CHAPTER – 3

INVERSION FORMULAE

The "real" and the "complex" inversion formulae are given in this chapter.

3.1 Real Inversion Formulae :-

The real inversion formulae for the generalized function Stieltjes transformation were given by Pandey [14], Pathak [15] & Erdelyi [4].

We give here the real inversion formula for the T_{r+1} transformation. We shall use the following differential operators from [4]. From section 1.4, relation (1.4.2) with $P = 1+n$, $q = 1+n$ & $r+1+k$ instead of ρ.

$$\angle_{n,r,k,s} = \frac{(-1)^{n+1}\Gamma(r+1+k)}{(n+1)!\Gamma(n+r+k+1)} D^{n+1} x^{2n+r+k+2} D^{n+1}, n \in IN_0$$

where D is the ordinary derivative.

The operators $\angle_{n,r,k,x}, n \in N_0$ are formally self – adjoint.

Also, we shall use modified operators.

$$L_{n,r,k,x} = \frac{(-1)^{n+1}\Gamma(r+k+2)}{(n+1)!\Gamma(n+r+k+2)(r+1)_{k+1}} D^{n+k+2} x^{2n+r+k+3} D^{n+1}, n \in IN_0$$

As we are going to use these operators in the distributional sense with D – distributional derivative. We use the notation $\angle_{n,k,x}, L_{n,k,x}$ if D is understood in the distributional sense and $\angle_{n,r,k,x}$, $L_{n,k,x}$. if D is understood in the ordinary sense.

We shall use the following two formulae (1.4.3) & (1.4.4)

$$\angle_{n,r,k,x}(x+t)^{-r-k-1} = \frac{\Gamma(2n+r+k+3)}{(n+1)!\,\Gamma(n+r+k+1)} \frac{x^{n+r+k+1}t^{n+1}}{(x+t)^{2n+r+k+3}}, x \geq 0, t > 0 \tag{3.1.1}$$

and $$\int_0^\infty \angle_{n,r,k,x}(x+t)^{-r-k-1}dt = 1, x > 0, n+r+k > -1 \tag{3.1.2}$$

Lemma 3.1.1 :-

The function

$$\frac{\angle_{n,r,k+1,x}\left(\Gamma(r+k+2)T_{r+k+2}F_1\right)(x) - F_1(x)}{xB(x)},$$

where $B(x) = (1+x)^{r+k-\epsilon}$ if $r+k-\epsilon > 0$ and $B(x) = 1$ if $r+k-\epsilon < 0$, $x \in (0,\infty)$, converges uniformly to zero on $(0, \infty)$ as $n \to \infty$.

Proof :-

Using (3.1.1)

$$\angle_{n,r,k,x}(x+t)^{-r-k-1} = \frac{\Gamma(2n+r+k+3)}{(n+1)!\;\;\Gamma(n+r+k+1)} \frac{x^{n+r+k+1}t^{n+1}}{(x+t)^{2n+r+k+3}}, x \geq 0, t > 0$$

and (3.1.2)

$$\int_0^\infty \angle_{n,r,k,x}(x+t)^{-r-k-1}dt = 1, x > 0, n+r+k > -1$$

and also let us put $F_1(x) = \int_0^x F(t)dt.\ x \in R$. (3.1.3)

The function F_1 is continuous $F_1(0) = 0$ and $F_1' = F$. It follows

from

$|F(x)| \le C_1(1+x)^{r+k-\epsilon}$ that for some $C_1>0$

$$|F_1(x)| \le \int_0^x |F(t)|dt \le C_1(1+x)^{r+1+k-\epsilon}, x \ge 0 \qquad (3.1.4)$$

we get ,

$$\angle_{n,r,k+1,x}\left(\Gamma(r+k+2)T_{r+k+2}F_1\right)(x) - F_1(x)$$

$$= \frac{\Gamma(2n+r+k+4)}{(n+1)!\Gamma(n+r+k+2)}\left|\int_0^\infty \frac{(F_1(t)-F_1(x))x^{n+r+k+2}t^{n+1}}{(x+t)^{(2n+r+k+4)}}.dt\right|$$

$$\le \frac{\Gamma(2n+r+k+4)}{(n+1)!\Gamma(n+r+k+2)}\left|\int_0^\infty \frac{F_1(ux)-F_1(x)u^{n+1}}{(1+u)^{2n+r+k+4}}.du\right|$$

$$\le \frac{\Gamma(2n+r+k+4)}{(n+1)!\Gamma(n+r+k+2)} x\int_0^\infty \frac{u^{n+1}}{(1+u)^{(2n+r+k+4)}}\left|\int_1^u |F_1'(xs)|ds\right|du \qquad (\because F'_1 = F)$$

$$\le \frac{\Gamma(2n+r+k+4)}{(n+1)!\Gamma(n+r+k+2)} x\int_0^\infty \frac{u^{n+1}}{(1+u)^{(2n+r+k+4)}}\left|\int_1^u |F(xs)|ds\right|du$$

$$\le C\frac{\Gamma(2n+r+k+4)}{(n+1)!\Gamma(n+r+k+2)}\int_0^\infty \frac{u^{n+1}A(x,u)|u-1|}{(1+u)^{(2n+r+k+4)}}du$$

where for r + k-∈ >0

$$A(x,u)=\begin{cases}(1+x)^{r+k-\epsilon}, u\in(0,1)\\(1+xu)^{r+k-\epsilon}, u\in[1,\infty)\end{cases} \qquad (\because 3.1.4)$$

$$A_1(u)=\begin{cases}1, u\in(0,1)\\(1+u)^{r+k-\epsilon}, u\in[1,\infty)\end{cases}$$

and for r + k-∈<0 , A(x,u)= 1 , $A_1(u)$ = 1 x, u ∈ (0, ∞).

If we put $\wedge(u)=\dfrac{uA_1(u)|u-1|}{(1+u)^{r+k+4}}, u\in(0,\infty)$, then ∧ satisfies the

conditions of Widder's theorem ([30],P. 344 Theorem 8.C) which

implies.

$$\frac{(2n-1)!}{n!(n-2)!}\int_0^\infty \frac{u^n \wedge(u)}{(1+u)^{2n}}du \to \wedge(1)=0 \text{ as } n\to\infty .$$

Since $\dfrac{\Gamma(2n+r+k+4)n!(n-2)!}{(2n-1)!(n+1)!\Gamma(n+r+k+2)} \to 2^{r+k+4}$ as $n\to\infty$.

the assertion of Lemma 3.1.1 follows .

$$\text{i.e } \frac{\angle_{n,r,k+1,x}(\Gamma(r+k+2)T_{r+k+2}F_1)(x)-F_1(x)}{xB(x)}=0 \text{ as } n\to\infty .$$

$$\therefore \angle_{n,r,k+1,x}(\Gamma(r+k+2)T_{r+k+2}F_1)(x)-F_1(x)\to 0 \text{ as } n\to\infty .$$

$$\therefore \angle_{n,r,k+1,x}(\Gamma(r+k+2)T_{r+k+2}F_1)(x)\to F_1(x) \text{ as } n\to\infty .$$

Hence the proof of Lemma 3.1.1.

Lemma 3.1.2 :-

$x^{r+k+1}\left(\Gamma(r+1)T_{r+1}f\right)(x), x \in (0,\infty)$ is a bounded function in any interval (0,B) B>0.

Proof :-

For x>0 we have

$$\left|x^{r+k+1}\int_0^{\infty}\frac{F_1(t)}{(x+t)^{r+k+2}}dt\right| \le x^{r+k+1}\left(\int_0^{B}\frac{|F_1(t)|}{(x+t)^{r+k+2}}dt + \int_B^{\infty}\frac{|F_1(t)|}{(x+t)^{r+k+2}}dt\right)$$

$$\le x^{r+k+1}\int_0^{B}\frac{|F_1(t)|}{(x+t)^{r+k+2}}dt + x^{r+k+1}\int_B^{\infty}\frac{|F_1(t)|}{(x+t)^{r+k+2}}dt\ .$$

Thus we have to prove that

$$x^{r+k+1}\int_0^{B}\frac{|F_1(t)|}{(x+t)^{r+k+2}}dt \quad \text{is bounded when } x \to 0.$$

From (3.1.4) it follows $|F_1(x)| \le C_1, x \in (0,B)$, where, C_1 is a suitable constant . So we have,

$$x^{r+k-1}\int_0^{B}\frac{|F_1(t)|}{(x+t)^{r+k+2}}dt \le \ C_1 x^{r+k+1}\int_0^{B}\frac{1}{(x+t)^{r+k+2}}.dt$$

By the direct computation of the last integral the assertion follows.

The proof of Lemma is complete.

Let us denote by [r+k+1] the greatest integer not exceeding r+k+1

Since [r+k+1] – (r+k+1) > -1 and

$$\left|x^{[r+k+1]}\left(\Gamma(r+1)T_{r+1}f\right)_1(x)\right| = \left|x^{[r+k+1]-(r+k+1)}x^{(r+k+1)}\left(\Gamma(r+1)T_{r+1}f\right)_1(x)\right|$$

$\leq Cx^{[r+k+1]-(r+k+1)}$

where $x \in (0,B)$, B>0 & C is a suitable constant , it follows that

$x^{[r+k+1]}\left(\Gamma(r+1)T_{r+1}f\right)_+(x)$ is locally integrable on IR.

We put $l = [r+k]$.

Obviously, the cases,

$$\left(\Gamma(r+1)T_{r+1}f\right)_+ \in L'_{loc}(IR) \text{ or } x^s\left(\Gamma(r+1)T_{r+1}f\right)_+ \in L'_{loc}(IR)$$

are possible for the some s, 0<s < l+1 which depend on f.

Let us suppose that $x^l\left(\Gamma(r+1)T_{r+1}f\right)_+ \notin L'_{loc}(IR)$. In this case

$\overline{\left(\Gamma(r+1)T_{r+1}f\right)_+}$ denotes the following regularization of the function

$\left(\Gamma(r+1)T_{r+1}f\right)_+$

$$\left\langle\overline{\left(\Gamma(r+1)T_{r+1}f\right)_+}(x),\phi(x)\right\rangle = \int_0^1 \left(\Gamma(r+1)T_{r+1}f\right)(x)\left\{\phi(x)-\phi(0)-----\frac{x^l}{l!}\phi^{(l)}(0)\right\}.dx$$
$$+\int_1^{\infty}\left(\Gamma(r+1)T_{r+1}f\right)(x)\phi(x).dx, \phi \in S.$$

(3.1.5)

If we put

$$\left(\Gamma(r+k+2+i)T_{r+k+2+i}F_1\right)_+(x) = \begin{cases}\left(\Gamma(r+k+2+i)T_{r+k+2+i}F_1\right)(x), & x>0\\ 0 & x \leq 0, i \in N_0,\end{cases}$$

from the proof of Lemma 3.1.1 we have to put

$$\left\langle\left(\Gamma(r+k+2+i)T_{r+k+2+i}F_1\right)(x),\phi(x)\right\rangle = \int_0^1 \Gamma(r+k+2+i)T_{r+k+2+i}F_1)(x)$$

$$\times\left\{\phi(x)-\phi(0)-\cdots-\frac{x^{l+i}}{(l+i)!}\phi^{(l+i)}(0)\right\}dx$$

$$+\int_1^\infty\left(\Gamma(r+k+2+i)T_{r+k+2+i)}F_1)\right)(x)\phi(x)dx. \qquad (3.1.6)$$

Let us find the relation between $\left(\Gamma\overline{(r+1)T_{r+1}f}\right)_+^{(m)}$ and

$\left(\Gamma(r+2+m+k)T_{r+k+2+m}F_1\right)$, $m\in IN$. For m = 1, & $\phi\in S$

we have

$$\left\langle\left(\Gamma\overline{(r+1)T_{r+1}f}\right)_+'(x),\phi(x)\right\rangle=-\left\langle\left(\overline{\Gamma(r+1)T_{r+1}f}\right)_+(x),\phi'(x)\right\rangle$$

$$=-\int_0^1\left(\Gamma(r+1)T_{r+1}f\right)(x)\left\{\phi'(x)-\phi'(0)-\frac{x'\phi''(0)}{l!}-\cdots-\frac{x^l\phi^{(l+1)}(0)}{l!}\right\}dx$$

$$-\int_1^\infty\left(\Gamma(r+1)T_{r+1}f\right)(x)\phi'(x)dx. \qquad (\text{by } \because 3.1.6)$$

$$=-(r+1)_{k+1}\int_0^1\left(\Gamma(r+k+2)T_{r+k+2}F_1\right)(x)\left\{\phi(x)-\phi(0)-\frac{x^{l+1}}{(l+1)!}\phi^{(l+1)}(0)\right\}'dx$$

$$-(r+1)_k\int_1^\infty\left(\Gamma(r+k+2)T_{r+k+2}F_1\right)(x)\phi'(x)dx$$

$$=-(r+1)_{k+1}\left(\Gamma(r+k+2)T_{r+k+2}F_1\right)(1)\left\{\phi(1)-\phi(0)\ldots\ldots-\frac{\phi^{(l+1)}(0)}{(l+1)!}\right\}$$

$$-(r+1)_{k+2}\int_0^1\left(\Gamma(r+k+3)T_{r+k+3}F_1\right)(x)\times$$

$$\left\{\phi(x)-\phi(0)-\cdots-\frac{x^{l+1}\phi^{(l+1)}(0)}{(l+1)!}\right\}dx+(r+1)_{k+1}\left(\Gamma(r+k+2)T_{r+k+2}F_1\right)(1)\phi(1)$$

$$-(r+1)_{k+2}\int_1^\infty\left(\Gamma(r+k+3)T_{r+k+3}F_1\right)(x)\phi(x)dx.$$

Thus we obtain

$$\left(\Gamma(r+1)T_{r+1}f\right)_{+}' = -(r+1)_{k+2}\left(\overline{\Gamma(r+k+3)T_{r+k+3}F_1}\right)_{+}(x)$$

$$+(r+1)_{k+1}\left(\left(\Gamma(r+k+2)T_{r+k+2}F\right)(1)\sum_{i=0}^{l+1}(-1)^i\delta^{(i)}\right)$$

and by repeating the preceding arguments we obtain

$$\left(\Gamma\overline{(r+1)T_{r+1}f}\right)_{+}^{(m)} = (-1)^m(r+1)_{k+1+m}\left(\Gamma\overline{(r+k+m+2)T_{r+k+m+2}F_1}\right)_{+} + A_{m,k}(\delta) \tag{3.1.7}$$

where

$$A_{m,k}(\delta) = \sum_{i=0}^{l+m} C_{m,k,i}\delta^{(i)} \tag{3.1.8}$$

$C_{m,k,i}$, i =0 ,1,2 ----- l+m are suitable constants.

If $j \in \mathbb{N}$ (3.1.7) (3.1.8) imply

$$\left(\Gamma\overline{(r+1)T_{r+1}f}\right)_{+}^{(m+j)} = \left(\left(\overline{\Gamma(r+1)T_{r+1}f_1}\right)_{+}^{(m)}\right)^{(j)}$$

$$= (-1)^m(r+1)_{k+1+m+j}\left(\overline{\Gamma(r+k+2+m)T_{r+k+2+m}F_1}\right)_{+}^{(j)}(A_{m,k}(\delta))^{(j)}$$

$$= (-1)^{m+j}(r+1)_{k+1+m+j}\left(\overline{\Gamma(r+k+2+m+j)T_{r+k+2+m+j}F_1}\right) + A_{m+j,k}(\delta).$$

So we obtain

$$\left(\overline{\Gamma(r+k+2+m)T_{r+k+2+m}F_1}\right)_{+}^{(j)} = (-1)^{(j)}(r+k+2+m)_j$$

$$\times\left(\overline{\Gamma(r+k+2+m+j)T_{r+k+m+j+2}F_1}\right)_{+} + A_{m,k,j}(\delta), \tag{3.1.9}$$

$$A_{m,k},j(\delta) = \sum_{i=1}^{l+m+j} C_{m,k,j,i}(\delta)^{(i)}$$

where

$$C_{m,k,j,i}, i = 0,1,2,----l+m+j \text{ are suitable constants .}$$

If $\left(\Gamma(r+1)T_{r+1}f\right)_+ \in L'_{loc}$, then $\left(\Gamma\overline{(r+1)T_{r+1}f}\right)_+$ is a regular distribution.

Lemma 3.1.3 :-

If $\phi \in S$ and $n \to \infty$ we have

$$(r+1)_k \left\langle L_{n,r,k,x}\, x_+^{-r-k-1}, \phi(x) \right\rangle \to \left\langle \delta^{(k)}(x), \phi(x) \right\rangle$$

Proof :-

Since -r-k>1 and $n \to \infty$, there exists $s \in IN_0$ such that in the expression $D^{n+1}x_+^{-r-k-1}$(n is sufficiently large) the regularizations of $x_+^{-r-k-j-1}, j = s,----,n+1$ are to appear . From the fact that $x^p\delta^{(j)}(x) = 0$ if p>j , we obtain

$$\frac{(r+1)_k(-1)^{n+1}\Gamma(r+k+2)}{(n+1)!\,\Gamma(n+r+k+2)(r+1)_{k+1}}\left\langle D^{n+1}x^{2n+r+k+3}D^{n+1}x^{-r-k-1}, \phi^{(k+1)}(x)\right\rangle$$

$$= \frac{(-1)^{n+k}\Gamma(r+k+2)}{(n+1)!\,\Gamma(n+r+k+2)(r+k+1)}\int_{-\infty}^{\infty} D^{n+1}\left((-1)^{n+1}(r+k+1)_{n+1}x_+^{n+1}\right)\phi^{(k+1)}(x)dx$$

$$= \frac{(-1)^n\Gamma(r+k+2)(r+k+2)n}{\Gamma(n+r+k+2)}\left\langle \vartheta(x), \phi^{(k+1)}(x)\right\rangle = \left\langle \vartheta^{(k+1)}(x), \phi(x)\right\rangle, \phi \in S.$$

Hence the proof of the lemma 3.1.3 .

Lemma 3.1.4 :-

$$\frac{\angle_{n,r,k+1,x}\left(\Gamma(r+k+2)T_{r+k+2}F_1\right)(x)-F_1(x)}{x}$$ converges uniformly to zero

cn $(0,\infty)$ as $n\to\infty$.

Real Inversion Theorem 3.1.1 :-

Let $f \in L'(r)$. Then for any $\phi \in S$

$$\lim_{n\to\infty}\left\langle L_{n,r,k,x}\left(\overline{\Gamma(r+1)T_{r+1}f}\right)_+(x),\phi(x)\right\rangle = \left\langle f(x),\phi(x)\right\rangle,$$

where $k \in \mathbb{N}_0, x>0, n \in \mathbb{N}_0, r \in \mathbb{R}\setminus(-\mathbb{N}_0)$.

Proof :-

By $(\overline{\Gamma(r+1)T_{r+1}f})_+$ we denote the distribution which corresponds to the function $(\Gamma(r+1)T_{r+1}f)_+$ where

$$(\Gamma(r+1)T_{r+1}f)_+(x)=\begin{cases}(\Gamma(r+1)T_{r+1}f)(x), & x>0\\ 0 & x\le 0.\end{cases}$$

If $(\Gamma(r+1)T_{r+1}f)_+$ is a function from $L'_{loc}(\mathbb{R})$, then $(\overline{\Gamma(r+1)T_{r+1}f})_+$ denotes the regular distribution which corresponds to $(\Gamma(r+1)T_{r+1}f)_+$ and if $(\Gamma(r+1)T_{r+1}f)_+$ does not belong to $L'_{loc}(\mathbb{R})$ then $(\overline{\Gamma(r+1)T_{r+1}f})_+$ is an appropriate regularization of $(\Gamma(r+1)T_{r+1}f)_+$

In proving the theorem ,we observe separately the cases r+k>-1 and r+k < -1 .

<u>The CASE r+k>-1 :-</u>

Let us put .

$F_1(x)=\int_0^x F(t)dt, x\in \mathrm{IR}$. The function F_1 is continuous $F_1(0)=0$

and $F'_1=F$.

It follows from equation (1.2.3) that for some $C_1>0$

$$|F_1(x)|\le \int_0^x |F(t)|dt \le C_1(1+x)^{r+1+k-\epsilon}, x\ge 0 \qquad (3.1.4)$$

(we choose $\in$ such that $r+k+1-\in>0$).

Since $f=F_1^{(k+1)}$ from equation (3.1.4) by partial integration we obtain

$$(\Gamma(r+1)T_{r+1}f)(s)=(r+1)_{k+1}(\Gamma(r+k+2)T_{r+k+2}f)(s), s\in ¢\setminus(-\infty,0].$$

In the same way as in ([4],P.239) (proof of $L_{n,x}\phi\to\phi(x)$ as $n\to\infty$),

we can prove that for x>0

$$\angle_{n,r,k+1,x}(\Gamma(r+k+2)T_{r+k+2}F_1)(x)\to F_1(x) \text{ as } n\to\infty . \qquad (3.1.10)$$

By Lemma 3.1.1 we get complete proof of (3.1.10).

We shall give the proof for the extreme case

i e we suppose

$$x^1(\Gamma(r+1)T_{r+k+2}f)_+ \notin L'_{loc}(\mathrm{IR}).$$

The other cases discussed above can be proved in a similar way in

lemma(3.1.2) . By equation (3.1.7) , we obtain ($\phi\in S$)

$$\left\langle L_{n,r,k,x}\left(\Gamma(r+1)\overline{T_{r+1}f}\right)_{t}(x),\phi(x)\right\rangle$$

$$=\frac{(-1)^{n+k}\Gamma(r+k+2)}{(n+1)!\Gamma(n+r+k+2)(r+1)_{k+1}}\left\langle D^{n+1}x^{2n+r+k+3}D^{n+1}\left(\Gamma\overline{(r+1)T_{r+1}f}\right)_{t}(x),\phi^{(k+1)}(x)\right\rangle$$

$$=\frac{(-1)^{n+k}\Gamma(r+k+2)}{(n+1)!\Gamma(n+r+k+2)(r+1)_{k+1}}\left\langle D^{n+1}x^{2n+r+k+3}\left((-1)^{n+1}(r+1)_{k+n+2}\right)\right.$$

$$\left.\times\overline{\left(\Gamma(r+k+n+3)T_{r+k+n+3}F_{1}\right)}_{t}(x)+A_{n+1,k}(\delta)(x),\phi^{(k+1)}(x)\right\rangle$$

Using Leibniz formula , (3.1.9) , the fact that

$x^{p}\delta^{(k)}(x)=0$, if p>k & (3.1.5) (3.1.7)

we have

$$\left\langle D^{n+1}x^{2n+r+k+3}D^{n+1}\left(\Gamma\overline{(r+1)T_{r+1}f}\right)_{t}(x),\phi^{(k+1)}(x)\right\rangle$$

$$=\left\langle\sum_{i=0}^{n+1}\binom{n+1}{i}(x^{2n+k+r+3})^{(n+1-i)}\left((-1)^{n+1}(r+1)_{k+n+2}\left(\Gamma\overline{(r+k+n+3)T_{r+k+n+3}F_{1}}\right)_{t}^{(i)}(x)\right.\right.$$

$$\left.+\left(A_{n+1,k}(\delta)\right)^{(i)}(x)\right),\phi^{(k+1)}(x)\right\rangle$$

$$=\left\langle\sum_{i=0}^{n+1}\binom{n+1}{i}\left(x^{2n+r+k+3}\right)^{(n+1-i)}(-1)^{n+1+i}(r+1)_{k+n+i+2}\times\right.$$

$$\left.\left(\Gamma(r+k+n+i+3)T_{r+k+n+i+3}F_{1}\right)_{t}(x),\phi^{(k+1)}(x)\right\rangle$$

$$=\left\langle\sum_{0}^{n+1}(-1)^{n+1+i}\binom{n+1}{i}(2n+r+k+i+3)--(n+r+k+3+i)(r+1)_{k+n+i+2}\right.$$

$$\times\int_0^\infty \left(\Gamma(r+k+n+i+3)T_{r+k+n+i+3}F_1\right)(x)x^{n+r+k+i+2}\phi^{(k+1)}(x)dx$$

thus, we obtain

$$\left\langle L_{n,r,k,x}\,\overline{\left(\Gamma(r+1)T_{r+1}f\right)}_{+}(x),\phi(x)\right\rangle$$

$$=\sum_{i=0}^{n+1}\frac{(-1)^{i+k+1}\binom{n+1}{i}\Gamma(r+k+2)(n+r+k+i+3)_{n+1-i}(r+1)_{k+n+i+2}}{(n+1)!\,\Gamma(n+r+k+2)(r+1)_{k+1}}$$

$$\times\int_0^\infty \left(\Gamma(r+k+n+i+3)T_{r+k+n+i+3}F_1\right)(x)x^{n+r+k+i+2}\phi^{(k+1)}(x)dx\,.$$

For $x > 0$ we have

$$\sum_{i=0}^{n+1}\frac{\binom{n+1}{i}(-1)^i\,\Gamma(r+k+2)(r+k+2)_{n+1+i}(n+r+k+i+3)_{n+1-i}\,x^{n+r+k+i+2}}{(n+1)!\,\Gamma(n+r+k+2)}$$

$$\times\left(\Gamma(r+k+n+i+3)T_{r+k+n+i+3}F_1\right)(x)=\angle_{n,r,k+1,x}\left(\Gamma(r+k+2)T_{r+k+2}F_1\right)(x)\,.$$

$$\therefore\left\langle \tilde{L}_{n,r,k,x}\left(\overline{\Gamma(r+1)T_{r+1}F}\right)_{+}(x),\phi(x)\right\rangle=\angle_{n,r,k+1,x}\left(\Gamma(r+k+2)T_{r+k+2}F_1\right)(x)\,.$$

Now Lemma 3.1.1 implies

$$\left\langle L_{n,r,k,x}\left(\overline{\Gamma(r+1)T_{r+1}f}\right)_{+}(x)-f(x),\phi(x)\right\rangle$$

$$=(-1)^{k+1}\int_0^\infty\left(\angle_{n,r,k+1,x}(\Gamma(r+k+2)T_{r+k+2}F_1)(x)-F_1(x)\right)\phi^{k+1}(x)dx$$

$$=(-1)^{k+1}\int_0^\infty\frac{\angle_{n,r,k+1,x}(\Gamma(r+k+2)T_{r+k+2}F_1)(x)-F_1(x)}{xB(x)}\times\left(xB(x)\phi^{(k+1)}(x)\right)dx\to 0$$

as $n \to \infty$.

$$\therefore \tilde{L}_{n,r,k,x}\left(\Gamma(\overline{r+1)T_{r+1}f}\right)_+(x) - f(x) \to 0, \text{ as } n \to \infty .$$

$$\therefore L_{n,r,k,x}\left(\Gamma(r+1)T_{r+1}f\right)_+(x) = f(x) \text{ as } x \to \infty .$$

This completes the proof of case r + k >-1.

Case r + k < -1 :-

Let us remark that $r + k \notin - \mathbb{N}$.

We put $F_1(x) = \begin{cases} -\int_x^{\infty} F(t)dt, x > 0 \\ 0, x \le 0 \end{cases}$

Since we suppose that $f \in L'(r)$, (1.2.3) implies that there exists

$B_1 > 0$ such that

$$|F_1(x)| \le \int_0^{\infty} |F(t)|dt \le B_1(1+x)^{r+1+k-\epsilon}, x \ge 0 .$$

In the sense of the distributional derivative , we have $F_1' = F - \alpha_0 \delta$,

where $\alpha_0 = \int_0^{\infty} F(t)dt$ (3.1.11)

By partial integration, we obtain (x > 0)

$$\left(\Gamma(r+1)T_{r+1}f\right)(x) = (r+1)_k \int_0^{\infty} \frac{F(t)}{(x+t)^{r+k+1}} dt$$

$$= (r+1)_k \left((r+k+1) \int_0^{\infty} \frac{1}{(x+t)^{r+k+2}} dt + \alpha_0 \frac{1}{x^{r+k+1}} \right). \quad (3.1.12)$$

Let

$$x_+^{-r-k-1} = \begin{cases} x^{-r-k-1} & ,x>0 \\ 0 & ,x<0 \end{cases}.$$

Let $\phi \in S$. If we prove,

$$\lim_{n\to\infty}\left\langle L_{n,r,k,x}(r+1)_{k+1}\left(\overline{\Gamma(r+k+2)T_{r+k+2}F_1}\right)_+(x),\phi(x)\right\rangle$$

$$=\left\langle F_1^{(k+1)}(x),\phi(x)\right\rangle \qquad (3.1.13)$$

then (3.1.12) , lemma 3.1.4 & (3.1.11) imply

$$\lim_{n\to\infty}\left\langle L_{n,r,k,x}\left(\overline{\Gamma(r+1)T_{r+1}f}\right)_+(x),\phi(x)\right\rangle$$

$$=\lim_{n\to\infty}\left\langle L_{n,r,k,x}(r+1)_k\left(\overline{\Gamma(r+k+2)T_{r+k+2}f_1}\right)_+(x),\phi(x)\right\rangle$$

$$+\alpha_0\lim_{n\to\infty}\left\langle L_{n,r,k,x}(r+1)_k x_+^{-r-k-1},\phi(x)\right\rangle$$

$$=\left\langle\left(F_1^{(1)}+\alpha_0\delta\right)^{(k)},\phi\right\rangle=\left\langle F^{(k)},\phi\right\rangle=\left\langle f,\phi\right\rangle.$$

For the proof of equation (3.1.13) , we have to repeat the arguments of the proof of the Inversion theorem for r + k >-1 . The regularizations of $\left(\Gamma(r+k+2)T_{r+k+2}F_1\right)_+^{(j)}$ will occur for $0<s\le j\le n+1$, where s does not depend on n. At the end , instead of lemma 3.1.1 we have to use lemma 3.1.4 .

Thus the Inversion theorem is completely proved.

3.2 Complex Inversion Formula :-

3.2.1 Introduction :-

In this section we shall prove the complex inversion theorem for the classical Stieltjes transformation given by Sumner in [26] is also valid, with the convergence , in S' instead of the pointwise convergence , for the $\Gamma(r+1)T_{r+1}f, r > -1$ where f belongs to a subspace of the space of T_{r+1} - transformable tempered distributions .

First, we give some preliminary results. Let $\eta>0$, $t \in IR$, we denote $C_{\eta t}$, the contour in $\mathbb{C}$ which starts at the point, $-t-i\eta$, proceeds along the straight line Imz = $-\eta$ to the point $-i\eta$, then along the semicircle $|z| = \eta$, Rez ≥ 0 to the point $i\eta$ and finally along the line Imz = η to the point $-t + i\eta$. We notice that these contours were observed in [26] only for $t > 0$.

Let,

$$K(u,t) = K_1(t-u),\ t,u \in IR, t \neq u.$$

where,

$$K_1(x) = x^{-1}\left((-\eta - ix)^{-r-1} - (\eta + ix)^{-r-1}\right),\ x \in R \setminus \{0\}, \eta > 0 .$$

For convenience we take that determination of $(s+t)^{-r-k-1}$ which occurs in this section for which

$$\arg(s+t)^{-r-k-1}\left(\arg z^{-r-s-1}, s \in IN_0\right) \quad \text{has its principal value.}$$

Using the identities

$$\int_{C_{\eta t}} \frac{(z+t)^r}{(z+t)^{r+2}}.dz = \frac{\eta^{r+1}}{r+1} k(u,t) \ , \tag{3.2.1}$$

$u > 0, t \in IR, t \neq u, \eta > 0$,

$\partial^i K(u,t)/\partial t^i = (-1)^i \partial^i K(u,t)/\partial u^i, t, u \in IR, u \neq t, \eta > 0.$

By Leibniz formula we have

$$\left|\partial^i K(u,t)/\partial t^i\right| \leq C_1 \sum_{p=0}^{i} \left|t-u^{-i-1+p}\right| \left(\eta^2 + (t-u)^2\right)^{-(r+1+p)/2} ,$$

where $C_1 = 2\max\left\{\binom{i}{p}(i-p)!(r+1)_p ; 0 \leq p \leq i\right\}$.

Thus with $C_0 = (i+1)C_1$, it holds

$$\left|\partial^i K(u,t)/\partial t^i\right| \leq C_0 |t-u|^{-i-1} \left(\eta^2 + (t-u)^2\right)^{(r+1)/2} . \tag{3.2.2}$$

We shall suppose that r > -1 . We denote by $L'(r)(r > -1)$ a subset of L'(r+1) such that

$f \in L'(r)$, if $f = t^{-r} D^k F, k \in IN_0$, F is continuous and if instead of (3.2.3) it holds $|F(x)| \leq C(1+x)^{r+1-\epsilon}, x \geq 0$ (3.2.3)

for some C > 0 and some $\epsilon > 0$.

Lemma 3.2.1 :-

Let F be a continuous function on IR with $\operatorname{supp} F \subset [0,\infty)$ & let (3.2.3) hold. Then for every $k \in IN_0$ and $t_0 \in IR$

$$\int_{-\infty}^{\infty} F(u)\left(\partial^k K(u,t)/\partial t^k\right)\Big|_{t=t_0} du = \frac{d^k}{dt^k}\left(\int_{-\infty}^{\infty} F(u)K(u,t)du\right)\Bigg|_{t=t_0} . \quad (3.2.4)$$

<u>Lemma 3.2.2 :-</u>

Let F satisfy the conditions of Lemma 3.2.1 and let

$$\phi_i(t) = \int_{-\infty}^{\infty} \left|F(u)\partial^i K(u,t)/\partial t^i\right| du, t \in IR, i \in IN_0. \quad (3.2.5)$$

(i) There exists constants $k(i,\eta)$ and polynomials $P_i(t)$ such that

$\phi_i(t) \le k(i,\eta) P_i(t), t \in IR, i \in IN.$

(ii) There exist a constant k_0 (which does not depend on η) and a polynomial $P_0(t)$ such that

$$\eta^{r+1}\phi_0(t) \le k_0 p_0(t), t \in IR.$$

<u>Proof :-</u>

(i) Let t>1, we have

$$\phi_i(t) \le \left(\int_0^{t-1} + \int_{t-1}^{t+1} + \int_{t+1}^{\infty}\right)\left(\left|F(u)\partial^i K(u,t)/\partial t^i\right|\right) du = J_1 + J_2 + J_3$$

By (3.2.2) & (3.2.3) we have

$$J_1 \le CC_0 \int_0^{t-1} \frac{(1+u)^{r+1-\epsilon}}{|t-u|^{i+1}\left(\eta^2 + (t-u)^2\right)^{(r+1)/2}} . du \le CC_0 t(1+t)^{r+1-\epsilon}$$

because for t >0 and $u \in (0, t-1), (t-u)^{i+1}\left(\eta^2 + (t-u)^2\right)^{(r+1)/2} > 1$.

$$J_3 \le CC_0 \int_{t+1}^{\infty} \frac{(1+u)^{r+1-\epsilon} du}{|t-u|^{i+1}\left(\eta^2+(t-u)^2\right)^{(r+1)/2}}$$

$$= C_0 C \int_0^{\infty} \frac{(2+t+v)^{r+1-\epsilon}}{(v+1)^{i+1}\left(\eta^2+(v+1)^2\right)^{(r+1)/2}} . dv \ .$$

Since r- $\in$ > -1 , from $(2+t+v)^{r+1-\epsilon} \le 2^{r+1-\epsilon}\left((2+t)^{r+1-\epsilon} + v^{r+1-\epsilon}\right)$,

($\because$ putting u = t+v+1 $\therefore |t-u| = |v+1|$)

v>0 , we obtain

$$J_3 \le 2^{r+1-\epsilon} C_0 C\left((2+t)^{r+1-\epsilon} \int_0^{\infty} \frac{dv}{(v+1)^{i+1}\left(\eta^2+(v+1)^2\right)^{(r+1/2)}}\right.$$

$$\left. + \int_0^{\infty} \frac{v^{r+1-\epsilon} dv}{(v+1)^{i+1}\left(\eta^2+(v+1)^2\right)^{(r+1/2)}}\right).$$

For J_2 we have

$$J_2 \le \sup_{t-1 \le u \le t+1} \{|F(u)|\} \int_{t-1}^{t+1} \left|\partial^i K(u,t)/\partial t^i\right| du \le C(t+2)^{r+1-\epsilon} \int_{-1}^{1} \left|\partial^i K(0,s)/\partial s^i\right| ds.$$

Since the function H(0,s), s$\in$[-1,1]

where

$$H(u,t) = \begin{cases} K(u,t), u \ne t \\ \dfrac{2(r+1)i}{\eta^{r+2}}, u = t, (u,t) \in R^2 \end{cases}$$

is a smooth one . We obtain that for some constant M_i which depends on

η.

$$J_2 \le M_i (t+2)^{r+1-\epsilon} .$$

Estimations for J_1, J_2 and J_3 imply that the assertion holds if $t > 1$.

Let $0 \le t \le 1$. Then we have

$$\phi_i(t) \le \left(\int_0^2 + \int_2^\infty\right)\left(\left|F(u)\partial^i K(u,t)/\partial t^i\right|\right)du.$$

and by the similar arguments as above we can prove that the assertion holds.

If $t < 0$ there is no need to divide the integral in (3.2.5) and assertion (i) follows by arguments given by above.

(ii) Let $t > 1$. From the first part of this lemma we conclude that only in the calculation of the integral J_2, the constant $k(i,\eta)$ depends on η. But on setting

$$s = \eta tg\phi \ \text{ in } \int_1^1 \left|\partial^i K(0,s)/\partial s^i\right| ds,$$

in the same way as in the proof of Lemma 4.b from [26] , we prove the assertion.

For $t \le 1$, we have to use arguments given above.

Hence the proof of Lemma.

<u>Lemma 3.2.3 :-</u>

Let F be a continuous function on IR with supp $F \subset [0,\infty)$ and let $|F(x)| \le |C(1+x)^{r+1-\epsilon}, x \ge 0$ hold. Then $\lim_{\eta \to 0^+} \frac{\eta^{r+1}}{2\pi i} \int_{-\infty}^{\infty} F(u)K(u,t)du = F(t), t \in IR$

Proof :-

For t ≥0 the proof follows from ([26],Lemma 4.C)since for enough large R

$$\lim_{\eta\to 0^+}\frac{\eta^{r+1}}{2\pi i}\int_{IR} F(u)K(u,t)du = 0.$$

Since $r > -1$ and $\int_0^\infty |F(u)K(u,t)|dt \le \int_0^\infty |F(u)|(|t|+u)^{-r-2}.du < \infty, t < 0$

we obtain

$$\lim_{\eta\to 0^+}\frac{\eta^{r+1}}{2\pi i}\int_{-\infty}^{\infty} F(u)K(u,t)du = 0, t < 0$$

The proof is complete .

If $f \in L'(r)$, then for $t \in IR$ we have

$$(r+1)\int_{c_{\eta t}}(z+t)^r(\Gamma(r+2)\Gamma_{r+2}F)(z)dz = (r+1)_{k+1}$$

$$\int_{c_{\eta t}}(z+t)^r(\Gamma(r+k+2)\Gamma_{r+k+2}F)(z)dz$$

$$= (r+1)_{k+1}\int_{C_{\eta t}}(z+t)^r\left(\int_{-\infty}^{\infty}\frac{F(u)}{(z+u)^{r+k+2}}du\right)dz$$

$$= (r+1)_{k+1}\int_{-\infty}^{\infty}F(u)\left(\int_{C_{\eta t}}\frac{(z+t)^r}{(z+u)^{r+k+2}}dz\right)du.$$

The last equality holds on the basis of the uniform convergence of

$$\int_{-\infty}^{\infty} \frac{F(u)}{(z+u)^{r+k+2}} .du, \text{ for } z \in C_{\eta t}.$$

Thus we have by (3.2.1)

$$(r+1)\int_{C_{\eta t}} (z+t)^r (\Gamma(r+2)T_{r+2}f)(z)dz = (-1)^k \eta^{r+1} \int_{-\infty}^{\infty} F(u)(\partial^k K(u,t)/\partial t^k)du,$$

$t \in IR$.

Complex Inversion Theorem 3.2.1 :-

Let $f \in L'(r)$. Then for every $\phi \in S$.

$$\lim_{\eta \to 0^+}\left(\frac{r+1}{2\pi i}\left\langle \int_{C_{\eta t}} (z+t)^r (\Gamma(r+2)T_{r+2}f)(z)dz, \phi(t)\right\rangle\right) = \langle f(t), \phi(t)\rangle.$$

Proof :-

We have

$$\frac{r+1}{2\pi i}\left\langle \int_{C_{\eta t}} (z+t)^r (\Gamma(r+2)T_{r+2}f)(z)dz, \phi(t)\right\rangle$$

$$= \frac{1}{2\pi i}(r+1)_{k+1}\left\langle \int_{C_{\eta t}} (z+t)^r (\Gamma(r+k+2)T_{r+k+2}F)(z)dz, \phi(t)\right\rangle$$

($\because$ by3.2.5)

$$= \frac{(r+1)_{k+1}}{2\pi i}\left\langle \int_{C_{\eta t}} (z+t)^r \left(\int_{-\infty}^{\infty} \frac{F(u)}{(z+u)^{r+k+2}} .dz\right)du, \phi(t)\right\rangle$$

$$= \frac{(r+1)_{k+1}}{2\pi i}\left\langle \int_{-\infty}^{\infty} F(u)\left(\int_{C_{\eta}} \frac{(z+t)^r}{(z+u)^{r+k+2}}.dz \right) du, \phi(t) \right\rangle$$

$$= \frac{(-1)^k}{2\pi i}(r+1)\left\langle \int_{-\infty}^{\infty} F(u)\left(\frac{\partial^k}{\partial u^k} \int_{C_{\eta}} \frac{(z+t)^r}{(z+u)^{r+2}}.dz \right) du, \phi(t) \right\rangle$$

$$= \frac{(-1)^k}{2\pi i}\eta^{r+1}\left\langle \int_{-\infty}^{\infty} F(u)\left(\partial^k K(u,t)/\partial u^k\right) du, \phi(t) \right\rangle \qquad (\text{by} \because (3.2.1))$$

$$= \frac{\eta^{r+1}}{2\pi i}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} F(u)\left(\partial^k K(u,t)/\partial t^k\right) du \right)\phi(t)dt.$$

$$= \frac{\eta^{r+1}}{2\pi i}\int_{-\infty}^{\infty}\frac{d^k}{dt^k}\left(\int_{-\infty}^{\infty} F(u)(K(u,t))du \right)\phi(t)dt. \qquad (\text{by Lemma 3.2.1})$$

$$= \frac{\eta^{r+1}}{2\pi i}(-1)^k\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} F(u)K(u,t)du \right)\phi^{(k)}(t)dt.$$

(by partial integration)

Thus , by lemma 3.2.3 , lemma 3.2.2 (ii) and the Lebesgue theorem , we obtain

$$\lim_{\eta \to 0^+} \frac{r+1}{2\pi i}\left\langle \int_{C_{\eta}} (z+t)^r \left(\Gamma(r+2)T_{r+2}f\right)(z)dz, \phi(t) \right\rangle$$

$$= (-1)^k \left\langle F(t), \phi^{(k)}(t) \right\rangle = \langle f, \phi \rangle .$$

∴ The proof is complete .