CHAPTER II

C H A P T E R - 2

Source For Electromagnetic Fields

§ 1. Introduction

The utility of the NP spin coefficient formalism is profusely demonstrated in electromagnetic field theory by Debney and Zund (1971, 1972 -a,b) Zund (1973, 1974, 1977). Tariq and Tupper (1976, 1977), Wallace and Zund (1979). The novelty of this formalism can be realized through thorough exploitation of Bianchi Identities.

In this chapter by exploiting the eleven Bianchi identities we have characterized the source term for non-null and null electromagnetic field of Ruse-Synge classification when interacting with different types of free-gravitational field. It is shown, for the non-null electromagnetic field interacting with the free gravitational field of Petrov-type N, that the current vector J^a is necessarily a zero vector. It is proved to be space-like or null accordingly when purely real ($\text{Im}\ \phi_1 = 0$) or purely imaginary ($\text{Re}\ \phi_1 = 0$) non-null electromagnetic field interacts with the Petrov-type III field. It is also observed that when purely real non-null field interacts with the Petrov type D field then the current vector is either space-like or zero vector, however, it is proved to be time-like when purely imaginary non-null electromagnetic field interacts with Petrov type D gravitational field. In case of null electromagnetic field of type B ($\phi_2 \neq 0$) coupled with various types of free-gravitational fields (except N, Ψ_4 (l^a) $\neq 0$) it is computed that the current vector J^a is necessarily null, while for N N field (null electromagnetic field and null free-gravitational field with the common propagation vector l^a) it is space-like. The results obtained in this chapter are summarized in the following table -

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Gravitational fields → / Electromagnetic fields ↓		$\Psi_0 \neq 0$	$\Psi_1 \neq 0$	$\Psi_2 \neq 0$	$\Psi_3 \neq 0$	$\Psi_4 \neq 0$
Non-null $\phi_1 \neq 0$	$\mathrm{Im}\ \phi_1 = 0$	$J^a = 0$	$J_a J^a < 0$	$J_a J^a < 0$ or $J^a = 0$	$J_a J^a < 0$	$J^a = 0$
	$\mathrm{Re}\ \phi_1 = 0$	$J^a = 0$	$J_a J^a = 0$	$J_a J^a > 0$	$J_a J^a = 0$	$J^a = 0$
Null $\phi_2 \neq 0$		$J_a J^a = 0$	$J_a J^a = 0$	$J_a J^a = 0$	$J_a J^a = 0$	$J_a J^a < 0$
$\phi_o \neq 0$		$J_a J^a < 0$	$J_a J^a = 0$	$J_a J^a = 0$	$J_a J^a = 0$	$J_a J^a = 0$

Here J^a is the current vector with conventions

(i) $J^a = 0 \Longrightarrow J^a$ is zero vector

(ii) $J_a J^a = 0 \Longrightarrow J^a$ is null vector

(iii) $J_a J^a < 0 \Longrightarrow J^a$ is space-like

(iv) $J_a J^a > 0 \Longrightarrow J^a$ is time like

§ 2. Electromagnetic Field Equations :

The stress energy momentum tensor T^{ab} of electromagnetic field satisfies the equation

$$T^{ab}_{;b} = F^{a}{}_{b}\, J^{b} \qquad \ldots (2.1)$$

Here the conservation of stress energy momentum tensor

$$T^{ab}{}_{;b} = 0 \Longrightarrow F^{a}{}_{b}\, J^{b} = 0\,, \qquad \ldots (2.2)$$

where the electromagnetic bivector F_{ab} and the stress energy momentum tensor T_{ab} of electromagnetic field are defined in (9.10) and (9.15) of Chapter 1 respectively as:

$$F_{ab} = -2\, R_e\, \phi_1\, l_{[a}n_{b]} + 2i\, \mathrm{Im}\, \phi_1\, m_{[a}\bar{m}_{b]} + \phi_2\, l_{[a}m_{b]} + + \bar{\phi}_2\, l_{[a}\bar{m}_{b]} - \bar{\phi}_o\, n_{[a}m_{b]} - \phi_o\, n_{[a}\bar{m}_{b]} \qquad \ldots (2.3)$$

$$T_{ab} = \frac{1}{2}\left\{\phi_{22}\, l_a l_b + \phi_{oo}\, n_a n_b + \phi_{20}\, m_a m_b + \phi_{02}\, \bar{m}_a \bar{m}_b\right\} + + \phi_{11}\left[l_{(a}n_{b)} + m_{(a}\bar{m}_{b)}\right] - \phi_{21}\, l_{(a}m_{b)} - - \phi_{12}\, l_{(a}\bar{m}_{b)} - \phi_{10}\, n_{(a}m_{b)} - \phi_{01}\, n_{(a}\bar{m}_{b)}. \qquad \ldots (2.4)$$

The source term J^a is given by Debney and Zund (1971) as

$$J^a = I_o\, n^a + I_2\, l^a - \bar{I}_1\, m^a - I_1\, \bar{m}^a \qquad \ldots (2.5)$$

Here I_o, I_1, I_2 are the source scalars and the magnitude of current vector is

$$J_a J^a = 2(I_o I_2 - I_1 \bar{I}_1) \qquad \ldots (2.6)$$

The scalar form of Maxwell equations

$$F_{[ab;c]} = 0$$

and $$F^{ab}{}_{;b} = \frac{1}{2} J^a$$

can be worked out in following equations

$$D\phi_1 - \bar{\delta}\phi_o = (\pi - 2\alpha)\phi_o + 2\rho\phi_1 - k\phi_2 + I_o,$$

$$\delta\phi_2 - \Delta\phi_1 = -\nu\phi_o + 2\mu\phi_1 + (\tau - 2\beta)\phi_2 + I_2,$$

$$D\phi_2 - \bar{\delta}\phi_1 = -\lambda\phi_o + 2\pi\phi_1 + (\rho - 2\epsilon)\phi_2 - \bar{I}_1,$$

$$\delta\phi_1 - \Delta\phi_o = (\mu - 2\nu)\phi_o + 2\tau\phi_1 - \sigma\phi_2 - I_1. \qquad \ldots (2.7)$$

§ 3. Non-null Electromagnetic field Interacting with free-gravitational field :

The nature of the source J^a of electromagnetic field is examined under the choice of particular fields.

Theorem 1 : For the non-null electromagnetic when coupled with the Petrov-type N gravitational field, the current vector J^a is necessarily a zero vector.

Proof : We consider the amalgamation of the Petrov-type N gravitational field characterized by $\Psi_4\ (l^a) \neq 0$ or $\Psi_o\ (n^a) \neq 0$ with propagation vector l^a and n^a respectively and non-null electromagnetic field characterized by $\phi_1 \neq 0$. For such fields the Bianchi identities are respectively (vide, appendix A)

$$\rho = \mu = \sigma = \tau = k = 0, \qquad \ldots (3.1)$$

and

$$\rho = \mu = \lambda = \tau = \pi = \nu = 0. \qquad \ldots (3.2)$$

The Maxwell equations (2.7) for the non-null electromagnetic field provide

$$D\phi_1 = 2\rho\phi_1 + I_o ,$$

$$\bar{\delta}\phi_1 = -2\pi\phi_1 + \bar{I}_1 ,$$

$$\delta\phi_1 = 2\tau\phi_1 - I_1 ,$$

$$\Delta\phi_1 = -2\mu\phi_1 - I_2. \qquad \ldots (3.3)$$

Consequently, the energy balance equations $T^{ab}{}_{;b} = 0$ for the non-null fields yield

$$R_e\, \phi_1\, I_o = 0\,,$$

$$R_e\, \phi_1\, I_2 = 0\,,$$

$$Im\, \phi_1\, I_1 = 0,$$

$$Im\, \phi_1\, \bar{I}_1 = 0\,. \qquad \ldots (3.4)$$

(i) If the Maxwell scalar ϕ_1 is complex then we have Re $\phi_1 \neq 0$, Im $\phi_1 \neq 0$ and hence from (3.4) we obtain

$$I_o = I_1 = I_2 = 0. \qquad \ldots (3.5)$$

This shows from (2.5) that the current vector $J^a = 0$. This means that J^a is zero vector field.

(ii) If ϕ_1 is purely real, we have $R_e\, \phi_1 \neq 0$ and Im $\phi_1 = 0$. In this case equation (3.4) yields

$$I_o = 0, \quad I_2 = 0, \quad I_1 \neq 0 \qquad \ldots (3.6)$$

The value of the component I_1 is obtained from

$$F^{ab}{}_{;b} = \frac{1}{2}\, J^a \qquad \ldots (3.7)$$

as $\quad I_1 = -\, 2(\tau + \bar{\pi})\, \phi \qquad \ldots (3.8)$

For non-vanishing current J^a, we must have $(\tau + \bar{\pi}) \neq 0$. But by using the set of Bianchi identities (3.1) or (3.2) we obtain $I_1 = 0$. Hence (3.6) implies $I_o = I_1 = I_2 = 0$. This proves that J^a is a zero vector.

(iii) If the Maxwell scalar ϕ_1 is purely imaginary we have $R_e\, \phi_1 = 0$ and $I_m\, \phi_1 \neq 0$, then for such case equations (3.4) give,

$$I_1 = 0 \text{ and } I_o \neq 0,\ I_2 \neq 0 \qquad \dots (3.9)$$

The nonvanishing components I_o and I_2 of J^a are obtained from equation (3.7) as

$$I_o = -2(\rho - \bar{\rho})\,\phi_1$$

$$I_2 = -2\,(\mu - \bar{\mu})\,\phi_1 \qquad \dots (3.10)$$

By the use of Bianchi identities (3.1) or (3.2) we see that $I_o = 0$ and $I_2 = 0$.
Hence this with (3.9) shows that the current vector J^a is zero vector.
Here the proof of the theorem is completed.

By using the same process we study the characterization of the source term for the non-null electromagnetic field interacting with Petrov-type III and D fields. The results obtained are listed below :

I) If the two fields are such that $\Psi_1 \neq 0$, $\phi_1 \neq 0$; then Bianchi identities (vide, A 2) imply

$$\mu = \lambda = \pi = \gamma = 0 \qquad \dots (3.11)$$

Case i) When $R_e\, \phi_1 \neq 0$ and $I_m\, \phi_1 = 0$.

This yields from the equations (3.4) and (3.7) that

$$I_o = 0, \quad I_2 = 0 \text{ and } I_1 = -\tau\, \phi_1 \qquad \ldots (3.12)$$

We note that as $\tau \neq 0$ the component $I_1 \neq 0$ and hence $J^a = \bar{\tau}\, \bar{\phi}_1\, m^a + \tau\, \phi_1\, \bar{m}^a$. It follows therefore $J_a\, J^a < 0$.

Proving that the current vector J^a is space-like.

Case ii) When $I_m\, \phi_1 \neq 0$ and $R_e\, \phi_1 = 0$

For this case the equations (3.4) and (3.7) provide

$$I_1 = I_2 = 0 \text{ and}$$

$$I_o = -2\,(\rho - \bar{\rho})\,\phi_1 \qquad \ldots (3.13)$$

It follows from the equation (2.5) therefore that $J^a = -2(\rho - \bar{\rho})\phi_1\, n^a$.

Hence the current vector is a null vector.

II) When the fields are such that $\Psi_2 \neq 0$ and $\phi_1 \neq 0$

If we use these conditions in the set of Bianchi identities, we get (vide, appendix A)

$$\tau = \pi = k = \nu = 0,$$

$$\rho + \bar{\rho} = \mu + \bar{\mu} = 0. \qquad \ldots (3.14\ a)$$

or $\sigma = \lambda = k = \nu = 0.$ ... (3.14 b)

Discussion :

In the case i) ($R_e\ \phi_1 \neq 0$ and $I_m\ \phi_1 = 0$) we have $I_o = 0$, $I_2 = 0$ and I_1 is either zero (due to 3.14a) or $-2(\tau + \bar{\pi})\ \phi_1$ (due to 3.14 b). Accordingly it follows from (2.6) that the current vector J^a is either a zero vector or space like vector.

In the case ii) ($R_e\ \phi_1 = 0$ and $I_m\ \phi_1 \neq 0$) we have $I_1 = 0$ and $I_o = -2(\rho - \bar{\rho})\ \phi_1 \neq 0$, $I_2 = -2(\mu - \bar{\mu})\ \phi_1 \neq 0$.

Consequently, $J^a = -2(\rho - \bar{\rho})\phi_1\ n^a - 2(\mu - \bar{\mu})\ \phi_1\ l^a$.

This shows that $J_a\ J^a > 0$ and consequently J^a is time like vector field.

III) When the two fields are such that $\Psi_3 \neq 0$ and $\phi_1 \neq 0$

The Bianchi identities are,

$\rho = \sigma = \tau = k = 0.$ ... (3.15)

In the case i) ($R_e\ \phi_1 \neq 0$ and $I_m\ \phi_1 = 0$) we have

$I_o = 0$, $I_2 = 0$ and $I_1 = -\pi\ \phi_1$ (due to 3.15)

This gives from (2.5) that $J^a = \bar{\pi}\ \phi_1\ m^a + \pi\ \phi_1\ \bar{m}^a$, and hence $J_a\ J^a < 0$. This proves that J^a is a space like vector.

In the case ii) ($R_e\ \phi_1 = 0$ and $I_m\ \phi_1 \neq 0$) we get

$$I_1 = 0, \quad I_o = 0, \quad I_2 = -2(\mu - \bar{\mu})\, \phi_1.$$

This shows that J^a is null vector since $J_a J^a = 0$, where $J^a = -2(\mu - \bar{\mu})\, \phi_1 l^a$.

§ 4. Null electromagnetic field coupled with free-gravitational field.

In this section, we prove that the current vector field J^a is necessarily null $(J_a J^a = 0)$ for the null electromagnetic field with propagation vector l^a when interacting with different types of free gravitational field except for type N field with propogation vector l^a. If the propagation vector for both the null fields is same, the source term for such fields is shown to be space-like.

Theorem 2 : If the null electromagnetic field $(\phi_2 \neq 0)$, interacts with Petrov-type D free-gravitational field $(\Psi_2 \neq 0)$, the source term for such fields is null.

Proof : We recall the Bianchi identities for fields under consideration are (*vide* appendix B)

$$\sigma = \lambda = k = 0,$$

$$\bar{\delta}\, \phi_{22} = 3 \nu\, \Psi_2 + (\bar{\tau} - 2\bar{\beta} - 2\alpha)\, \phi_{22} \qquad \ldots (4.1)$$

The Maxwell equations (2.7) for the null electromagnetic field direct that

$$D\phi_2 = (\rho - 2\epsilon)\phi_2 - \bar{I}_1 .$$

$$k\phi_2 = I_o ,$$

$$\delta\phi_2 = (\tau - 2\beta)\phi_2 + I_2,$$

$$\sigma\phi_2 + I_1 = 0. \qquad \ldots (4.2)$$

The mixing of equations (4.1) and (4.2) give

$$I_o = 0, \quad I_1 = 0 \text{ and}$$

$$I_2 = \phi_2^{-1}\left[\, 3\nu\,\phi_2 - 2\alpha\,\phi_{22} - \bar{\phi}_2\,\bar{\delta}\,\phi_2 \,\right] \qquad \ldots (4.3)$$

For any choice of I_2, we have from (2.6) that $J_a J^a = 0$.

Hence the source for null field coupled with type D field is null. Here the proof is complete.

Note : In the case, when both the null fields ($\phi_2 \neq 0$, $\Psi_4 \neq 0$) have the common propagation vector l^a, the Bianchi identities (B 5) give

$$k = 0, \quad \rho\,\phi_{22} = \sigma\,\Psi_4,$$

$$\delta\,\Psi_4 - \bar{\delta}\,\phi_{22} = (\tau - 4\beta)\,\Psi_4 - (\bar{\tau} - 2\alpha - 2\bar{\beta})\,\phi_{22},$$

$$D\,\Psi_4 = \bar{\sigma}\,\phi_{22} - (4\,\epsilon - \rho\,)\,\Psi_4. \qquad \ldots (4.4)$$

By using (4.2) and (4.4) we then obtain

$$I_o = 0, \quad I_1\,\phi_2 + \bar{I}_1\,\bar{\phi}_2 = 0,$$

and $I_2 = \phi_2^{-1} [\delta \Psi_4 - \bar{\phi}_2 \bar{\delta} \phi_2 - 2 \alpha \phi_{22} -$

$$- (\bar{\tau} - 4\beta) \Psi_4] \qquad \ldots (4.5)$$

Consequently

$$J^a = \phi_2^{-1} (\delta \Psi_4 - \bar{\delta} \phi_2 \bar{\phi}_2 - (\bar{\tau} - 4\beta) \Psi_4 - 2\alpha\phi_{22}) l^a.$$

We observe that $J_a J^a < 0$ and hence the source term for such fields is space-like.

The nature of the source term for other fields can be studied in similar fashion, and the results are discussed below :

1) <u>For the field $\phi_2 \neq 0$ and $\Psi_o \neq 0$</u>;

The Bianchi identities for these fields are (<u>vide</u> appendix B),

$$\rho = \sigma = \lambda = k = \nu = 0,$$

$$D \phi_{22} = 0,$$

$$\delta \phi_{22} = (\bar{\tau} - 2\alpha - 2\bar{\beta}) \phi_{22} \qquad \ldots (4.6)$$

On combining (4.2) and (4.6), we have

$$I_o = 0, \qquad I_1 = 0,$$

$$I_2 = - (2\alpha\phi_{22} + \bar{\phi}_2 \bar{\delta} \phi_2) \qquad \ldots (4.7)$$

Hence it follows from (2.5) that $J^a = - (2\alpha \phi_{22} + \bar{\phi}_2 \bar{\delta}\phi_2) l^a$, and consequently $J_a J^a = 0$.

This shows that the current vector is a null.

2) <u>For the field $\phi_2 \neq 0$ and $\Psi_1 \neq 0$;</u>

We have the Bianchi identities (*vide,* B 2)

$$\sigma = k = \lambda = 0,$$

$$D\,\phi_{22} = (\rho + \bar{\rho})\phi_{22},$$

$$\bar{\phi}_2\,\bar{\delta}\,\phi_2 = -\,2\,\alpha\,\phi_{22} \qquad \ldots (4.8)$$

We derive from (4.2) and (4.8) that

$$I_o = I_1 = 0,$$

and $\quad I_2 = \delta\phi_2 - (\tau - 2\beta)\phi_2.$

Consequently, the equation (2.6) yields

$$J_a J^a = 0,$$

where $J^a = (\,\delta\phi_2 - (\tau - 2\beta)\phi_2\,)1^a$.

This proves J^a is a null vector.

3) <u>For the field $\phi_2 \neq 0$ and $\Psi_3 \neq 0$:</u>

We deduce from (2.6) and the Bianchi identities given in the appendix B (B 4) that $J_a J^a = 0$. Hence the current vector is a null.

Results of the similar type are also hold good for the null electromagnetic field $\phi_0 \neq 0$ when interacts with various types of free-gravitational field. These results are summarised in the table cited above.

Appendix A

When Petrov-type classification of free gravitational field coupled with source free non-null (and with null electromagnetic field in appendix B and appendix C) electromagnetic field, the simplified forms of the eleven Bianchi identities are found by Katkar (1989) and are cited below for our ready references. While simpli-- fying the eleven Bianchi identities (7.7) it was assumed without loss of generality that the vector field l^a is geodesic ($l_{a;b}\, l^b = 0$) i.e. $k = \epsilon + \bar{\epsilon} = 0$.

1) $\Psi_o \neq 0, \quad \phi_1 \neq 0,$

$$\rho = \mu = \lambda = \tau = \pi = \nu = 0,$$

$$\bar{\delta}\, \Psi_o = 4\alpha\, \Psi_o,$$

$$\Delta\, \Psi_o = 4\, \gamma\, \Psi_o + 2\, \sigma\, \phi_{11}\, ,$$

$$D\phi_{11} = \Delta\phi_{11} = \delta\, \phi_{11} = 0. \qquad \text{... (A 1)}$$

2) $\Psi_1 \neq 0, \quad \phi_1 \neq 0$

$$\mu = \lambda = \pi = \nu = 0,$$

$$D\, \Psi_1 = 2\, (2\, \rho + \epsilon)\, \Psi_1\, ,$$

$$\delta\, \Psi_1 = 2(2\tau + \beta)\, \Psi_1 - 2\, \sigma\, \phi_{11},$$

$$\bar{\delta}\, \Psi_1 = 2\alpha\, \Psi_1 - 2\, \rho\, \phi_{11},$$

$$\Delta \Psi_1 = 2 \gamma \Psi_1 + 2 \tau \phi_{11},$$

$$D\phi_{11} = 2 (\rho + \bar{\rho}) \phi_{11},$$

$$\delta\phi_{11} = 2 \tau \phi_{11},$$

$$\Delta\phi_{11} = 0. \qquad \ldots \text{(A 2)}$$

3) $\Psi_2 \neq 0, \quad \phi_1 \neq 0.$

$$k = \nu = \tau = \pi = 0,$$

$$\rho + \bar{\rho} = \mu + \bar{\mu} = 0,$$

$$\Psi_2 = - \frac{2}{3} \phi_{11},$$

$$D \phi_{11} = D \phi_{11} = \delta \phi_{11} = 0.$$

or,

$$\sigma = \lambda = \nu = k = 0,$$

$$D \Psi_2 = \rho (3 \Psi_2 + 2\phi_{11}),$$

$$\bar{\delta} \Psi_2 = - \pi (3 \Psi_2 - 2 \phi_{11}),$$

$$\delta \Psi_2 = \tau (3 \Psi_2 - 2 \phi_{11}),$$

$$\Delta \Psi_2 = - \mu(3 \Psi_2 + 2 \phi_{11}),$$

$$D \phi_{11} = 2 (\rho + \bar{\rho}) \phi_{11},$$

$$\delta\phi_{11} = -2 (\bar{\pi} - \tau) \phi_{11},$$

$$\Delta \phi_{11} = - 2(\mu + \bar{\mu}) \phi_{11}. \qquad \ldots \text{(A 3)}$$

4) $\Psi_3 \neq 0, \ \phi_1 \neq 0.$

$$\rho = \sigma = k = \tau = 0,$$

$$D\Psi_3 + 2 \epsilon \Psi_3 + 2\pi \phi_{11} = 0,$$

$$\bar{\delta} \Psi_3 + 2(2\pi + \alpha) \Psi_3 - 2 \lambda \phi_{11} = 0,$$

$$\delta \Psi_3 + 2 \beta \Psi_3 - 2\mu \phi_{11} = 0,$$

$$\Delta \Psi_3 + 2 \gamma \Psi_3 - 2 \nu \phi_{11} = 0,$$

$$\Delta \phi_{11} = - 2(\mu + \bar{\mu}) \phi_{11},$$

$$\delta \phi_{11} = - \bar{\pi} \phi_{11} ,$$

$$D \phi_{11} = 0. \qquad \text{... (A 4)}$$

5) $\Psi_4 \neq 0, \ \phi_1 \neq 0.$

$$\rho = \sigma = k = \tau = \mu = \pi = 0,$$

$$D \Psi_4 + 4 \epsilon \Psi_4 + 2 \lambda \phi_{11} = 0,$$

$$\delta \Psi_4 + 4 \beta \Psi_4 - 2 \nu \phi_{11} = 0,$$

$$D \phi_{11} = 0, \quad \Delta \phi_{11} = 0, \quad \delta\phi_{11} = 0. \qquad \text{... (A 5)}$$

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Appendix B

When null electromagnetic field ($\phi_2 \neq 0$) interacts with various types of free-gravitational fields the Bianchi identities are

1) $\Psi_o \neq 0, \quad \phi_2 \neq 0.$

$$\rho = \lambda = k = \sigma = \nu = 0,$$

$$\Delta \Psi_o = (4\gamma - \mu)\, \Psi_o,$$

$$\bar{\delta}\, \phi_{22} = (\bar{\tau} - 2\alpha - 2\bar{\beta})\, \phi_{22},$$

$$\bar{\delta}\, \Psi_o = (4\alpha - \pi)\, \Psi_o \, . \qquad \text{... (B 1)}$$

2) $\Psi_1 \neq 0, \quad \phi_2 \neq 0.$

$$\sigma = k = \lambda = 0,$$

$$D\, \Psi_1 = 2(2\rho + \epsilon)\, \Psi_1,$$

$$\bar{\delta}\, \Psi_1 + 2(\pi - \alpha)\, \Psi_1 = 0,$$

$$\delta\, \Psi_1 = 2(2\tau + \beta)\, \Psi_1,$$

$$\Delta \Psi_1 = 2(\gamma - \mu)\, \Psi_1 \, ,$$

$$2\nu\, \Psi_1 = \rho\, \phi_{22},$$

$$\bar{\delta}\, \phi_{22} = (\bar{\tau} - 2\alpha - 2\bar{\beta})\, \phi_{22}. \qquad \text{... (B 2)}$$

3) $\Psi_2 \neq 0, \quad \Phi_2 \neq 0.$

$$k = \sigma = \lambda = 0,$$

$$D \Psi_2 = 3 \rho \Psi_2,$$

$$\bar{\delta} \Psi_2 = - 3 \pi \Psi_2,$$

$$\delta \Psi_2 = 3 \tau U_2,$$

$$\bar{\delta} \Phi_{22} = (\bar{\tau} - 2\alpha - 2\bar{\beta}) \Phi_{22} + 3 \nu U_2,$$

$$3\mu \Psi_2 + \rho \Phi_{22} = 0. \qquad \ldots (B\ 3)$$

4) $\Psi_3 \neq 0, \quad \Phi_2 \neq 0.$

$$\sigma = k = 0,$$

$$D \Psi_3 + 2 (\epsilon - \rho) \Psi_3 = 0,$$

$$\bar{\delta} \Psi_3 + 2(\pi + \alpha) \Psi_3 = 0,$$

$$\delta \Psi_3 = \rho \Phi_{22} - 2 (\beta - \tau) \Psi_3,$$

$$\Delta \Psi_3 = (\bar{\tau} - 2\alpha - 2\bar{\beta}) \Phi_{22} - \bar{\delta} \Phi_{22} -$$

$$- 2(\gamma + 2\mu) \Psi_3 \qquad \ldots (B\ 4)$$

5) $\Psi_4 \neq 0, \quad \Phi_2 \neq 0.$

$$k = 0, \quad \sigma U_4 = \rho \Phi_{22}$$

$$D \Psi_4 = \bar{\sigma} \phi_{22} - (4 \epsilon - \rho) \Psi_4.$$

$$\delta \Psi_4 = (\tau - 4\beta) \Psi_4 + \bar{\delta} \phi_{22} +$$

$$+ (\bar{\tau} - 2\alpha - 2\bar{\beta}) \phi_{22}$$

or

$$\delta \Psi_4 + (4\beta - \tau) \Psi_4 = \bar{\phi} \bar{\delta} \phi_2 + 2\alpha \phi_{22} \ldots. \quad (B\ 5)$$

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Appendix C

Bianchi identities for electromagnetic field ($\phi_o \neq 0$) when coupled with various types of free-gravitational fields

1) $\Psi_o \neq 0, \quad \phi_o \neq 0.$

$$\nu = 0,$$

$$\bar{\delta}\Psi_o - \delta\,\phi_{oo} = (4\alpha - \pi)\,\Psi_o + (\bar{\pi} - 2\bar{\alpha} - 2\beta)\phi_{oo},$$

$$\Delta\,\Psi_o = (4\gamma - \mu)\,\Psi_o - \bar{\lambda}\,\phi_{oo},$$

$$\lambda\,\Psi_o = \mu\,\phi_{oo}\,. \qquad \ldots (C\ 1)$$

2) $\Psi_1 \neq 0, \quad \phi_o \neq 0.$

$$\lambda = \nu = 0,$$

$$D\,\Psi_1 = 2(\,\epsilon + 2\,\rho\,)\,\Psi_1 - (\bar{\pi} - 2\bar{\alpha} - 2\bar{\beta})\,\phi_{oo} + \delta\,\phi_{oo},$$

$$\bar{\delta}\,\Psi_1 + 2(\,\pi - \alpha)\,\Psi_1 + \mu\,\phi_{oo} = 0,$$

$$\Delta\,\Psi_1 = 2(\,\gamma - \mu)\,\Psi_1$$

$$\delta\,\Psi_1 = 2(2\tau + \beta)\,\Psi_1 \qquad \ldots (C\ 2)$$

3) $\Psi_2 \neq 0, \quad \phi_0 \neq 0.$

$$\sigma = \lambda = \nu = 0,$$

$$3k\, \Psi_2 + \phi_{00}\, (\bar{\pi} - 2\bar{\alpha} - 2\beta) + \delta\, \phi_{00} = 0,$$

$$D\, \Psi_2 + 3\, \rho\, \Psi_2 + \mu\, \phi_{00} = 0,$$

$$\bar{\delta}\, \Psi_2 + 3\, \pi\, \Psi_2 = 0,$$

$$\Delta\, \Psi_2 + 3\mu\, \Psi_2 = 0,$$

$$\delta\, \Psi_2 = 3\tau\, U_2, \qquad \ldots \text{(C 3)}$$

4) $\Psi_3 \neq 0, \quad \phi_0 \neq 0.$

$$\sigma = \lambda = \nu = 0,$$

$$\delta\, \phi_{00} + (\bar{\tau} - 2\bar{\alpha} - 2\beta)\, \phi_{00} = 0,$$

$$\mu\, \phi_{00} - 2k\, \Psi_3 = 0,$$

$$D\, \Psi_3 + 2(\epsilon - \rho)\, \Psi_3 = 0,$$

$$\bar{\delta}\, \Psi_3 + 2(2\pi + \alpha)\, \Psi_3 = 0,$$

$$\delta\, \Psi_3 + 2(\beta - \tau)\, \Psi_3 = 0,$$

$$\Delta\, \Psi_3 + 2(\gamma + 2\mu)\, \Psi_3 = 0. \qquad \ldots \text{(C 4)}$$

5) $\Psi_4 \neq 0, \quad \phi_0 \neq 0$

$$\sigma = \lambda = k = \nu = \mu = 0,$$

$$\delta\, \phi_{00} + (\bar{\pi} - 2\bar{\alpha} - 2\beta)\, \phi_{00} = 0,$$

$$D\, \Psi_4 + (4\, \epsilon - \rho)\, \Psi_4 = 0,$$

$$\delta\, \Psi_4 = (\tau - 4\beta)\, \Psi_4. \qquad \ldots \text{(C 5)}$$

...