CHAPTER IV

CONFORMAL MOTIONS AND CURVATURE INHERITANCE (JOINT EFFECT)

INTERPLAY BETWEEN CKV AND CI :

In chapter II, we have studied the properties of RIM distribution admitting a group of conformal motions.

The chapter III is devoted to the study of symmetry group of conformal motions leading to curvature inheritance in RIM distribution. These two aspects of conformal motions and curvature inheritance are treated separately in earlier chapters.

This chapter IV throws light on the implications of RIM distribution admitting both group of conformal motions and curvature inheritance simultaneously.

Theorem : (Duggal, 1992)

A CI is CKV if

$$\nabla_{ab}\psi = \frac{\alpha}{2}\left[\frac{R}{3}g_{ab} - 2R_{ab}\right]. \qquad \ldots.(1.1)$$

By using the values [I,(4.5) and (4.6)] for R and R_{ab} respectively in above equation, we get

$$\nabla_{ab}\psi = \frac{\alpha}{2}\left[-\frac{\rho}{3}g_{ab} - 2[(\rho + \mu h^2)u_a u_b - \tfrac{1}{2}(\rho + \mu h^2)g_{ab} - \mu h_a h_b]\right]. \qquad \ldots.(1.2)$$

The contractions of this with $u^a u^b$, $h^a h^b$ and $u^a h^b$ after simplification give the following results.

$$(\nabla_{ab}\psi\ )u^a u^b = -\frac{\alpha}{2}\left[\frac{4\rho}{3} + \mu h^2\right], \qquad \ldots.(1.3)$$

$$(\nabla_{ab}\psi\ )h^a h^b = \frac{\alpha}{2}\left[\frac{(3\mu h^2 - 2\rho\ )h^2}{3}\right], \qquad \ldots.(1.4)$$

$$(\nabla_{ab}\psi\ )u^a h^b = 0. \qquad \ldots.(1.5)$$

Further, equation (1.2) after transvection with g^{ab} yields

$$(\nabla_{cd}\psi\ )g^{cd} = \frac{\alpha\rho}{3}, \qquad \ldots.(1.6)$$

This result when coupled with equation (II, 2.14) produces

$$L_\xi\,\rho = 2(\alpha\rho + R^a_b\,\nabla_a\,\xi^b). \qquad \ldots.(1.7)$$

By making use of (1.3) in equation (II, 2.15), we derive

$$L_\xi\,(\mu h^2) = -8(\nabla_{cd}\psi\ )g^{cd} + 4B\psi + \frac{\alpha}{2}\left[\frac{4\rho}{3} + \mu h^2\right] - 2R^a_b\nabla_a\,\xi^b\ .$$

This after using (1.6) and simplifying provides

$$L_\xi\,(\mu h^2) = \alpha(-\frac{4}{3}\rho + \frac{\mu h^2}{2}) + 4B\psi - 2R^a_b\nabla_a\,\xi^b. \qquad \ldots.(1.8)$$

Now by substituting the values from (1.4) and (1.6) in (II, 2.10), we get

$$L_\xi\,B = \alpha(\frac{\rho}{3} - \frac{1}{2}\mu h^2) - 2B\psi - 2Ch^a(L_\xi\,h_a). \qquad (1.9)$$

<u>Claim</u> : If RIM distribution admits CKV and CI both then

$$h^a L_\xi h_a = 0 \iff \rho = \frac{3}{8}\mu h^2.$$

The proof follows from the results (1.7), (1.8) and (1.9).

2. THE SPECIAL CONFORMAL MOTIONS

We recall the defining conditions for special conformal motions (Vide Chapter I, Note (ii))

$$\nabla_{cd}\psi = 0. \qquad .....(2.1)$$

These conditions with (1.6) imply

$$\alpha = 0. \qquad ....(2.2)$$

Hence we infer that the curvature inheritance degenerates into curvature collineation if CKV is special CKV.

Further the conditions (2.1) simplify equations (1.7) and (1.8) as follows.

$$L_\xi \rho = 2R^a_b \nabla_a \xi^b. \qquad ....(2.3)$$

$$L_\xi (\mu h^2) = -2(\rho + \mu h^2)\psi - 2R^a_b \nabla_a \xi^b. \qquad ....(2.4)$$

In particular if $\nabla_a \xi^b = \xi u_a h^b$ then (2.1) with (I, 1.9) gives

$$L_\xi (\mu h^2) = -2(\rho + \mu h^2)\psi \qquad ....(2.5)$$

This shows that the conformal factor ψ depends explicitly on the magnetic field only.

3. EXPLICIT RELATION BETWEEN α AND ψ (A CASE STUDY)

It is proved by Duggal, 1992 that "the spacetime admitting CKV and proper CI simultaneously is conformally flat". This result is used to develop the relation between ψ and α by considering a suitable example.

Any metric g_{ik} leading to conformally flat space can be expressed in terms of Minkowski metric η_{ik} as follows.

$$g_{ik} = X^2 \eta_{ik}, \qquad \ldots.(3.1)$$

where X is a function of co-ordinates.
The Scalar curvature R pertaining to the metric g_{ik} is related to the function X through the equation (Gidas, 1982).

$$(\nabla_{cd}X)g^{cd} + \frac{1}{6} RX^3 = 0, \qquad \ldots.(3.2)$$

$$\text{i.e. } (\nabla_{cd}X)g^{cd} - \frac{1}{6}\rho X^3 = 0, \quad \text{(Vide I, 4.6).} \qquad \ldots.(3.3)$$

from equation (3.1) we can write

$$L_\xi (g_{ik}) = 2\xi (\log X)\, g_{ik}, \qquad \ldots.(3.4)$$

This implies a functional relation

$$\psi X = \xi(X). \qquad \ldots.(3.5)$$

We recall the earlier result (Vide, 1.6)

$$(\nabla_{cd}\psi\ )g^{cd} = \frac{-\alpha R}{3}. \qquad \ldots.(3.6)$$

By using (3.3), (3.5) and (3.6) we derive

$$(\nabla_{cd}\psi\ )g^{cd} = \frac{1}{6}\ \psi\, RX^2 + (\frac{1}{X})\left[\nabla_{cd}\,\xi\,(X)\right]g^{cd},$$

This implies

$$\alpha = \psi\,(-\frac{1}{2}\,X^2) - \frac{3\ [\nabla_{cd}\,\xi\,(X)]g^{cd}}{RX},$$

i.e.

$$\alpha = (-\frac{1}{2}\,X^2)\ \psi + \frac{3}{\rho X}\left[\nabla_{cd}\,\xi\,(X)\right]g^{cd}. \qquad \ldots.(3.7)$$

Hence we got an explicit relation between α and ψ via known function X.

4. SPACETIME MODEL ADMITTING CONFORMAL MOTIONS

We develop here a static spherically symmetric space-time model concomitant with RIM distribution admitting a group of conformal motions.

The static spherically symmetric metric in Schwarzchild coordinates is given by the line element.

$$ds^2 = A^2(r)dt^2 - B^2(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad ..(4.1)$$

We choose comoving system so that the orghogonality condition between flow field and magnetic field imply

$$u^a = u^4\delta^a_4, \quad u^4 = \frac{1}{A}. \qquad \ldots.(4.2)$$

$$h^a = h^1\delta^a_1. \qquad \ldots.(4.3)$$

For this choice the line element (4.1) yields the field equations for the RIM distribution (I, 4.5) are given by

$$\mu h^2 = \frac{1}{B^2}\left[\frac{2A'}{Ar} + \frac{1}{r^2}\right] - \frac{1}{r^2}, \qquad \ldots.(4.4)$$

$$\frac{1}{2}\mu h^2 = \frac{1}{B^2}\left[\frac{A'B'}{AB} - \frac{A''}{A} - \frac{1}{r}\left(\frac{A'}{A} - \frac{B'}{B}\right)\right], \qquad \ldots.(4.5)$$

$$\left(\rho + \frac{1}{2}\mu h^2\right) = \frac{1}{B^2}\left[\frac{1}{r^2} + \frac{2B'}{Br}\right] - \frac{1}{r^2}. \qquad \ldots.(4.6)$$

(Primes denote differentiation w.r.t. r).

Further the Maxwell equation (I, 5.1) for the line element (4.1) and for the choice (4.2) and (4.3) we obtain

$$\frac{\frac{\partial}{\partial t}(u^4\sqrt{-9})}{\sqrt{-9}\, u^4} + \frac{h^1{}_{,4}}{h^1} = 0, \qquad \ldots.(4.7)$$

$$\frac{u^4{}_{,1}}{u^4} + \frac{\nabla_b h^b}{h^1} = 0. \qquad \ldots,(4.8)$$

On integrating these two equations, we get the value of h_1 in the form

$$h_1 = \frac{-Bf^2}{r^2}, \text{ where } f = f(r). \qquad \ldots(4.9)$$

Now we find the value of A, we impose an extra condision of conformal symmetry

$$L_X g_{ab} = 2\psi\, g_{ab}. \qquad \ldots(4.10)$$

These sixteen equations with the choice of arbitrary vector

$$\bar{X} = \lambda(r) \frac{\partial}{\partial r}, \qquad \ldots(4.11)$$

and using the metric symmetry (4.1) we get

$$B = \frac{n}{\psi}, \text{ where n is arbitrary constant of of integration.} \qquad \ldots(4.12)$$

Also the value of A is given by one of the integrals as

$$A = mr, \quad \text{with m as constant.} \qquad \ldots(4.13)$$

For these values of A and B the equations (4.4) to (4.6) generate the following results

$$\frac{1}{2}\mu h^2 = \frac{3\psi^2}{n^2}\left[\frac{1}{r^2}\right] - \frac{1}{r^2}. \qquad \ldots(4.14)$$

$$-\frac{1}{2}\mu h^2 = \frac{\psi^2}{n^2}\left[\frac{2\psi}{r\psi} + \frac{1}{r^2}\right]. \qquad \ldots(4.15)$$

$$(\rho + \mu h^2) = \frac{2}{n^2}\left[\frac{2\psi'}{2r\psi} + \frac{1}{r^2}\right] - \frac{1}{r^2}. \qquad \ldots(4.16)$$

Adding and subtracting (4.14) and (4.15) we get

$$\frac{2\psi^2}{n^2r^2} - \frac{1}{2r^2} + \frac{\psi\psi'}{n^2r} = 0. \qquad \ldots(4.17)$$

$$-2\mu h^2 = \frac{-4\psi^2}{n^2r^2} + \frac{2}{r^2} + \frac{4\psi\psi'}{n^2r},$$

$$\text{i.e. } \mu h^2 = \frac{2\psi^2}{n^2r^2} - \frac{1}{r^2} - \frac{2\psi\psi'}{n^2r}. \qquad \ldots(4.18)$$

Equations (4.17) and (4.18) imply

$$-2\rho = \frac{1}{r}\left[\frac{1}{r} - \frac{6\psi\psi'}{n^2}\right],$$

$$\text{i.e. } \rho = \frac{1}{r}\left[\frac{3\psi\psi'}{n^2} - \frac{1}{2r}\right]. \qquad \ldots(4.19)$$

To find the value of ψ^2

From equation (4.18) we write

$$2\psi\psi' + \frac{4\psi^2}{r} - \frac{n^2}{r} = 0. \qquad \ldots(4.20)$$

This provides an immediate integral in terms of integral constant C_1 in the form

$$\psi^2 = \frac{n^2}{4} + \frac{C_1}{4r^4}, \quad \ldots(4.21)$$

That implies that

$$2\psi\psi' = \frac{-C_1}{r^5}. \quad \ldots(4.22)$$

On substituting this value in (4.21) gives

$$\frac{C_1}{r^5} - \frac{4}{r}\left[\frac{n^2}{4} + \frac{C_1}{4r^4}\right] + \frac{n^2}{r} = 0. \quad \ldots(4.23)$$

Thus the static spherically symmetric spacetime model admitting a group of conformal motions corresponding RIM distribution is given by

$$ds^2 = m^2r^2dt^2 - 2n\left[n^2 + \frac{C_1}{r^4}\right]^{-\frac{1}{2}} dr^2 - r^2\,(d\theta^2 + \sin^2\theta d\Phi^2).$$

$$\ldots(4.24)$$

<u>Discussion</u> : We have successed in constructing a spacetime model compatible with RIM distribution admitting a group of conformal motions.

The value of parameter α describing the curvature inheritance is explicitly related with the value of ψ through equation (3.7). Therefore we infer that the above model also satisfies the CI symmetry property.

APPENDIX

A) THE VALUES OF KINEMATICAL PARAMETERS

i. Expansion Scalar (θ)

$$\theta = \nabla_a u^a \quad \text{(since by definition)} \qquad \ldots.(1)$$

$$\theta = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^a} (u^4 \sqrt{-g})$$

$$\theta = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial t} (u^4 \sqrt{-g}) \quad \text{(since only } u^4 \text{ is existing)}$$

$$\theta = \frac{1}{ABr^2} \frac{\partial}{\partial t} \left(\frac{1}{A} ABr^2 \right), \qquad \text{(Vide IV, 4.2)}$$

$$\theta = \frac{1}{ABr^2} \frac{\partial}{\partial t} (Br^2),$$

$$\theta = 0. \qquad \text{(Vide, IV 4.13)} \qquad (2)$$

2. Acceleration ($\overset{*}{u}{}^a$)

$$\overset{*}{u}{}^a = (\nabla_b u^a) u^b,$$

i.e. $\overset{*}{u}{}^a = (u^a{}_{,b} + u^c \Gamma^a_{cb}) u^b,$

i.e. $\overset{*}{u}{}^a = u^a{}_{b} u^b + u^c u^b \Gamma^a_{cb}.$

For $a = 1 \Longrightarrow$

$$\overset{*}{u}{}^1 = \nabla_b u^1 u^b + u^c u^b \Gamma^1_{cb},$$

i.e. $\overset{*}{u}^1 = 0 + \Gamma^1_{44}\, u^4 u^4$ (since only u^4 is existing)

i.e. $\overset{*}{u}^1 = 0 + \Gamma^1_{44}\, (u^4)^2$,

since $u^4 = \frac{1}{A}$ and $\Gamma^1_{44} = \frac{1}{2B^2}\frac{\partial}{\partial r}(A^2))$

we get, $$\overset{*}{u}^1 = \frac{1}{2B^2}\frac{\partial}{\partial r}(A^2)\left(\frac{1}{A}\right)^2$$

This implies

$$u^{*1} = \frac{A^1}{B^2 A}. \qquad \ldots.(3)$$

Also we get

$$u^{*2} = u^{*3} = u^{*4} = 0$$

(since only u^4 is existing and $u^{*4} = u^4{}_{,4}u^4 + \Gamma^4_{44}\, u^4 u^4$).

Therefore we get

$$\overset{*}{u}^a \overset{*}{u}_a = \overset{*}{u}^1 \overset{*}{u}_1 = g_{11}(\overset{*}{u}^1)^2$$

i.e. $$\overset{*}{u}^a \overset{*}{u}_a = -\frac{\psi^2}{n^2 r^2} \qquad \text{(Vide IV, 4.13, 4.14)} \qquad (4)$$

3) <u>Shear Tensor Components</u>

$$\sigma_{ab} = \tfrac{1}{2}[\nabla_b u_a + \nabla_a u_b] - \tfrac{1}{2}[(\nabla_c u_a)u^c u_b + u_a(\nabla_c u_b)u^c] - \frac{1}{3}\theta\, h_{ab}.$$

Since $\theta = 0$ and only u_4 is existing and $u^a u_a = 1$, $u'_{a,4} = 0$ we find

$$\sigma_{ab} = \tfrac{1}{2}\,[\nabla_b u_a + \nabla_a u_b] - u_4 \Gamma^4_{ab} + \tfrac{1}{2}\,[\,\Gamma^4_{a4}\, u_b + u_a \Gamma^4_{b4}]$$

Clearly

$$\sigma^a_a = 0$$

and $\sigma_{12} = 0$, ($u_1 = 0 = u_2$ and $\Gamma^4_{12} = 0$)

$\sigma_{13} = 0$, ($u_1 = 0 = u_3$ and $\Gamma^4_{13} = 0$)

$\sigma_{23} = 0$, ($u_2 = 0 = u_3$ and $\Gamma^4_{23} = 0$)

$\sigma_{24} = 0$, ($u_{4,3} = 0$, $u_3 = 0$, $\Gamma^4_{43} = 0 = \Gamma^4_{34}$)

$\sigma_{14} = \tfrac{1}{2}[u_4 - u_4\, \Gamma^4_{14}]$, (since $u_1 = 0$)

i.e. $\sigma_{14} = \tfrac{1}{2}\,[A' - A \dfrac{1}{2A^2}\,(2AA')]$, (since $u^4 = \dfrac{1}{A}$)

$\sigma_{14} = \tfrac{1}{2}\,(A' - A')$

$\sigma_{14} = 0.$

Hence $\sigma^2 = \sigma^{ab}\sigma_{ab} = 0.$ ....(5)

4. <u>Rotation Tensor Components</u>

$$w_{ab} = \tfrac{1}{2}[\nabla_b u^a - \nabla_a u^b] - \tfrac{1}{2}[(\nabla_c u_a)u^c u_b - u_a(\nabla_c u_b)u^c],$$

(since by definition).

By using $\Gamma^4_{ab} = \Gamma^4_{ba}$, $u_{a,4} = 0$, $u^a u_a = 1$.

and u^4 is existing we get

$$w_{ab} = \tfrac{1}{2}[u_{a,b} - u_{b,a}] - \tfrac{1}{2}[u_a \Gamma^4_{b4} - \Gamma^4_{a4} u_b]$$

$$w_{11} = 0$$

$$w_{12} = 0 \qquad \text{(since } u_1 = 0 = u_2\text{)}$$

$$w_{13} = 0 \qquad \text{(since } u_1 = 0 = u_3\text{)}$$

$$w_{23} = 0 \qquad \text{(since } u_1 = 0 = u_3\text{)}$$

$$w_{24} = 0 \qquad \text{(since } u_2 = 0 = u_{4,2} = 0,\ \Gamma^4_{24} = 0\text{)}$$

$$w_{43} = 0 \qquad u_{4,3} = 0,\ u_3 = 0,\ \Gamma^4_{34} = 0$$

$$w_{41} = \tfrac{1}{2}[u_{4,1} - u_4 \Gamma^4_{14}] \qquad \text{(since } u_{14} = 0\text{)}$$

i.e. $$w_{41} = \tfrac{1}{2}[A' - A \frac{1}{2A^2}(2AA')]$$

i.e. $$w_{41} = \tfrac{1}{2}(A' - A')$$

$$w_{41} = 0.$$

Thus we get

$$w^2 = w^{ab} w_{ab} = 0. \qquad \ldots.(6)$$

Further the relative anisotrophy

$$\Longrightarrow \frac{\sigma^2}{\rho} = \frac{0}{\rho} = 0 \Longrightarrow \sigma = 0 \qquad (7)$$

This proves that the flow of RIM distribution admitting conformal motion is essentially accelerating.

B) **THE VALUES OF DYNAMICAL VARIABLES**

We recall the equation (IV, 4.20)

$$\rho = \frac{1}{r}\left[\frac{3\psi\psi'}{n^2} - \frac{1}{2r}\right],$$

using the equation (IV, 4.23) this yields

$$\rho = \frac{1}{r}\left[-\frac{3C_1}{2n^2r^5} - \frac{1}{2r}\right]. \qquad \ldots.(8)$$

Recall the equation (IV, 4.19)

$$\mu h^2 = \frac{2\ ^2}{n^2r^2} - \frac{1}{r^2} - \frac{2\psi\psi'}{n^2r}$$

By using the equations (IV, 4.22 and 4.23) this yields

$$\mu h^2 = \frac{3C_1}{2n^2r^6} - \frac{1}{r^2}. \qquad \ldots.(9)$$