$\Rightarrow x \underset{m}{\in} (A \underset{m}{\cup} B)$

Similarly $x \underset{m}{\in} B$ implies $x \underset{m}{\in} (A \underset{m}{\cup} B)$

Conversly, let $x \underset{m}{\in} (A \underset{m}{\cup} B)$

$\Rightarrow (A \underset{m}{\cup} B)\, x \geqslant m$ .... (1)

In view of definition of $A \underset{m}{\cup} B$, if $(A \underset{m}{\cup} B)x = \min(Ax, Bx)$

then $A(x), B(x) \geqslant m$.

i.e. $x \underset{m}{\in} A$ and $x \underset{m}{\in} B$

If $(A \underset{m}{\cup} B)\, x = \max(A(x), B(x))$ then by 1

either $A(x) \geqslant m$ or $B(x) \geqslant m$

i.e. either $x \underset{m}{\in} A$ or $x \underset{m}{\in} B$

This result is analogous to the result

$x \in (A \cup B) \Leftrightarrow x \in A$ or $x \in B$

in ordinary set theory.

<u>Theorem 3.3.</u> : $x \underset{m}{\in} (A \underset{m}{\cap} B)$ iff $x \underset{m}{\in} A$ and $x \underset{m}{\in} B$

Proof : (a) Assume that both $x \underset{m}{\in} A$ and $x \underset{m}{\in} B$ hold.

Then, $A(x) \geqslant m$ and $B(x) \geqslant m$. Hence

$(A \underset{m}{\cap} B)x = \max(A(x), B(x)) \geqslant m$

i.e. $x \underset{m}{\in} (A \underset{m}{\cap} B)$

(b) Let $x \underset{m}{\in} (A \underset{m}{\cap} B)$

$\Rightarrow \quad (A \underset{m}{\cap} B)x \geqslant m \quad \ldots \quad (2)$

We must prove that $A(x) \geqslant m$ and $B(x) \geqslant m$.
Let on contrary $A(x) < m$. Then either $B(x) \geqslant m$ or $B(x) < m$.

If $B(x) \geqslant m$,

$(A \underset{m}{\cap} B)x = \min(A(x), B(x)) < m$ a contrudiction and if $B(x) < m$ then,

$(A \underset{m}{\cap} B)x = \max(A(x), B(x)) < m$ again contrudicting to (2) i.e. we must have both $A(x), B(x) \geqslant m$

i.e. $x \underset{m}{\in} A$ and $x \underset{m}{\in} B$

<u>Definition 3.3.</u> : Let $A, B, \in F$ and $m \in V$. Then we write $A \underset{m}{\subseteq} B$ iff $\forall x \in U$, $x \underset{m}{\in} A \Rightarrow x \underset{m}{\in} B$

<u>Definition 3.4.</u> : For $A, B, \in F$, $m \in V$, we write $A \underset{m}{=} B$ iff $A \underset{m}{\subseteq} B$ and $B \underset{m}{\subseteq} A$

Then clearly $\underset{m}{\subseteq}$ is reflexive.

Theorem 3.4. : For every A, B, $\in$ F,

$$A \underset{m}{\subseteq} B \quad \& \quad B \underset{m}{\subseteq} C \Rightarrow A \underset{m}{\subseteq} C$$

i.e $\underset{m}{\subseteq}$ is transitive.

Proof : Let $A \underset{m}{\subseteq} B$ & $B \underset{m}{\subseteq} C$. Then for $x \in U$,

$x \underset{m}{\in} A \Rightarrow x \underset{m}{\in} B$ & $x \underset{m}{\in} B \Rightarrow x \underset{m}{\in} C$

Hence $x \underset{m}{\in} A \Rightarrow x \underset{m}{\in} C$

i.e. $A \underset{m}{\subseteq} C$ #

Theorem 3.5 : $A, B \underset{m}{\subseteq} (A \underset{m}{\cup} B)$ & $(A \underset{m}{\cap} B) \underset{m}{\subseteq} A, B$.

Proof : Let $x \underset{m}{\in} A$ Then by theorem 3.2,

$x \underset{m}{\in} (A \underset{m}{\cup} B)$

Hence $A \underset{m}{\subseteq} (A \underset{m}{\cup} B)$ Analogously $B \underset{m}{\subseteq} (A \underset{m}{\cup} B)$

Also if $x \underset{m}{\in} (A \underset{m}{\cap} B)$ then by theorem 3.3. $x \underset{m}{\in} A$ &

$x \underset{m}{\in} B$, i.e.

Hence $(A \underset{m}{\cap} B) \underset{m}{\subseteq} A$ and $(A \underset{m}{\cap} B) \underset{m}{\subseteq} B$. #

Theorem 3.6. : $A \underset{m}{\subseteq} C$ & $B \underset{m}{\subseteq} C \Rightarrow (A \underset{m}{\cup} B) \underset{m}{\subseteq} C$;

$$C \underset{m}{\subseteq} A \quad \& \quad C \underset{m}{\subseteq} B \Rightarrow_m C \underset{m}{\subseteq} (A \underset{m}{\cap} B)$$

Proof : Let $A \underset{m}{\subseteq} C$ and $B \underset{m}{\subseteq} C$

Then, $x \underset{m}{\in} (A \underset{m}{\cup} B)$

$x \underset{m}{\in} A$ or $x \underset{m}{\in} B$ by thm 3.2

or both

But since $A \underset{m}{\subseteq} C$ and $B \underset{m}{\subseteq} C$.

$x \underset{m}{\in} C$ Hence the conclusion.

Second part can be analogusly proved. #

Theorem 3.7 : $A \underset{m}{\subseteq} B$ iff $\bar{B} \underset{1-m}{\subseteq} \bar{A}$

<u>Proof</u> : We have $\bar{B} \underset{1-m}{\subset} \bar{A}$

$$\Leftrightarrow \quad x \underset{1-m}{\in} \bar{B} \quad \Rightarrow \quad x \underset{1-m}{\in} \bar{A}$$

$$\Leftrightarrow \quad \bar{B}(x) \geqslant 1-m \quad \Rightarrow \quad \bar{A}(x) \geqslant 1-m$$

$$\Leftrightarrow \quad 1-B(x) \geqslant 1-m \quad \Rightarrow \quad 1-A(x) \geqslant 1-m$$

$$\Leftrightarrow \quad Bx \leqslant m \quad \Rightarrow \quad A(x) \leqslant m \Leftrightarrow A(x) > m \Rightarrow B(x) > m$$

$$\Leftrightarrow \quad x \underset{m}{\in} A \quad \Rightarrow \quad x \underset{m}{\in} B \qquad \Leftrightarrow A \underset{m}{\subseteq} B$$

Hence proof #

Thus a full theory of m - membership can be constructed for each grade m.

Further Zenzo clearifies that Zadeh's definitions of inclusion and equality of two fuzzy sets are connected with two-valued logic only. The definition of m - inclusion and m - equality given by him are weaker than those of Zadeh, as can be seen in the following theorem. Here the notation $\subseteq_z$ indicates zadeh's inclusion and a ' $=_z$ ' zadeh's equality.

<u>Theorem 3.8</u> : Zadeh's inclusion implies m - inclusion and zadeh's equality implies m-equality.

i.e. $A \subseteq_z B \Rightarrow A \subseteq_m B$ for any $m \in V$

and $A =_z B \Rightarrow A =_m B$ for any $m \in V$

<u>Proof</u> : Let $A \subseteq_z B$

$\Rightarrow A(x) \leqslant B(x) \quad \forall x \in U.$

Now let $x \in_m A$ Then $A(x) \geqslant m$. But as $A(x) \geqslant B(x)$, this implies $B(x) \geqslant m$. Thus $x \in_m B$

Thus $x \in_m A \quad x \in_m B$

Hence $A \subseteq_m B \quad (m \in V)$

Now let $A =_z B$

According to zadeh's definition this implies $A(x) = B(x)$ for all x Now we have to prove that $A \subseteq_m B$ & $B \subseteq_m A$ for every $m \in V$

Let $x \in_m A \Rightarrow A(x) \geqslant m$

$\Rightarrow B(x) \geqslant m \quad (\text{as } A(x) = B(x))$

$= x \underset{m}{\in} B$

Thus $A \underset{m}{\subset} B$ Similarly $B \underset{m}{\subset} A$ can be proved.

Hence $A \underset{m}{=} B$

Remark : Converse of above theorem is not true. That is, if cardinality of V is greater than 2, for no m does m - equality reduce to zadeh's equality, or m - inclusion reduce to ~~zadeh~~ zadeh's inclusion. Thus Zenzo's definitions are more 'general '.

2. SUB MULTIPLE BOOLEAN ALGEBRA :

In any new concept, the sub structure play an important role. One reason behind this may be that they 'keep' the ' identity ' of the original structure. Here we have tried to define the sub-multiple Boolean algebra in the usual manner, analogous to sub-Boolean algebra.

Definition 3.5. : Let E be a multiple Boolean algebra of order p with u as the fundamental isomorphism, $\underline{0}, \underline{1}, \ldots, \underline{p-1}$ as the operations, $e_0, e_1, \ldots, e_{p-1}$ identities. A non empty subset E_0 of E is said to be a sub multiple Boolean algebra of E provide Eo is closed under each operation $\underline{m}$ and the bijection u, and it preserves identities in E. That is, if

(a) For all $x, y \in$ Eo, $x \, \underline{m} \, y \in$ Eo
for each $m = 0, 1, 2, \ldots, p-1$

(b) For all $x \in$ Eo, $u(x) \in$ Eo.

(c) $e_m \in$ Eo for each m.

Thus each sub multiple Boolean algebra Eo of E is also a multiple Boolean algebra under the operatio-ns of E and the isomorphism u restricted to Eo.

Remark 1 : Each sub algebra contains all the absorbing elements of E. For, let $x \in$ Eo then

$x \, \underline{m} \, u(x) \, \underline{m} \, u^2(x) \ldots \underline{m} \, u^{p-1}(x)$ Eo for each m.

$= a_m \in$ Eo for each m. (by MBA 9).

Hence if the set of identities and that of absorbing elements are equal (as in Boolean algebra), then every sub multiple Boolean algebra will automaticaly preserve the identities and hence, then the condition (c) in the definition (3.5) may be deleted. In this respect our definition of multiple Boolean algebra viz def. 1.3 seems more justified in which $a_m = e_{m+1}$ is assumed.

Remark 2 : Also the condition (a) can be reduced for ' any one m ' , because if

$x \, \underline{m} \, y \in$ Eo for some m and for all $x, y \in$ Eo then $x \ \underline{m+1} \ y = u \left(u^{p-1}(x) \ \underline{m} \ u^{p-1}(y) \right)$

as $u^{p}(x) = x$; and as $u^{p-1}(x)$ and $u^{p-1}(y)$ are members of Eo (by b) we have,

$$\left(u^{p-1}(x) \ \text{M} \ \underline{m} \ u^{p-1}(y) \right) \in Eo$$

and hence $u \left(u^{p-1}(x) \ \underline{m} \ u^{p-1}(y) \right) \in Eo$

Thus , $x \ \underline{m+1} \ y \in Eo$ for all $x, y \in Eo$
