CHAPTER I

CHAPTER I
BASIC CONCEPTS AND PREREQUISTIES

SECTION (I) : INTRODUCTION

The key to the problem of determining numerically the solution of the differential equation lies in the ability to develop accurate function approximation methods. It can happen, that either there is no solution of the classical statement of the problem because some of the data are not smooth, or if a smooth solution exists it cannot be found in a closed form due to the complexity of the domain, coefficients and boundary conditions.

To overcome this difficulty, we reformulate the boundary value problem in a way that will admit weaker conditions on the solution and its derivatives. Such reformulations are called weak or variational formulations of the problem and are designed to accommodata data and irregular solutions. Hence we can consider problems with quite irregular shapes.

The mathematical formulation of a variational principle is that the integral of some typical function has a smaller (or larger) value for the actual performance of the system than for any virtual performance subject to the general conditions of the system. The integrand is a function of co-ordinates, field amplitudes, and their derivatives and the integration is over a region governed by the co-ordinates of the system which may include the time.

Consider the problem of finding a function

$$u = u(x),\ 0 \le x \le 1$$

which satisfies the following differential equation.

$$-u'' + u = x \qquad 0 < x < 1 \qquad \ldots . \ (1.1)$$
$$u(0) = 0, \quad u(1) = 0$$

The data of the problem consists of all the information given in advance, the domain of the solution ($0 \le x \le 1$), the "nonhomogeneous part " of the differential equation ($f(x) = x$), the coefficients of various derivatives of (-1, +1) and boundary values ($u(0) = 0$, $u(1) = 0$). In this example, we can determine the exact solution i.e. $u(x) = x - (\sin hx/\sin h1)$.
The weak statement of the model problem is given as follows : find the function u such that the differential equation, together with the boundary conditions are satisfied in the sense of weighted averages. We require that

$$\int_0^1 (- u'' + u) \, dx = \int_0^1 xv \, dx \qquad \ldots . \ (1.2)$$

for all members v, of a suitable class of functions.

In many applications, there is no solution or quite irregular solution exists
Consider the differential equation

$$- u'' + u = \delta(x - 1/2) \qquad 0 < x < 1 \qquad \ldots .. \ (1.3)$$
$$u(0) = 0 = u(1)$$

where $\delta(x - 1/2)$ is the dirac delta; the unit " impulse " or "point source " concentrated at $x = 1/2$ defined by

$$\delta(x - 1/2) \, \phi(x) = \phi(1/2) \qquad \ldots . \ (1.4)$$

for any smooth function ϕ satisfying the boundary conditions. Here the second derivative does not exists at $x = 1/2$ because of the very irregular data of the problem. The difficulty is that, our requirement, that a solution is to the differential equation at every point x, $0 < x < 1$ is too strong. Therefore we reformulate the boundary value problem in a way that will admit the weaker conditions on the solution and its derivatives.

Whenever a smooth classical solution to a problem exists, it is also the solution of the weak problem. Thus we lose nothing by reformulating a problem in a weaker way and we gain the significant advantage of being able to consider problems with quite irregular solutions.

Weak or variational boundary value problems are precisely, the formulations we use to construct finite element approximations of the solutions. The variational formulations with the weaker continuity requirements lends itself naturally to approximate methods of solution usually referred to as direct methods. Such methods transform the problem into one involving the stationary points of a function of a finite number of real variables.

Finite Element Method belongs to the family of direct methods.

SECTION (II): **NOTATIONS AND DEFINITIONS**

1) Polynomial of degree n

A polynomial of degree n in x and y over the complex field is of the form.

$$P_n(x,y) = \sum_{\substack{r+s <= n \\ r,s = 0,1,2,...}} (a_{rs}x^r y^s)$$

Where the a_{rs} are members of the complex field K and there is at least one nonzero coefficient for which r+s = n

2) Plane algebraic curve of order n

The set of points on which P_n (x,y) = 0 is a plane algebraic curve of order n.

3) Irreducible Polynomial

A polynomial is irreducible if and only if it cannot be factored into a product cannot be factored into a product of polynomials of lower positive degree.

4) Irreducible or nondegenerate curve

The curve of each irreducible factor of a polynomial is a simple component of the curve of the polynomial and is called a nondegenerate or irreducible curve.

5) Simple Point of Curve P

Let P_x and P_y denote the partial derivatives of polynomial P with respect to x and y respectively. A simple point of curve P is a point, where either P_x or P_y is nonzero.

6) Singular Point

When both partial derivatives P_x and P_y of polynomial P with respect to x and y vanish, the point is said to be a singular point.

7) Intersection point

An intersection point of two curves is a point common to both curves.

8) P.Q

Two curves P and Q which do not have a common component intersect at a finite number of points. This set of points is denoted by the symbol P.Q

9) Equivalent Polynomials

Two polynomials which differ only in the normalization have the same curve and are said to be equivalent.

10) $P \equiv Q$ Mod R

If, for given polynomials P, Q, R, there is a b in such that p = bq at all points on curve R, we say that P is congruent to Q modulo R and is written as $P \equiv Q$ mod R.

11) Triangulation

Let $\Omega \subseteq R^2$ be a polygonal domain. A finite collection of triangles T_h satisfying the following conditions is called a triangulation.

(i) $\overline{\Omega} = \bigcup_{K \in T_h} \overline{K}$, $\overline{K}$ denotes a triangle with boundary.

(ii) $K_1 \cap K = \emptyset$, for $K_1, K \in T_h$, $K_1 \neq K$

(iii) $\overline{K}_1 \cap \overline{K}$ = a vertex or a side, i.e, if we consider two different triangles, their boundaries may have one vertex common or one side common.

12) Barycentric co-ordinates

The barycentric co-ordinates $\lambda_j = \lambda_j(x)$, $1 \le j \le n+1$ of any point $X \in R$, with respect to the $(n+1)$ points a_j, are defined to be the unique solution of the linear system

$$\sum_{j=1}^{n+1} a_{ij} \lambda_j = x_i \qquad 1 \le i \le n$$

whose matrix is defined by

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots\cdots & a_{1n+1} \\ a_{21} & a_{22} & a_{23} & \cdots\cdots & a_{2n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots\cdots & a_{nn+1} \\ 1 & 1 & 1 & \cdots\cdots & 1 \end{vmatrix}$$

13) Polycon

A polycon is a closed figure in the real plane bounded by segments of lines and conics. The polynomials which define these segments have real coefficients.

14) Polygon

When all the boundary segments are lines, a polygon is said to be a polygon.

15) Vertices

The intersection point of adjacent segments are called vertices.

16) Well-set polycon

A polycon is well set if and only if it is convex.

17) Ill-set Polycon

A polygon that is not well set is said to be ill-set.

18) Exterior Intersection Points (EIP)

Points at which the extensions of boundary segments intersect are called exterior intersection points.

19) Order of a Polycon

The order of a polycon is the order of its boundary curve.

20) Finite Element (Ciarlet 1975)

Let K be a polyhedron in R^n , P_K the space of polynomials with dimension m and Σ_K a set of distributions with cardinality m. Then the triplex (K, P_K, Σ_K) is called a finite element if

$$\Sigma_K = \{ L_i \in D^* / i = 1, 2, 3, \ldots\ldots\ldots, m \}$$

is such that for a given $\alpha \in R$, $1 \le i \le m$, the system of equations

$$L_i (p) = \alpha_i \qquad \text{for} \qquad 1 \le i \le m$$

has a unique solution $p \in P_K$

21) Degrees of freedom

The elements L_i, $i = 1,2, \ldots\, ,$ m are called degrees of freedom of P_K

22) Triangular finite element

The triplex $(K, P_1, \{a_i\}\ i = 1, 2, 3)$ where K is a triangle,

P_1 a space of polynomials of degree ≤ 1 and a_1, a_2, a_3 the vertices of K, is called the triangular finite element.

23) Rectangular finite element

The triplex (K, P_1, $\{a_i\}$ i = 1, 2, 3, 4) where K is a rectangle with the sides parallel to the axes and a_1, a_2, a_3, a_4 are the corners is called a rectangular finite element.

24) Lagrange finite elements

A finite element (K, P_K, Σ_K) is called a Lagrange finite element if Σ_K contains only Dirac masses and no derivatives of Dirac masses.

25) Hermite finite Element

A finite element (K, P_K, Σ_K) is called a Hermite finite element if Σ_K contains at least one directional derivative.

26) Stiffnes Matrix

The M x M rectangular array of numbers $K = [K_{lm}]$ is called stiffness matrix for the basis functions N_m

27) Load Vector

The M x 1 column vector $F = \{f_l\}$ is called load vector.

28) Finite Element Mesh

The collection of elements and nodal points making up the domain of the approximation problem is called finite element mesh.

29) OPPOSITE FACTOR

A polygon wedge is of the form $N_i\ (x,y) = k_i * P^i * R^i / Q$.

Polynomial P is called opposite factor which is the product of the linear and quadratic forms which vanish on the sides opposite node i. P^i is of degree m-2 when the polygon is of order m for vertex nodes at the intersection of two linear sides.

30) ADJACENT FACTOR

Polynomial R^i is adjacent factor which is unity for all side nodes and for all vertex nodes at the intersection of two linear sides. A vertex node at the intersection of a linear and a conic side has an opposite factor of degree m-2.

31) ABSOLUTE MESH DENSITY

The absolute mesh density represents the number of elements per unit area. Let D be a planar domain divided into triangles and A its area.

Consider an elementary area ΔA containing ΔN elements

Absolute Mesh Density at point P

$$\delta_a = \lim_{\Delta A \to 0} \Delta N / \Delta A$$

Total Number of elements in D is

$$N = \int \delta_a \, dA$$

32) Density Coefficient (δ_u)

It is a dimensionless number proportional to the desired mesh density at the point considered

$$\delta_a = K_a \delta_u \quad \text{where}$$

$$K_a = \frac{N}{\int_A \delta_u \; dA} \qquad \text{and} \qquad \delta_a = \frac{N}{\int_A \delta_u \; dA} \delta_u$$

33) Completeness

Consider a set of linearly independent functions, denoted by ϕ_r. Such a set is said to be complete if the linear combination $\sum_{r=1}^{M} \alpha_r \phi_r$ converges in some specified sense to arbitrary function f as M tends to infinity, where the α_r are suitably chosen constants.

$$\lim_{M \to \infty} \sum_{r=1}^{M} \alpha_r \phi_r \quad .$$

34) Complete Polynomial

A polynomial series, in one or more variables with all the terms in the sequence present is said to be complete.