CHAPTER III

CURVATURE INHERITANCE IN RIM DISTRIBUTION

The critical study of the work due to Maartens and Maharaj (1986) suggest that there do exist proper CKV in FRW models and perfect fluid distributions. This suggested a clue to Duggal (1992) to modity the concept of curvature collineation suitable for proper CKV and other related proper symmetries. Accordingly, he introduced a new symmetry called as Curvature Inheritance (CI) defined by equation (I,3.3).

Our aim here is to explore the geometrical and dynamical properties of CI pertinent to RIM distribution.

<u>Theorem</u>(1) : If the spacetime admits CI, then

$$L_{\xi} R_{ab} = 2\alpha\, R_{ab}, \qquad \ldots.(1)$$

where α is scalar function of coordinates (Duggal, 1992). Here we call the symmetry vector ξ satisfying (1) as the vector generating Ricci Inheritance (RI). We can rewrite the condition (1) for CI in the following explicit form as

$$(\nabla_c R_{ab})\, \xi^c + R_{ab}(\nabla_a\, \xi^d) + R_{ab}(\nabla_b\, \xi^d) = 2\alpha R_{ab}. \qquad \ldots(2)$$

We study this expression with the following two specific types of the vector ξ .

<u>Case (i) : The Choice $\xi^a = u^a$</u>

For this case, equation (2) becomes

$$(\nabla_c R_{ab})u^c + R_{db}\nabla_a u^d + R_{ab}\nabla_b u^d = 2\alpha\, R_{ab}. \qquad \ldots.(3)$$

The concentration of this equation with g^{ab} yields.

$$(\nabla_c R)u^c + R^a_d(\nabla_a u^d) + R^b_d(\nabla_a u^d) = 2\alpha\, R,$$

i.e. $(\nabla_c R)u^c + 2R^{ad}(\nabla_a u_d) = 2\alpha\, R.$ ....(4)

For the RIM distribution $R = -\rho$ and hence equation (4) reduces to

$$(\nabla_c \rho)u^c - 2R^{ad}(\nabla_a u_d) = 2\alpha\rho\,. \qquad \ldots.(5)$$

By using the equation (I,4.5) this result produces

$$(\nabla_c\rho)u^c - 2\left[(\rho + \mu h^2)u^a u^d - \tfrac{1}{2}(\rho + \mu h^2)g^{ad} - \mu h^a h^d\right]\nabla_a u_d = $$
$$= 2\alpha\rho\,.$$

If we substitute $u^d(\nabla_a u_d) = 0$ and $\nabla_a u^a = \theta$, then the above equation leads to

$$\overset{*}{\rho} + \rho\theta + \mu h^2\theta + 2\mu h^a h^d\, \nabla_a u_d = 2\alpha\rho\,.$$

This, with the help of continuity equation (I,6.4), generates

$$\mu h^2\theta + 2\mu h^a h^d\, \nabla_a u_d = 2\alpha\rho\,.$$

After using Maxwell equation (I,5.3), we get

$$\mu h^2\theta + \mu \overset{*}{h}{}^2 = -2\alpha\rho\,. \qquad \ldots.(6)$$

i.e. $L_u(\mu h^2) = -2\alpha\rho - \mu h^2\theta$. ....(7)

Further, innermultiplying equation (3) with $u^a u^b$ gives

$$u^a u^b(\nabla_c R_{ab})u^c + u^a u^b R_{db}\nabla_a u^d + u^a u^b R_{ad}\nabla_a u^d$$

$$= 2\alpha R_{ab}u^a u^b. \quad ....(8)$$

We simplify each term of this equation in following manner.

$$u^a u^b(\nabla_c R_{ab})u^c =$$

$$= u^a u^b\Big[\nabla_c[(\rho + \mu h^2)u_a u_b - \tfrac{1}{2}(\rho + \mu h^2)g_{ab} - \mu h_a h_b]\Big]u^c,$$

(Vide I,4.5).

If we substitute $(\nabla_c u_a)u^a = 0$, $u^a h_a = 0$ and $(\nabla_c g^{ab}) = 0$, then the above equation leads to

$$u^a u^b(\nabla_c R_{ab})u^c = \tfrac{1}{2}\Big[\nabla_c(\rho + \mu h^2)\Big]u^c. \quad ....(9)$$

Now we take the second term of L.H.S. of equation (8).

$$u^a u^b R_{db}\nabla_a u^d =$$

$$= u^a u^b\Big[(\rho + \mu h^2)u_d u_b - \tfrac{1}{2}(\rho + \mu h^2)g_{db} - \mu h_a h_b\Big]\nabla_a u^d,$$

(Vide I,4.5).

After using $u_d\nabla_a u^d = 0$, this reduces to

$$u^a u^b R_{db}\nabla_a u^d = 0. \quad ....(10)$$

Similarly we observe that the third term on L.H.S. of equation (8) also becomes

$$u^a u^b R_{ad} \nabla_a u^d = 0. \qquad \ldots.(11)$$

Also the R.H.S. of equation (8) is simplified as

$$2\alpha R_{ab} u^a u^b = \alpha(\rho + \mu h^2), \quad \text{(Vide I,4.7)}. \qquad \ldots.(12)$$

Thus by means of equations (9), (10), (11) and (12) the equation (8) is reduced to

$$\left[\nabla_c(\rho + \mu h^2)\right] u^c = 2\alpha(\rho + \mu h^2),$$

i.e. $L_u(\rho + \mu h^2) = 2\alpha(\rho + \mu h^2)$. ....(13)

Further, by the contraction of (3) with $h^a h^b$ yields

$$h^a h^b (\nabla_c R_{ab}) u^c + h^a h^b R_{db} \nabla_a u^d + h^a h^b R_{ad} \nabla_b u^d =$$

$$= 2\alpha R_{ab} h^a h^b. \qquad \ldots.(14)$$

We simplify each term of this equation as follows

$$h^a h^b (\nabla_c R_{ab}) u^c =$$

$$= h^a h^b \left[\nabla_c [(\rho + \mu h^2) u_a u_b - \tfrac{1}{2}(\rho + \mu h^2) g_{ab} - \mu h_a h_b]\right] u^c,$$

(Vide I,4.5).

By recalling $u^a h_a = 0$, $h^a h_a = -h^2$,

$$(\nabla_c h_a) u^c = \overset{*}{h}_a, \ \overset{*}{h}_a h^a = -\tfrac{1}{2}\overset{*}{h}{}^2 \text{ and } (\nabla_c \rho) u^c = \overset{*}{\rho},$$

we get

$$h^a h^b (\nabla_c R_{ab}) u^c = \frac{h^2}{2} \left(\overset{*}{\rho} - \tfrac{1}{2}\mu \overset{*}{h}{}^2 \right). \qquad \ldots.(15)$$

Now, we simplify the second term on L.H.S. of (14)

$$h^a h^b R_{db} \nabla_a u^d =$$

$$= h^a h^b \left[(\rho + \mu h^2) u_d u_b - \tfrac{1}{2}(\rho + \mu h^2) g_{db} - \mu h_d h_b \right] \nabla_a u^d,$$

(Vide I,4.5).

If we substitute $h^a u_a = 0$ in this, then we get

$$h^a h^b R_{db} \nabla_a u^d = \tfrac{1}{2} h^a h_d (\nabla_a u^d)(\mu h^2 - \rho),$$

Using the Maxwell equation (I,5.3) we get,

$$h^a h^b R_{db} \nabla_a u^d = \tfrac{1}{2}(-\tfrac{1}{2}\overset{*}{h}{}^2 - h^2 \theta)(\mu h^2 - \rho). \qquad \ldots.(16)$$

Similarly we observe that the third term on L.H.S. of equation (14) also provides

$$h^a h^b R_{ad} (\nabla_b u^d) = \tfrac{1}{2}(-\tfrac{1}{2}\overset{*}{h}{}^2 - h^2 \theta)(\mu h^2 - \rho). \qquad \ldots.(17)$$

Further the R.H.S. of equation (14) gives

$$2\alpha \, R_{ab} h^a h^b = \alpha(\rho - \mu h^2) h^2, \qquad \text{(Vide I,4.8).} \qquad \ldots(18)$$

Thus employing equations (15), (16), (17) and (18) in equation (14) we obtain

$$\tfrac{1}{2}\overset{*}{\rho}h^2 - \tfrac{1}{2}\mu\overset{*}{h}{}^2h^2 + (-\tfrac{1}{2}\overset{*}{h}{}^2 - h^2\theta)(\mu h^2 - \rho) =$$

$$= \alpha(\rho - \mu h^2)h^2,$$

This when simplified gives

$$\overset{*}{\rho}h^2 - 2\mu\overset{*}{h}{}^2h^2 - 2\mu\theta h^4 + \rho\overset{*}{h}{}^2 + 2h^2\theta\rho = 2\alpha(\rho - \mu h^2)h^2. \quad ..(19)$$

Further, innermultiplying equation with u^ah^b, we get

$$u^ah^b(\nabla_cR_{ab})u^c + u^ah^bR_{db}(\nabla_au^d) + u^ah^bR_{ad}(\nabla_bu^d) =$$

$$= 2\alpha R_{ab}u^ah^b. \quad ....(20)$$

We simplify each term of this equation in following manner

$$u^ah^b(\nabla_cR_{ab})u^c =$$

$$= u^ah^b\Big[\nabla_c[(\rho + \mu h^2)u_au_b - \tfrac{1}{2}(\rho + \mu h^2)g_{ab} - \mu h_ah_b]\Big]u^c,$$

(Vide I,4.5).

If we substitute $u^ah_a = 0$, $\overset{*}{u}_b = (\nabla_cu_b)u^c$ and $\overset{*}{h}_a = (\nabla_ch_a)u^c$ and knowing that $\overset{*}{h}_au^a = 0$, $h_a\overset{*}{u}{}^a = 0$, the above equation reduces to

$$u^ah^b(\nabla_cR_{ab})u^c = 0. \quad ....(21)$$

The second term on L.H.S. of (20) is also can be simplified as follows

$$u^a h^b R_{ad}(\nabla_b u^d) =$$

$$= u^a h^b \left[(\rho + \mu h^2)u_a u_d - \tfrac{1}{2}(\rho + \mu h^2)g_{ab} - \mu h_a h_d\right]\nabla_b u^d,$$

(Vide I,4.5).

By using the results $u^a h_a = 0$ and $u_a \nabla_b u^a = 0$, we get

$$u^a h^b R_{ad} \nabla_b u^d = 0. \qquad \ldots.(22)$$

In similar way the third term on the L.H.S. of equation (20) provides

$$u^a h^b R_{db} \nabla_a u^d = 0. \qquad \ldots.(23)$$

Further the R.H.S. of equation (20) becomes

$$2\alpha R_{ad} u^a h^b = 0. \qquad \ldots.(24)$$

By making use of these values (21), (22), (23) and (24) it is observed that the equation (20) is identically satisfied.

<u>Claim</u> : For the RIM distribution obeying the Ricci Inheritance property along the flow vector u^a implies Ricci Collineation iff $\theta = 0$.

<u>Proof</u> : On subtracting equation (6) from equation (13), we get

$$\overset{*}{\rho} - \mu h^2 \theta = 4\alpha\rho + 2\alpha\mu h^2.$$

By using continuity equation (Vide I, 6.4) in this equation, we get

$$-\rho\theta - \mu h^2\theta = 2\alpha(2\rho + \mu h^2),$$

i.e. $$\alpha = \frac{-(\rho + \mu h^2)\theta}{2(2\rho + \mu h^2)} . \qquad \ldots.(25)$$

We know that the Ricci curvature Inheritance implies Ricci Collineation iff $\alpha = 0$.

But $\alpha = 0 \Longleftrightarrow \theta = 0$.

This completes the proof [Vide, (25)].

Case (ii) : The choice $\xi^a = h^a$

For this case equation (2) becomes

$$(\nabla_c R_{ab})h^c + R_{db}\nabla_a h^d + R_{ad}\nabla_a h^d = 2\alpha R_{ab}. \qquad \ldots.(26)$$

The contraction of this with g^{ab} yields

$$(\nabla_c R)h^c + 2R^a_d \nabla_a h^d = 2\alpha R. \qquad \ldots.(27)$$

For the RIM distribution $R = -\rho$ and hence (27) reduces to

$$(\nabla_c \rho)h^c - 2R^{ad}(\nabla_a h_d) = 2\alpha\rho . \qquad \ldots.(28)$$

By using the equation (I, 4.5), this result becomes

$$(\nabla_c \rho)h^c - 2\left[(\rho + \mu h^2)u^a u^d - \tfrac{1}{2}(\rho + \mu h^2)g^{ad} - \mu h^a h^d\right] \nabla_a h_d =$$

$$= 2\alpha\rho .$$

If we substitute I, (6.6) and (6.7) then the above result leads to

$$[\nabla_c(\rho - \tfrac{1}{2}\mu h^2)]\, h^c = 2\alpha\rho \,. \qquad \ldots.(29)$$

Further, innermultiplying equation (26) with $u^a u^b$ gives

$$u^a u^b(\nabla_c R_{ab})h^c + u^a u^b R_{db}(\nabla_a h^d) + u^a u^d R_{ad}\nabla_b h^d = 2\alpha u^a u^b R_{ab}$$

$$\ldots.(30)$$

We simplify each term of this equation in following manner

$u^a u^b(\nabla_c R_{ab})h^c =$

$$= u^a h^b\Big[\nabla_c[(\rho + \mu h^2)u_a u_b - \tfrac{1}{2}(\rho + \mu h^2)g_{ab} - \mu h_a h_b]\Big]h^c,$$

(Vide I,4.5).

If we substitute $u^a h_a = 0$ and $u^a \nabla_c u_a = 0$ the above equation leads to

$$u^a u^b(\nabla_c R_{ab})h^c = \tfrac{1}{2}\Big[\nabla_c(\rho + \mu h^2)\Big]h^c. \qquad \ldots.(31)$$

Now we take the second term of L.H.S. of equation (30)

$u^a u^b R_{db}\nabla_a h^d =$

$$= u^a u^b\Big[(\rho + \mu h^2)u_d u_b - \tfrac{1}{2}(\rho + \mu h^2)g_{db} - \mu h_d h_b\Big]\nabla_a h^d,$$

(Vide I, 4.5).

After using $u_a h^{*a} = 0$ and $u^a h_a = 0$, this reduces to

$$u^a u^b R_{db}\nabla_a h^d = 0. \qquad \ldots.(32)$$

Similarly, we observe that the third term on L.H.S. of equation (30) also simplified as

$$u^a u^b R_{ad} \nabla_b h^d = 0. \qquad \ldots\ldots(33)$$

Further, R.H.S. of equation (30) is simplified as

$$2\alpha u^a u^b R_{ab} = \alpha(\rho + \mu h^2), \quad \text{(Vide I, 4.7).} \qquad \ldots\ldots(34)$$

Thus by utilising equations (31), (32), (33) and (34) in equation (30) we get

$$\left[\nabla_c(\rho + \mu h^2)\right] h^c = 2\alpha(\rho + \mu h^2). \qquad \ldots\ldots(35)$$

Further innermultiplying equation (26) with $h^a h^b$, we have

$$h^a h^b (\nabla_c R_{ab}) h^c + h^a h^b R_{db} \nabla_a h^d + h^a h^b R_{ad} \nabla_b h^d = 2\alpha\, h^a h^b R_{ab}.$$

$$\ldots\ldots(36)$$

We simplify each term of this equation in the following manner.

$$h^a h^b (\nabla_c R_{ab}) h^c =$$

$$= h^a h^b \left[\nabla_c [(\rho + \mu h^2) u_a u_b - \tfrac{1}{2}(\rho + \mu h^2) g_{ab} - \mu h_a h_b]\right] h^c,$$

(Vide I,4.5).

If we substitute $u^a h_a = 0$, $h^a h_a = -h^2$, $\nabla_c g_{ab} = 0$ and

$(\nabla_c h_a) h^a = -\tfrac{1}{2} \nabla_c h^2$, we get

$$h^a h^b \nabla_c R_{ab} h^c = \tfrac{1}{2}(\nabla_c \rho - \mu \nabla_c h^2) h^c h^2. \qquad \ldots.(37)$$

Now the second term on R.H.S. of equation (36) is simplified as follows

$$h^a h^b R_{db}(\nabla_a h^d) =$$

$$= h^a h^b \left[(\rho + \mu h^2) u_d u_b - \tfrac{1}{2}(\rho + \mu h^2) g_{ab} - \mu h_a h_b\right] \nabla_a h^d,$$

(Vide I, 4.5).

If we substitute $h^a u_a = 0$, $h_a h^a = -h^2$ and $(\nabla_a h^d) h_d = -\tfrac{1}{2}\nabla_a h^2$, we get

$$h^a h^b R_{db} \nabla_a h^d = \frac{1}{4}(\rho - \mu h^2) h^a \nabla_a h^2. \qquad \ldots.(38)$$

Similarly, we can observe that the third term on L.H.S. of (36) is simplified as

$$h^a h^b R_{ad} \nabla_b h^d = \frac{1}{4}(\rho - \mu h^2) h^b \nabla_b h^2. \qquad \ldots.(39)$$

Further the R.H.S. of equation (36) gives

$$2\alpha h^a h^b R_{ab} = \alpha(\rho - \mu h^2) h^2, \quad \text{(Vide I, 4.8)} \qquad \ldots.(40)$$

Thus utilising equations (37), (38), (39) and (40) in equation (36), we get

$$\left[\nabla_c(\rho - \mu h^2)\right] h^c h^2 + (\rho - \mu h^2)(\nabla_c h^2) h^c =$$

$$= 2\alpha(\rho - \mu h^2) h^2. \qquad \ldots.(41)$$

Further, innermultiplying equation (26) with $u^a h^b$, we get

$$u^a h^b(\nabla_c R_{ab})h^c + u^a h^b R_{db}\nabla_a h^d + u^a h^b R_{ad}\nabla_a h^d =$$

$$= 2\alpha\, u^a h^b R_{ab}. \qquad \ldots.(42)$$

We can simplify each term of this equation in following manner

$$u^a h^b(\nabla_c R_{ab})h^c =$$

$$= u^a h^b\Big[\nabla_c[(\rho + \mu h^2)u_a u_b - \tfrac{1}{2}(\rho + \mu h^2)g_{ab} - \mu h_a h_b]\Big]h^c,$$

(Vide I,4.5).

By using the results $u^a h_a = 0$, $\nabla_c g_{ab} = 0$ and $\overset{*}{h}{}^a u_a = 0$ we get

$$u^a h^b(\nabla_c R_{ab})h^c = \rho(\nabla_c u_b)h^b h^c). \qquad \ldots.(43)$$

The second term on L.H.S. of equation (42) is simplified in following manner.

$$u^a h^b R_{db}\nabla_a h^d = u^a h^b\Big[(\rho + \mu h^2)u_d u_b - \tfrac{1}{2}(\rho + \mu h^2)g_{ab} - \mu h_d h_b\Big]\nabla_a h^d$$

If we use the results $u^a h_a = 0$, $(\nabla_a h^d)h_d = -\frac{1}{2}\nabla_a h^2$, we get

$$u^a h^b R_{db}(\nabla_a h^d) = \frac{1}{4}(\rho - \mu h^2)(\nabla_a h^2)u^a. \qquad \ldots.(44)$$

The third term on L.H.S. of (42) is simplified as follows.

$$u^a h^b R_{ad} \nabla_b h^d =$$

$$= u^a h^b \left[(\rho + \mu h^2) u_a u_d - \tfrac{1}{2}(\rho + \mu h^2) g_{ad} - \mu h_a h_d \right] \nabla_b h^d,$$

(Vide I, 4.5).

Now, using the result $u^a h_a = 0$, we get

$$u^a h^b R_{ad} \nabla_b h^d = -\tfrac{1}{2}(\rho + \mu h^2) u_d h^b \nabla_b h^d. \qquad \ldots.(45)$$

Further the R.H.S. of equation (42) becomes

$$2\alpha \; u^a h^b R_{ab} = 0. \qquad \ldots.(46)$$

Thus by using the values (43), (44), (45) and (46) in (42), we get

$$\rho(\nabla_c u_b) h^b h^c + \tfrac{1}{2}(\rho - \mu h^2)(\nabla_a h^2) u^a - \mu h^2 (\nabla_b u_d) h^b h^d = 0. \qquad ..(47)$$

<u>THEOREM</u> : For RIM distribution, CI along vector $\bar{h}$ degenerates into CC.

<u>Proof</u> : On subtracting equation (29) from (31) we get

$$3\,(\nabla_c h^2) h^c = 4\,\alpha\, h^2. \qquad \ldots.(48)$$

Further multiplying (29) by two and adding in equation (35) we get

$$3(\nabla_c \rho) h^c = 2\alpha(3\rho + \mu h^2). \qquad \ldots.(49)$$

Substituting the values (48) and (49) in equation (41) we get

$$4\,\alpha\,\rho\,h^2 = 0,$$

i.e. $\alpha = 0,$ since $\rho \neq 0,\ h^2 \neq 0.$ ··(50)

Hence

$$L_\xi R^a_{bcd} = 2\,\alpha\,R^a_{bcd}$$

$\Longrightarrow L_\xi R^a_{bcd} = 0$ which describes CC. Here the proof of the theorem is complete.

Corollary : If RIM distribution admits CI along the magnetic field $\bar{h}$, then

$$\rho_{,c}h^c = 0 = \mu(\nabla_c h^2)h^c.$$

The proof follows from the equations (48), (49) and (50).

Remark : In case of homogeneous magnetic field we observe that Ricci Inheritance symmetry $\Longrightarrow$ CC.

Conclusion : In this chapter, we have examined the implications of curvature inheritance symmetry with reference to the spacetime of RIM distribution.

In Case (i), we have found that the curvature inheritance degenerates into curvature collineation along the symmetry vector $\bar{u}$ if either expansion vanishes or the matter density ρ is balanced by magnetic field ($\theta = 0$).

In second case, dealing with the curvature inheritance where the magnetic field vector $\bar{h}$ acts as symmetry vector, we have shown that it leads to curvature collineation. Moreover, the matter density and the magnitude of magnetic field remain invariant along this vector.