CHAPTER – 1

STIELTJES TRANSFORMATION OF GENERALIZED FUNCTIONS

1.1 INTRODUCTION :-

Mathematics plays an important role in knowledge of science. The range of mathematical tools expanded considerable through the use of theory of operators, theory of generalized functions, theory of functions of complex variables, topological & algebraic methods, computational mathematics & computers. Modern mathematical physics makes brief use of the latest mathematics one of which is the theory of generalized functions. However, the concept is a convenient links connecting many aspects of analysis, functional analysis, the theory of differential equations, the theory of probability & statistics. The new mode of thinking gives birth to theory of generalized functions which put the wheel of research in several branches of mathematics in rapid motion. The impact of generalized functions on the integral transformation has recently revolutionalized the theory of "Generalized Integral Transformations".

Integral transformation are used in pure as well as applied mathematics for solving certain boundary value problem and certain type of integral equations. Improper integral of the form

$$F(z) = \int_{-\infty}^{\infty} k(z,x).f(x)dx . \qquad (1.1)$$

This type of integral is called integral transform.

k(z, x) is called kernel of the integral transform. Here we shall discussed integral transform namely modified Stieltjes transforms.

The Stieltjes transformation is introduced for the first time in 1894 by J. J. Stieltjes in connection with this work on continued fraction.

The Stieltjes transformation was studied by Benedetto [1], Zemanian [32], Misra [12], Pathak [15], Pandey [14], Lavoine & Misra [5], [6], [7], Erdelyi [4], McClure & Wong [10], [11], Takaci [27], Nikolic – De Spotovic & Pilipovics [13], Pilipovic & Stankovic [22], [17], Pilipovic [16], [18], [20], [21].

For the classical Stieltjes transformation we refer the famous book of Widder [30] & for recent results we used Burne & Love [2], [3].

While modified Stieltjes transformation is introduced by Marichev.

As in the case of integral transformations of generalized function. There are two methods, namely direct one and the method of adjoints.

The first method is used more frequently. Both methods are discussed in Erdelyi [4].

There are several definitions of the modified Stieltjes transformation of generalized functions. In this & further chapter we have used the definition of the distributional modified Stieltjes transformation of index r, $r \in \mathbb{R} \setminus (-\mathbb{N})$, T_{r+1} transformation given by Lavoine & Misra [6], [7], Pilipovic & Stankovic [22] Slightly generalized T_{r+1} transformation. This generalized transformation is denoted by $\tilde{T}_{r+1}$ transformation. The T_{r+1} transformation & $\tilde{T}_{r+1}$ transformation are defined for suitable subspaces of S'_{+} while in [1], [32], [14], [15] & [4] the Stieltjes transformation is defined for the elements of appropriate space S.

In the section 1.2, we shall give the definitions of the T_{r+1} & $\tilde{T}_{r+1}$ transformations. We shall give in section 1.4 Erdelyi's approach.

1.2 THE STIELTJES TRANSFORMATION :-

1.2.1 Spaces M'(r) and L'(r) :-

Lavoine and Misra defined the modified Stieltjes transformation by introducing space D' in [6] and M'(r) in [7]. We use here the second space. Further, we give the definition of the space L'(r) ([16]) which will be of great use in chapter three.

Definition 1.2.1 :-

Let M'(r), $r \in \mathbb{R} \setminus (-\mathbb{N})$ denote the space of all distributions $f \in S'_+(\mathbb{R})$ such that $f = t^{-r} D^k F$ (1.2.1)

for some $k \in \mathbb{N}_0$ and locally integrable function F, supp$F \subset [0,\infty)$ and

$$\int_{\mathbb{R}} | F(t) | (t+\beta)^{-r-1-k} dt < \infty \text{ for } \beta > 0. \quad (1.2.2)$$

Definition 1.2.2 :-

Let L'(r), $r \in \mathbb{R} \setminus (-\mathbb{N})$ denote the space of all distributions $f \in S'_+(\mathbb{R})$ for which there holds (1.2.1)and there exist $C = C(F)$ and $\in = \in(F) > 0$ such that

$$|F(x)| \leq C(1+x)^{r+k-1}, x \geq 0. \quad (1.2.3)$$

Definition 1.2.3 :-

Classical Stieltjes Transformation :-

The $S_r(f)(s)$, $r \in \mathbb{R} \setminus (-N)$ is complex valued function defined by

$$S_r(f(t))(s) = \int_0^\infty \frac{f(t)}{(s+t)^{r+1}} dt \quad (1.2.4)$$

$s \in \mathbb{C} \setminus (-\infty, 0]$, $0 < t < \infty$, $r \in \mathbb{R} \setminus (-\mathbb{N})$

Definition 1.2.4 :-

Modified Stieltjes Transformation :-

It is introduced by Marichev as

$$T_\alpha(f(x)) = \frac{1}{\Gamma(\alpha)}\int_0^\infty (1+\frac{x}{y})^{-\alpha}.\frac{1}{y}f(y)dy \qquad (1.2.5)$$

$x \in \mathbb{C}\backslash(-\infty,0]$, $0 < y < \infty$

it can be written as,

$$T_\alpha(f)(x) = \frac{1}{\Gamma(\alpha)}\int_0^\infty \frac{y^{\alpha-1}f(y)}{(x+y)^\alpha}dy \qquad (1.2.6)$$

by putting , $r = \alpha - 1$, $f(t) = y^{\alpha-1}f(y)$ in (1.2.4)

we get,

$$S_{\alpha-1}(y^{\alpha-1}f(y))(x) = \int_0^\infty \frac{y^{\alpha-1}f(y)}{(x+y)^\alpha}dy . \qquad (1.2.7)$$

By equation (1.2.6) & (1.2.7)

$$T_\alpha(f)(x) = \frac{1}{\Gamma(x)}S_{\alpha-1}(y^{\alpha-1}f(y))(x) .$$

By interchanging x by z & α by r+1

$$\Gamma(r+1)T_{r+1}(f)(z) = S_r(y^r f)(z) . \qquad (1.2.8)$$

Definition 1.2.5. :-

Modified Stieltjes Transformation T_{r+1} :-

Let us define the T_{r+1} – transformation, $r \in IR \setminus (-N)$. Assume that $f \in M'(r)$ and that

$f = t^{-r} D^k F$ and $\int_{IR} |F(t)|\,(t+\beta)^{-r-1-k}$ as $< \infty$

for $\beta > 0$, holds .

Now definition is as follows

$T_{r+1}(f)$, $r \in IR \setminus (-N)$ is complex valued function defined by

$$\Gamma(r+1)T_{r+1}(f)(s) = (r+1)_k \int_0^{\infty} \frac{F(t)}{(s+t)^{r+1+k}}\, dt . \qquad (1.2.9)$$

$r \in IR \setminus (-IN), s \in ¢ \setminus (-\infty, 0], \qquad 0 < t < \infty .$

Definition 1.2.6 :-

Generalized modified Stieltjes transformation $\tilde{T}_{r+1}$:-

The T_{r+1} - transformation of a distribution $f \in S'_+(R)$ is complex valued function $\tilde{T}_{r+1}(f)$ defined by

$$\Gamma(r+1)\tilde{T}_{r+1}(f)(s) = \lim_{w \to 0} < f(t), \eta(t)(s+t)^{-r-1} \exp(-wt) > . \qquad (1.2.10)$$

$w \in IR, s \in \wedge \subset (¢ \setminus (-\infty, 0]), \eta \in A(s) .$

where $\wedge$ is the set of complex number for which this limit exists and A(s) is the family of all smooth functions , defined on R for which there exists $\in = \in_{\eta,s} > o$ such that $0 \le \eta(t) \le 1$, $t \in R$, $\eta(t) = 1$ if t belongs to the $\in$ - of neighbourhood $\overline{R}$, $\eta(t) = 0$ if it belongs to the complement s of the 2 $\in$ - neighbourhood of R_+, where $\in > 0$ is arbitrary if $Ims \neq 0$,and $0 < 2\in < \max Res$, if for somes $Ims = 0$ and $|D^p \eta(t)| \le C_p$, $t \in IR$. If $\eta(t) \in A(s)$, $s \in (\not{c} \setminus R_-)$

$(R_- = (-\infty, 0])$, then $\eta(t)(s+1)^{-r-1} \exp(-wt) \in S(IR)$ for $w \in R_+$, $r \in R$.

1.3 Existence of the Tr_{+1} – transformation :-

Proposition 1.3.1 :-

Let $f \in S'_+(IR)$, there exist $p \in INo$ and locally integrable function F, $\operatorname{supp}F \subset [0,\infty)$such that $f = t^{-r} D^p F$ and

$$\int_{|t| > To} | F(t)| \, t^{r_0 - 1 - p} . dt < \infty, \qquad (1.3.1)$$

for some To>0 & $r_0 \in R$ then $Tr_{+1}(f)$ exists for $r \ge r_0$ and

$$\Gamma(r+1)\tilde{T}_{r+1}(f)(s) = (r+1)_p \int_0^\infty F(t)(s+t)^{-r-1-p} dt , \; s \in (\not{c} \setminus R_-) \qquad (1.3.2)$$

Proof :-

Let $s \in ¢ \setminus (-\infty,0]$ be fixed and $r \geq r_0$ we have by the definition of generalized modified Stieltjes transformation.

$$\Gamma(r+1)\tilde{T}_{r+1}(f)(s) = \lim_{w \to 0} < F(t), \eta(t)(s+t)^{-r-1}\exp(-wt) >$$

Taking P^{th} derivative on R.H.S ,

we have

$$\Gamma(r+1)\tilde{T}_{r+1}(f)(s) = (-1)^{p} \lim_{w \to 0^+} < F(t), \frac{D^{p}(\eta(t)\exp(-wt))}{(s+t)^{r+1}} > \quad (1.3.3)$$

By using the Leibniz formula , r.h.s of (1.3.3) can be expressed as sum of the member of the form.

$$(r+1)_P \int_0^\infty F(t)(s+t)^{-r-1-p}\exp(-wt)dt \qquad (1.3.4)$$

and

$$C_k w^{p-k} \int_0^\infty F(t)(s+t)^{-r-1-k}\exp(-wt)dt \qquad k \leq p \qquad (1.3.5)$$

such that in eq^n. (1.3.5) for at least one i_o, $1 \leq i_o \leq n$, $k_{i_0} \leq p_{i_0 - 1}$.

When $w \to 0^+$, equation (1.3.4) converges to the integral (1.3.2.)

We divide the integral (1.3.5) into two integrals

$$C_k w^{p-k}\Big(\int_{|t|<T} + \int_{|t|>T}\Big)F(t)(s+t)^{-r-1-k}\exp(-wt)dt$$

since ,

$$w^{p-k}\int_{|t|<T} F(t)(s+t)^{-r-1-k}\exp(-wt)dt = 0\,.$$

As $w \to 0^+$ $\forall$ T >0.

We have to prove that there exist $T(\in) > 0$ such that for w belonging to some neighbourhood of $0 (w \in IR_+)$.

$$w^{p-k}\int_{|t|>T(\in)} |F(t)|(s+t)^{-r-1-k}\exp(-wt)dt < \in . \qquad (1.3.6)$$

This follows from the fact that

$$\max_{w>0}\{w^{p-k}\exp(-wt)\} = (p-k)^{k-p}t^{p-k}e^{p-k}$$

We have

$$\left|w^{p-k}\int_{|t|>T}\frac{F(t)\exp(-wt)}{(s+t)^{r+1+k}}dt\right| \le C\int_{|t|>T}\left|\frac{f(t)}{(s+t)^{r_0+p+1}}\right|dt \quad \text{(by (1.3.1) \& } r \ge r_0\text{)}.$$

We obtain the existance of T ($\in$).

This implies that all the members of the form (1.3.5) tend to zero when $w \to 0^+$ $(w \in \mathbb{IR}_+)$.

By (1.3.3) & (1.3.4) we get.

$$\Gamma(r+1)T_{r+1}(f)(s) = (r+1)_p \int_0^\infty F(t)(s+t)^{-r-1-p} \exp(-wt)dt \text{, as } w \to 0^+ \text{ .}$$

which is required results .

<u>Proposition 1.3.2 :-</u>

If $f \in S'_+(\mathbb{IR})$ and has the quasiasymptotic at ∞ related to $k^a L(k)$ with the limit g, then f has the $\tilde{T}_{r+1}$ transformation for $r>a$ and there exist a continuous function F, with the support in $\mathbb{IR}_+$ and $p \in \mathbb{IN}_0$, so that

$$\Gamma(r+1)\tilde{T}_{r+1} f(s) = (r+1)_p \int_{\mathbb{IR}_+} \frac{f(t)}{(s+t)^{r+p+1}} dt$$

$$s \in \mathbb{C} \setminus \overline{\mathbb{IR}_-}$$

where, $f = t^{-r} D^p F$ and F has the asymptotic at ∞ related to $k^{a+p} L(k)$, $a>0$, $a+p>0$ with the limit $C f_{a+p+1}$.

Proof :-

If $f \in S'_+$ (IR) and the quasiasymptotic at ∞ related to k^a L (k), then Proposition 0.14.2 implies that there exists p$\in$ INo such that p+a >O that $D^{-p}(t^r f) = F(t)$ is continuous function with the support in the $\overline{IR_+}$ and F(t) has also asymptotic at ∞ related to t^{a+p} L (t) with limit C f_{a+p+1}. Now for r>a , we have

$$F(t)t^{-(r+p+1)} \in L' \quad (|t| \geq T_0) \qquad \& \qquad t^r f = D^p F.$$

We obtain proposition 1.3.2 follows from proposition 1.3.1.

1.4 ERDELYI'S APPROACH :-

The material in this section is taken from [4], we only indicate which notions are from [32].

Definition 1.4.1 :-

Let a, b, $\in$ IR fixed .The space of all $\phi \in C^\infty(R_+)$for which

$$\mu_{a,b,k}(\phi) = \sup\left\{t^{1-a+k}(1+t)^{a-b}\left|\phi^{(k)}(t)\right|; t \in IR_+\right\} < \infty \qquad (1.4.1)$$

$k \in No$ is denoted by $M_{a,b}$. $M_{a,b}$ is a complete countable multinormed space ([32], 4.2)

If $c \geq a$ and $d \leq b$ then $M_{a,b} \supset M_{c,d}$ we assume that ρ is a fixed real number.

α, β are real numbers such that $\alpha > O$, $\beta > \rho$.

$M_{\alpha,\beta} \ni \phi \rightarrow \phi \in M_{a,b}$ is continuous

Let us quote the inversion formula obtained as an analogue of the real inversion formula for conventional function.

Let

$$L_{p,q,\rho,x} = \frac{(-1)^q \Gamma(\rho)}{p! \Gamma(q+\rho-1)} \left(\frac{d}{dx}\right)^p x^{p+q+\rho-1} \left(\frac{d}{dx}\right)^q \quad , p,q \in N_0 . \quad (1.4.2)$$

This operator maps $M_{\alpha,\beta}$ continuously into $M_{\alpha+\rho-1,\beta+\rho-1}$ and $M(\alpha, \beta)$ continuously into $M(_{\alpha+\rho-1,\beta+\rho-1})$. Also it can be extended to generalized functions ,and maps $M'_{a,b}$ and $M'(a,b)$continuously into M' $a+1-\rho,b+1-\rho$ and M' $(a+1-\rho, b+1-\rho)$ respectively.

In the chapter three we need the following identities.

$$L_{p,q,\rho,x}(x+t)^{-\rho} = \frac{\Gamma(p+q+\rho)}{P!\Gamma(q+\rho-1)} \quad \frac{x^{q+p-1}t^{p}}{(x+t)^{p+q+\rho}}, t>0, x>0 \qquad (1.4.3)$$

$$\int_0^\infty L_{p,q,\rho,x}(x+t)^{-\rho}dt = 1 \qquad p+q>1,(x>0). \qquad (1.4.4)$$

Set $L_n, x = L_{p+n,q+n,\rho,x}$.

Inversion of the stieltjes transformation of generalized functions can be obtained by means of

$$<\bar{f},\phi> = <f,\phi>.$$