CHAPTER I

BASIC CONCEPTS AND SYSTEM EQUATIONS

1. DYNAMICAL IDENTITIES

The general idealisation of space-time demands that it will be filled with relativistic perfect fluid described by the well-known stress-energy tensor

$$T_{ab} = (\rho + p)\, u_a u_b - p g_{ab}. \qquad \ldots.(1.1)$$

Here ρ is the matter energy density, p is the isotropic pressure and u_a is unit flow vector ($u^a u_a = 1$).

An important case of this distribution is always considered by choosing $p = 0$. This distribution is called as Relativistic dust distribution, which has the stress energy tensor

$$T_{ab} = \rho\, u_a u_b. \qquad \ldots.(1.2)$$

The stress tensors exhibiting coupled fields occur at many places in the literature of General Relativity. Accordingly, the Relativistic magneto-hydrodynamical theory comprises the stress-energy tensor emboding the perfect fluid tensor and the electromagnetic field tensor with infinite electrical conductivity and constant magnetic permeability. This system is known as magnetofluid system and is introduced by Lichnerowicz in 1967.

It is given by the stress-energy tensor

$$T_{ab} = (\rho + p + \mu h^2)\, u_a u_b - (p + \tfrac{1}{2}\mu h^2) g_{ab} - \mu h_a h_b. \quad (1.3)$$

Here μ is the constant magnetic permeability and h^a is the magnetic field vector with magnitude field vector with magnitude $h^a h_a = -h^2$. Moreover it is orthogonal to flow vector u_a ($u_a h^a = 0$).

If this system is pressurefree then we call it as Relativistic Incoherent Magnetofluid distribution (RIM distribution), which will be given by the stress energy tensor

$$T_{ab} = (\rho + \mu h^2)\, u_a u_b - \tfrac{1}{2}\mu h^2 g_{ab} - \mu h_a h_b. \quad \ldots(1.4)$$

This form of stress energy tensor yields the following results.

$$T_{ab} u^a = (\rho + \tfrac{1}{2}\mu h^2) u_b, \quad \ldots(1.5)$$

$$T_{ab} u^a u^b = (\rho + \tfrac{1}{2}\mu h^2) u_b, \quad \ldots(1.6)$$

$T_{ab} h^a$

$$T_{ab} u^a = -\tfrac{1}{2}\mu h^2 u_b, \quad \ldots(1.7)$$

$$T_{ab} h^a h^b = -\tfrac{1}{2}\mu h^4, \quad \ldots(1.8)$$

$$T_{ab} u^a h^b = 0, \quad \ldots(1.9)$$

$$T_{ab} g^{ab} = T = \rho. \quad \ldots(1.10)$$

It follows from these results that

(i)u^a is the time-like eigen vector for the RIM distribution with the eigen value $e_1 = \rho + \frac{1}{2}\mu h^2$.

(ii)h^a is the space-like eigen vector with the eigen value $e_2 = -\frac{1}{2}\mu h^4$.

This implies that the stress energy tensor for RIM distribution has two distinct eigen values e_1 and e_2.

Also we have the rest mass for the RIM distribution as given by

$$T = \rho \qquad \ldots(1.11)$$

we recall the theorem given by Hawking and Ellis 1973, "Any stress-energy tensor with distinct eigen values has distinct eigen vectors which are orthogonal to each other".

We observe that this theorem is valid for RIM distribution given by equation (1.4).

2. ENERGY CONDITIONS

According to Hawking and Ellis, 1973, the stress-energy tensor has to satisfy the following energy conditions.

(i)Weak Energy Condition : This condition is described by the inequality

$$T_{ab}\, u^a u^b \geq 0 . \qquad \ldots.(2.1)$$

This for RIM distribution takes the form

$$(\rho + \tfrac{1}{2}\mu h^2) \geq 0 . \qquad \ldots.(2.2)$$

(ii) The Strong Energy Condition : This is given by the inequality

$$T_{ab}\, u^a u^b - \tfrac{1}{2} T \geq 0 , \qquad \ldots.(2.3)$$

i.e. $(\rho + \mu h^2) \geq 0$, (Vide 1.4). ....(2.4)

(iii) The Dominent Energy Condition : This is described by the inequalities

(1) $T_{ab}u^a u^b \geq 0$

and

(2) $T_{ab}u^a$ is non-space like vector.

These inequalities with (1.4) yield

$$\rho + \tfrac{1}{2}\mu h^2 \geq 0$$

and

$(\rho + \tfrac{1}{2}\mu h^2)u_b$ which is time-like vector.

The validity of above three energy conditions (i), (ii) and (iii) justify that the stress-energy tensor (1.4) is physically transparent.

3. GEOMETRICAL SYMMETRIES

We present below definitions leading to geometrical symmetries.

(a) A group of conformal motions :

According to Olivar and Davis 1977, a one parameter group of continuous infinitesimal transformations described by

$$\bar{x}^a = x^a + \xi^a \delta t, \qquad \ldots.(3.1)$$

is said to exhibit a group of conformal motions if

$$L_\xi g_{ab} = 2\psi g_{ab}, \qquad \ldots.(3.2)$$

where ψ is scalar function of space-time co-ordinates and L_ξ denotes the lie derivative along vector ξ.

Note (i) : If ψ = constant, then (3.2) will lead to a group of homothetic motions, where $\psi = 0$ leads to a group of motions (isometries).

Note (ii) : A conformal group of motions ~~is~~ given by (3.2) generates a group of special conformal motions provided $\nabla_{ab}\psi = 0$ and $\nabla_a \psi = 0$. Hence the special conformal vector ξ is a particular case of the conformal vector ξ

(b) The transformations (3.1) lead to curvature inheritance (CI) if

$$L_\xi R^a{}_{bcd} = 2\alpha R^a_{bcd}, \qquad \ldots.(3.3)$$

where α is the scalar function of co-ordinates.

<u>Note</u> : In particular if $\alpha = 0$, then equation (3.3) describes curvature collineation given by

$$L_{\xi} \; R^{a}_{bcd} = 0. \qquad \ldots(3.4)$$

(c) <u>Spherical Symmetry</u> : The well-known geometrical symmetry known as spherical symmetry has a standard form of the fundamental quadratic metric as

$$ds^2 = A^2(r)dt^2 - B^2(r)dr^2 - r^2(d\theta^2+\sin^2\theta d\Phi^2). \qquad \ldots(3.5)$$

We have from equation (3.5)

$$g_{ab} = \begin{bmatrix} -B^2(r) & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2\sin^2\theta & 0 \\ 0 & 0 & 0 & A^2(r) \end{bmatrix}. \qquad \ldots(3.6)$$

For the choice of comoving frame the spherically symmetry yields

$$\left.\begin{aligned} u^a &= (0,\ 0,\ 0,\ u^4) \\ \text{and}\quad h^a &= (h^1,\ 0,\ 0,\ 0) \end{aligned}\right]. \qquad \ldots(3.7)$$

4. <u>FIELD EQUATIONS OF GRAVITATION</u>

The geometry and dynamics of the space-time is coupled with well-known Einstein's field equations through the tensor equation

$$R_{ab} - \tfrac{1}{2}\, Rg_{ab} = -k\, T_{ab}. \qquad \ldots(4.1)$$

Here the Right hand side describes the dynamics through the stress energy tensor of the RIM distribution (Vide, 1.4). The Left hand side involving Ricci tensor and Ricci scalar fully describes the geometry of the space-time with the signature (-2). We know that (4.1) are sixteen non-linear partial differential equations of order two. From (4.1) we can write

$$R = kT. \qquad \ldots(4.2)$$

This with (4.1) produces

$$R_{ab} = -k(T_{ab} - \tfrac{1}{2} Tg_{ab}). \qquad \ldots(4.3)$$

In particular if $k = -1$, then

$$R_{ab} = (T_{ab} - \tfrac{1}{2} Tg_{ab}). \qquad \ldots(4.4)$$

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for the RIM distribution (1.4), this yields

$$R_{ab} = (\rho + \mu h^2)u_a u_b - \tfrac{1}{2}(\rho + \mu h^2)g_{ab} - \mu h_a h_b. \qquad \ldots(4.5)$$

This implies

$$g^{ab}R_{ab} = R = -\rho, \qquad \ldots(4.6)$$

$$u^a u^b R_{ab} = \tfrac{1}{2}(\rho + \mu h^2), \qquad \ldots(4.7)$$

$$h^a h^b R_{ab} = \tfrac{1}{2}(\rho - \mu h^2)h^2, \qquad \ldots(4.8)$$

$$u^a h^b R_{ab} = 0. \qquad \ldots(4.9)$$

Note : Throughout the dissertation we use ∇ to represent covariant derivative and over head star (*) for covariant derivative along frow vector $\bar{u}$.

5. MAXWELL EQUATIONS

Further the magnetic field comprised in RIM distribution (1.4) satisfies the following Maxwell equations (Lichnerowicz, 1967).

$$\nabla_b(u^a h^b - u^b h^a) = 0 . \qquad ....(5.1)$$

This leads to the following results.

$$\overset{*}{u}{}^a h_a + \nabla_b h^b = 0. \qquad ....(5.2)$$

and

$$\tfrac{1}{2}\overset{*}{h}{}^2 + h^2 \theta + (\nabla_b u^a)\, h_a h^b = 0 \qquad ....(5.3)$$

Here we have the conventions

$$\overset{*}{u}_a = (\nabla_b u^a) u^b \quad : \text{ acceleration} \qquad ....(5.4)$$

$$\theta = \nabla_a u^a \quad : \text{ Expansion of } \bar{u} \qquad ....(5.5)$$

and

$$\overset{*}{h}{}^2 = (\nabla_b h^2) u^b \quad : \text{ Variation along } \bar{u} \qquad ....(5.6)$$

6. DIFFERENTIAL IDENTITIES

The conservation laws of energy and momentum are implied by Bianchi Identities as

$$\nabla_b T^{ab} = 0 . \qquad ....(6.1)$$

This with equation (1.4) provides

$$\left[\nabla_b(\rho + uh^2)\right] u^a u^b + (\rho + \mu h^2)(\nabla_b u^a)u^b +$$

$$+ (\rho + \mu h^2)u^a(\nabla_b u^b) - \tfrac{1}{2}\mu(\nabla_b h^2)g^{ab} - \mu(\nabla_b h^a)h^b -$$

$$- \mu h^a(\nabla_b h^b) = 0. \qquad \ldots.(6.2)$$

This when contracted with u_a, we get

$$(\nabla_b \rho)u^b + \rho\theta + \mu[\tfrac{1}{2}\overset{*}{h}{}^2 + h^2\theta + (\nabla_b u^a)h_a h^b] = 0, \quad ..(6.3)$$

i.e. $(\nabla_b \rho)u^b + \rho\theta = 0,$ (vide, 5.3) ....(6.4)

This is the continuity equation for RIM distribution. This exhibits the rate of change of matter density along the flow.

Further if we transvect equation (6.2) with h_a, we get

$$\rho \overset{*}{u}{}^a h_a + h^2\left[\mu(\overset{*}{u}{}^a h_a + \nabla_b h^b)\right] = 0 \qquad \ldots.(6.5)$$

This yields

$$\rho\, \overset{*}{u}{}^a h_a = 0, \qquad \text{(vide, 5.2)},$$

i.e. $\overset{*}{u}{}^a h_a = 0$, since $\rho \neq 0$. ....(6.6)

It follows from equations (5.2) and (6.6) that

$$\nabla_b(h^b) = 0. \qquad \ldots.(6.7)$$

By rearranging the terms in (6.2) and using the continuity equation (6.4) we get

$$(\rho + \mu h^2)\overset{*}{u}^a + \mu(\overset{*}{h}^2 + h^2\theta)u^a - $$
$$- \mu\left[\tfrac{1}{2}(\nabla_b h^2)g^{ab} - (\nabla_b h^a)h^b\right] = 0. \qquad \ldots.(6.8)$$

This represents the system of stream lines for RIM distribution which explains the deviation of actual path of fluid trajectories from geodesic path. It appears from (6.8) that this deviation is caused only because of the magnetic field.

Conclusions : For the RIM distribution we observe that

(i) the matter density is conserved along the flow lines iff they are expansionfree. (vide, 6.4).

(ii) the acceleration is normal to the divergencefree magnetic lines. [Vide (6.6) and (6.7)].

7. HOMOGENEOUS MAGNETIC FIELD

The homogeneous magnetic field is characterised by

$$\nabla_b h_a = 0. \qquad \ldots.(7.1)$$

This implies that

$$\overset{*}{u}_a h_a = -\overset{*}{h}_a u^a = 0, \qquad \ldots.(7.2)$$

$$\overset{*}{h}^2 = 0. \qquad \ldots.(7.3)$$

Consequently equations (5.3) and (5.4) produce

$$\nabla_b h^b = 0, \qquad \ldots\ldots(7.4)$$

$$\theta = 0. \qquad \ldots\ldots(7.5)$$

Similarly equations (6.4) and (6.8) generate the following results.

$$(\nabla_b \rho) u^b = 0 . \qquad \ldots\ldots(7.6)$$

$$\overset{*}{u}^a = 0, \text{ since } \rho + \mu h^2 \neq 0. \qquad \ldots\ldots(7.7)$$

Thus for RIM distribution with the homogeneous magnetic field we have

(i) Fluid flow lines are expansionfree. [Vide, (7.5)].

(ii) Magnetic lines are divergencefree. [Vide, (7.2)].

(iii) The matter energy density is conserved along geodesic path. [Vide equations (7.6) and (7.7)].