CHAPTRE III

Static Fluid Sphere In Einsein - Cartan Theory

STATIC FLUID SPHERE IN EINSTEIN - CARTAN THEORY

1. INTRODUCTION :

Recently Prasanna (1975) considered static fluid spheres in Einstein - Cartan theory and obtained three sets of solutions by adopting Hehl's (1973, 1974) approach and Tolman's (1934) technique. Naduka (1976) generalized the Prasann's work by considering static chaged fluid sphere in Einstein - Cartan theory. Krori and et al (1981) gave a singularity free solution for a static charged fluid sphere in Einstein - Cartan theory. Som and Bedran (1981) gave a class of solutions for the static spherical distribution. Kalyanshetti and Waghmode (1981) considered a static cosmological model in Einstein - Cartan theory based on modified Riemannian geometry.

Here we consider a static fluid sphere in Einstein-Cartan theory and obtain the solutions of the field equations with the equation of state $\bar{p} = \bar{\rho}$.

The equation of state $p = \rho$ was first given by Zeldovich, Novikov and Doroshkevich (1967) and is applicable in early stage of the universe.

2. FIELD EQUATIONS :

The Einstein – Cartan equations are

$$R^{i}_{j} - \frac{1}{2} R \delta^{i}_{j} = - K t^{i}_{j} \qquad \ldots \qquad \ldots \qquad (2.1)$$

$$Q^{i}_{jk} - \delta^{i}_{j} Q^{1}_{1k} - \delta^{i}_{k} Q^{1}_{j1} = - k S^{i}_{jk} \qquad \ldots \qquad (2.2)$$

where Q^{i}_{jk} is torsion tensor, t^{i}_{j} is the canonical energy momentum tensor and S^{i}_{jk} is the spin tensor and $K = 8\pi$.

The classical description of the spin is defined by the relation

$$S^{i}_{jk} = u^{i} S_{jk} \quad \text{with} \quad S_{ij} u^{j} = 0 \qquad \ldots \qquad (2.3)$$

where u_{i} is the four velocity vector. We consider a spherically symmetric metric in the form

$$ds^{2} = - e^{\lambda} dr^{2} - r^{2} (d\theta^{2} + \sin^{2}\theta \, d\phi^{2}) + e^{\nu} dt^{2} \ldots \qquad (2.4)$$

where λ and ν are fuctions of r only.

We assume that the spins of the particles composing the fluid are aligned in the radial direction only. Thus the non-vanishing components of S^{i}_{jk} are

$$S^{4}_{23} = - S^{4}_{32} = K$$

For the perfect fluid, we have

$$t^{1}_{1} = t^{2}_{2} = t^{3}_{3} = - P \quad \text{and} \quad t^{4}_{4} = \rho \qquad \ldots \qquad (2.5)$$

Hence the field equations for the metric (2.4) in Einstein – Cartan theory are

(following Prasanna)

$$e^{-\lambda}\left(\frac{\nu'}{r}+\frac{1}{r^2}\right)-\frac{1}{r^2}=8\pi\bar{P} \qquad (2.6)$$

$$e^{-\lambda}\left[\frac{\nu''}{2}+\frac{\nu'^2}{4}-\frac{\lambda'\nu'}{4}+\frac{(\nu'-\lambda')}{2r}\right]=8\pi\bar{P} \qquad (2.7)$$

and

$$e^{-\lambda}\left(\frac{\lambda'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2}=8\pi\bar{\rho} \quad \ldots \quad (2.8)$$

where we have followed Hehl by redefining pressure and density as

$$\bar{P}=\left(P-2\pi K^2\right), \quad \bar{\rho}=\left(\rho-2\pi K^2\right)$$

The equation of continuity now becomes

$$\frac{d\bar{P}}{dr}+\left(\bar{\rho}+\bar{P}\right)\frac{\nu'}{2}=0 \quad \ldots \quad (2.9)$$

3. SOLUTIONS OF THE FIELD EQUATIONS:

Equations (2.6), (2.8) and (2.9) are the same as obtained by Tolman and Prasanna solved these equations by Tolman's technique. Here we solve these field equations with the equation of state $\bar{P}=\bar{\rho}$ (Zeldovich fluid).

Taking $\bar{P} = \bar{\rho}$ we get from the equation (2.9)

$$\bar{P} = Ae^{-\nu} \qquad \ldots \qquad \ldots \qquad (3.1)$$

From the equations (2.6) and (2.8), we obtain

$$e^{-\lambda}(\ \nu' + \lambda'\ ) = 16\pi\bar{P}r \qquad \ldots \qquad (3.2)$$

At the centre ρ = constant = ρ_o and K = 0

Hence from the equation (2.8) we have

$$e^{-\lambda}(1 - r\lambda'\ ) = 1 - 8\pi\rho_o r^2$$

$$\text{i.e. } e^{-\lambda} = 1 - \frac{8\pi\rho_o}{3} r^2$$

$$= \left[1 - \frac{r^2}{R^2} \right] , \quad \text{(say)} \qquad (3.3)$$

where
$$\frac{1}{R^2} = \frac{8\pi\rho_o}{3} \qquad \ldots \qquad (3.4)$$

Now the equation (3.2) can be put in the form

$$T' + \frac{1}{\left[1 - \frac{r^2}{R^2} \right]} \frac{2r}{R^2} T = \frac{16\pi A r}{\left[1 - \frac{r^2}{R^2} \right]} \qquad (3.5)$$

where
$$T = e^{\nu} \qquad \ldots \qquad (3.6)$$

Equation (3.5) is a linear equation whose solution can be obtained readily.

Its solution is

$$T = e^{\nu} = 8\pi AR^2 + B\left(1 - \frac{r^2}{R^2}\right) \quad \ldots \quad (3.7)$$

where A and B are arbitrary constants. These constants A and B are evaluated by using the biundary conditions.

Thus we have

$$\left[e^{-\lambda}\right]_{r = r_o} = \left[e^{\nu}\right]_{r = r_o}$$

Hence $A = \dfrac{3}{16\pi R^2}\left(1 - \dfrac{r_o^2}{R^2}\right)$ and $B = \dfrac{-1}{2}$ ... (3.8)

Therefore

$$T = e^{\nu} = \left\{\frac{3}{2}\left(1 - \frac{r_o^2}{R^2}\right) - \frac{1}{2}\left(1 - \frac{r^2}{R^2}\right)\right\} \quad \ldots \quad \ldots \quad (3.9)$$

So, our solutions are

$$e^{-\lambda} = \left(1 - \frac{r^2}{R^2}\right) \quad \text{and}$$

$$e^{\nu} = \left\{\frac{3}{2}\left(1 - \frac{r_o^2}{R^2}\right) - \frac{1}{2}\left(1 - \frac{r^2}{R^2}\right)\right\} \quad \ldots \quad (3.10)$$

$$\rho = P = 2\pi A_1^2\, e^{-\nu} + \frac{3}{16\pi R^2}\left(1 - \frac{r_o^2}{R^2}\right) \times$$

$$\times \left\{\frac{3}{2}\left(1 - \frac{r_o^2}{R^2}\right) - \frac{1}{2}\left(1 - \frac{r^2}{R^2}\right)\right\}^{-1}$$

where the spin density K is given by

$$K = A_1 e^{-\nu/2} \qquad \ldots \qquad \ldots \qquad (3.11)$$

The constant A_1 can be evaluated in terms of central density ρ_o as

$$A_1 = \frac{\left[3\left(1 - \frac{r_o^2}{R^2}\right) - 1\right]^{1/2}}{\left(2\sqrt{2}\,\pi\right)^{1/2}} \left\{ \rho_o - \frac{3}{8\pi R^2} \frac{\left(1 - \frac{r_o^2}{R^2}\right)}{\left[3\left(1 - \frac{r_o^2}{R^2}\right) - 1\right]} \right\}^{1/2}$$

$$\ldots \qquad (3.12)$$

If A = 0, then we get $\overline{P} = \overline{\rho} = 0$, an empty model. If B = 0 then e^{ν} = constant, then our model reduces to static Einstein Universe.