CHAPTER 4

CHAPTER 4 FUZZY RELATION EQUATIONS ON UNION AND INTERSECTION PRESERVING OPERATORS

In this chapter we study union and intersection preserving operators defined on I and discuss fuzzy relation equations based on these operators.

4.1 UNION AND INTERSECTION PRESERVING OPERATORS:

Definition 4.1.1 [L]: A binary operation * on I is said to be union preserving operator if

1) $a * 0 = 0$

2) $a * (\sup_{i \in I} b_i) = \sup_{i \in I} (a * b_i), \forall\, a, b_i \in I.$

Definition 4.1.2[L]: A binary operation • on I is said to be intersection preserving operator if

1) $a \bullet 1 = 1$

2) $a \bullet (\inf_{i \in I} b_i) = \inf_{i \in I} (a \bullet b) \; \forall\, a, b_i \in I.$

Theorem 4.1.3 [L]: The union preserving operator * is an order preserving operator.

Proof: Let $a, b, c \in I$ and $b \le c$. Then $\mathrm{Sup}\{b, c\} = c$

Now $a * \sup\{b, c\} = \sup\{a * b, a * c\}$

i. e. $a * c = \sup\{a * b, a * c\}$

Therefore, $a * c \ge a * b$.

Theorem 4.1.4[L]: The intersection preserving operator • is an order preserving operator.

Proof: Let a, b, c $\in$ I and b $\leq$ c. Then inf{b, c} = b

Now a $\bullet$ inf{b, c}= inf{a $\bullet$ b, a $\bullet$ c}

i. e. a $\bullet$ b = inf{a $\bullet$ b,a $\bullet$ c}

Therefore, a $\bullet$ b $\leq$ a $\bullet$ c.

Theorem 4.1.5 [L]: If * is an union preserving operator, then there exists a unique intersection preserving operator $\bullet$ such that a * b $\leq$ c if and only if a $\bullet$ c $\geq$ b

Proof: Define a mapping *: I x I $\rightarrow$ I as follows

$$a \bullet c = \sup_{x \in B} \{x\}, \text{ where } B = \{ x \in I \mid a * x \leq c\}$$

Since a * 0 $\leq$ 0, a * 0 $\leq$ c, for all c $\in$ I

Therefore, 0 $\in$ B

i. e. B $\neq \phi$.

Therefore, B is a bounded subset of [0, 1] bounded by 0 and 1.

Hence, $\bullet$ is well defined.

Let a * b $\leq$ c. Then b $\in$ B

Therefore, $a \bullet c = \sup_{x \in B} \{x\} \geq b$

Thus, a $\bullet$ c $\geq$ b

Conversely let a $\bullet$ c $\geq$ b. Then a * b $\leq$ a * (a $\bullet$ c)

$$\begin{aligned} \text{Now } a * (a \bullet c) &= a * \sup_{x \in B} \{x\} \\ &= \sup_{x \in B} (a * x) \\ &\leq c \end{aligned}$$

Hence, a * (a $\bullet$ c) $\leq$ c

Therefore, a * b $\leq$ c if and only a $\bullet$ c $\geq$ b

Since $a * 1 \leq 1$, $1 \in B$

Therefore, $a * 1 \leq 1 \Rightarrow a \bullet 1 \geq 1$

Hence $a \bullet 1 = 1$

Next, we claim that,

$$a \bullet (\inf_{i \in I} b_i) = \inf_{i \in I}(a \bullet b_i)$$

Now $a \bullet (\inf_{i \in I} b_i) = a \bullet (\inf_{i \in I} b_i)$

Therefore, $a * (a \bullet \inf_{i \in I}\{ b_i\}) \leq \inf_{i \in I} \{b_i\} \leq b_i, \forall i \in I$

iff $a \bullet b_i \geq a \bullet (\inf_{i \in I} b_i)$

iff $\inf_{i \in I}(a \bullet b_i) \geq a \bullet (\inf_{i \in I} b_i)$

Now $\inf_{i \in I}(a \bullet b_i) \leq a \bullet b_i, \forall i \in I$

Therefore, $a * \inf_{i \in I}(a \bullet b_i) \leq b_i, \forall i \in I$.

iff $a * (\inf_{i \in I} (a \bullet b_i)) \leq \inf_{i \in I} \{ b_i\}$

iff $a \bullet \inf_{i \in I}\{ b_i\} \geq \inf_{i \in I} (a \bullet b_i)$

Hence, $a \bullet \inf_{i \in I}\{ b_i\} = \inf_{i \in I} (a \bullet b_i)$

Uniqueness: Since $a * b = a * b$, $a \bullet (a * b) \geq b$

Similarly $a * (a \bullet c) \leq c$

Suppose $\bullet_1$ is any other operator satisfying $a * b \leq c$ if and only if $a \bullet_1 c \geq b$.

Now $a * (a \bullet_1 c) \leq c$

Therefore, $a \bullet c \geq a \bullet_1 c$

Also $a * (a \bullet c) \leq c$

Hence, $a \bullet_1 c \geq a \bullet c$

Therefore, $a \bullet c = a \bullet_1 c$.

Similarly following result holds.

Theorem 4.1.6 [L]: If $\bullet$ is an intersection preserving operator, then there exists a unique union preserving operator * satisfying $a * b \leq c$ if and only if $a \bullet c \geq b$, $\forall$ a, b, c, $\in$ I.

Proof: Define a mapping $*: I \times I \to I$ as follows

$$a * b = \inf_{x \in C} \{x\}, \text{ where } C = \{x \mid a \bullet x \geq b\}.$$

Theorem 4.1.7 [L]: Let a, c $\in$ I. If the equation $a * x = c$ has at least one solution, then there exists an intersection preserving operator $\bullet$ such that $a \bullet c$ is the maximum solution of $a * x = c$.

Proof: Let x be a solution of the equation $a * x = c$. Then $a * x \leq c$.

Therefore, by above Theorem 4.1.5, there exists an intersection preserving operator $\bullet$ such that $a \bullet c \geq x$

Then $a * x \leq a * (a \bullet c)$

Therefore, $c \leq a * (a \bullet c) \leq c$

Hence, $a * (a \bullet c) = c$

Let $d \in I$ be a solution of $a * x = c$.

Then $a * d \leq c$

Therefore, $(a \bullet c) \geq d$.

Similarly we prove the following:

Theorem 4.1.8 [L]: Let $a, b \in I$. If the equation $a \bullet x = b$ has at least one solution, then there exists an union preserving operator $*$ such that $a * b$ is the minimum solution of $a \bullet x = b$.

Definition 4.1.9 [L]: Operators $*$ and $\bullet$ defined in the above Theorems 4.1.5 are called inverse operator of each other.

Theorem 4.1.10 [L]: If the equation $a * x = c$ has the minimum solution, $a \mathbin{\bar{\bullet}} c$, then following hold.

i) $a * (a \mathbin{\bar{\bullet}} c) = c$

ii) $a \bullet c \geq a \mathbin{\bar{\bullet}} c$

iii) $a \mathbin{\bar{\bullet}} (a * c) \leq c$

Proof: We only prove (iii)

iii) Since $a \mathbin{\bar{\bullet}} c$ is a minimum solution of $a * x = c$, $a \mathbin{\bar{\bullet}} (a * c)$ is a minimum solution of $a * x = a * c$

Also c is a solution of $a * x = a * c$

Hence, $a \mathbin{\bar{\bullet}} (a * c) \leq c$.

Similarly we prove the following:

Theorem 4.1.11 [L]: If the equation $a \bullet x = b$ has the maximum solution, $a \mathbin{\bar{*}} b$, then following hold.

i) $a \bullet (a \mathbin{\bar{*}} b) = b$

ii) $a * b \leq a \mathbin{\bar{*}} b$

iii) $a \mathbin{\bar{*}} (a \bullet b) \geq b$.

4.2 FUZZY RELATION EQUATIONS WITH UNION PRESERVING OPERATOR:

Definition 4.2.1: Let P(X, Y) and Q(Y, Z) be fuzzy relations. Let * be an union preserving operator. Then * - sup composition of P and Q is a fuzzy relation, R(X, Z), defined as follows:

$$P * Q\,(x, z) = \sup_{y \in Y} \{Q(x, y) * P(z, y)\}$$

Definition 4.2.2: Let P(X, Y), Q(Y, Z) and R(X, Z) be fuzzy relations. Then the equation P * Q = R is called * - sup fuzzy relation equation or fuzzy relation equation with union preserving operator *.

This problem, P * Q = R, can be partitioned into a set of simpler problems $p_i * Q = r_i$, $\forall$ i.

If $p = (p_1, p_2, \ldots, p_n)$, $Q = (q_{ij})_{n \times m}$ and $r = (r_1, r_2, \ldots, r_m)$. Let * be an union preserving operator. Then $r_j = \sup_{i}(q_{ij} * p_i)$

Let $p = (p_1, p_2, \ldots, p_n)$ and $r = (r_1, r_2, \ldots, r_n)$. Then $p \le r$ if and only if $p_i \le r_i$, for all i.

We will discuss above equation when Q and r are given.

Theorem 4.2.3 [L]: There exists a solution to fuzzy relation equation p * Q = r if and only if $t * Q \ge r$, where $t = (t_1, t_2, \ldots, t_n)$ and $t_i = \inf_{j}\{q_{ij} \bullet r_j\}$, • is inverse operator of *.

Proof: Suppose $t * Q \ge r$, where $t = (t_1, t_2, \ldots, t_n)$ and $t_i = \inf_{j}\{q_{ij} \bullet r_j\}$, • is inverse operator of *.

Now $\sup_{i}(q_{ij} * t_i) = \sup_{i}(q_{ij} * \inf\{q_{ik} \bullet r_k\})$

$\leq \sup_i (q_{ij} * \{q_{ij} \bullet r_j\})$

$\leq r_j$

Thus, $\sup_i (q_{ij} * t_i) \leq r_j, j = 1, 2, 3,..., m$

But $t * Q \geq r$. Therefore, $\sup_i (q_{ij} * t_i) \geq r_j$

Hence, $\sup_i (q_{ij} * t_i) = r_j, j = 1, 2, ...m$

Thus, $t * Q = r$

i. e. t is a solution of equation $p * Q = r$.

Conversely suppose that t is a solution of equation $p * Q = r$.

Therefore, $t * Q = r$

i. e. $\sup_i (q_{ij} * t_i) = r_j, j = 1, 2, ...m$.

Thus, $\sup_i (q_{ij} * t_i) \geq r_j, j = 1, 2, ...m$

i. e. $t * Q \geq r$

Theorem 4.2.4 [L]: Given Q and r if we define $t_i = \inf_j (q_{ij} \bullet r_j) \; \forall \; i = 1, 2, \ldots n$, then

$t = (t_1, t_2, \ldots, t_n)$ is the maximum solution of the equation $p * Q = r$.

Proof: Let $u = (u_1, u_2, \ldots, u_n)$ be a solution of $p * Q = r$

Then $r_j = \sup_i (q_{ij} * u_i), \forall j$

Now $t_i = \inf_j (q_{ij} \bullet r_j)$

$= \inf_j [q_{ij} \bullet \sup_k (q_{kj} * u_k)]$

$\geq \inf_j [q_{ij} \bullet (q_{ij} * u_i)]$

$\geq \inf_j u_i$

$\geq u_i$

Therefore, $t_i \geq u_i$, for all i = 1,2,3,...n.

Thus, $t \geq u$

Let us assume that there exists solutions to the equation $p * Q = r$ so that $t = (t_1, t_2, \ldots, t_n\}$ with $t_i = \inf_j (q_{ij} \bullet r_j)$ $\forall$ i = 1, 2, ... n, is the maximum solution . To obtain minimal solution of $p * Q = r$ for p, we shall introduce a matrix $D = (d_{ij})_{n \times m}$ as follows:

$$d_{ij} = \begin{cases} q_{ij} \,\bar{\bullet}\, r_j, & \text{if } q_{ij} * t_i = r_j \\ 0, & \text{otherwise} \end{cases}$$

Here $a \,\bar{\bullet}\, b$ is the minimum solution of $a * x = b$.

We assume that the minimum solution of $a * x = b$ exists.

Theorem 4.2.5 [L]: Define $D^* = (d_{ij}^*)_{n \times m}$ is a submatrix of matrix D, where $d_{ij}^* = d_{ij}$ or o. If $\sup_i d_{ij}^* > 0$, for $r_j \neq 0$, j = 1, 2, ..., m, then $A = (a_1, a_2, \ldots, a_n)$ where

$a_i = \sup_j d_{ij}^*$, i = 1, 2, 3, ... , n is a solution of the equation $p * Q = r$.

Proof: Let $r_j \neq 0$. Then there exists i_o such that,

$d_{ioj}^* = d_{ioj} > 0$

Now $\sup_i (q_{ij} * a_i) \geq q_{ioj} * a_{io}$

$\geq q_{ioj} * \sup_i d_{ioj}^*$

$\geq q_{ioj} * d_{ioj}^*$

$= q_{ioj} * (q_{ioj} \,\bar{\bullet}\, r_j)$

$= r_j$

Now $D = (d_{ij})_{n \times m}$

$$d_{ij} = \begin{cases} q_{ij} \,\bar{\bullet}\, r_j, & \text{if } q_{ij} * t_i = r_j \\ 0, & \text{otherwise} \end{cases}$$

Therefore, $q_{ij} \,\bar{\bullet}\, r_j = q_{ij} \,\bar{\bullet}\, (q_{ij} * t_i) \le t_i$

Hence, $a_i = \sup_j d_{ij}^* = \sup_j d_{ij}$

$= \sup_j (q_{ij} \,\bar{\bullet}\, r_j)$

$= \sup_j t_i$

$= t_i$

Thus, $\sup_i (q_{ij} * a_i) \le \sup_i (q_{ij} * t_i) = r_j, j = 1, 2, \ldots, m$

i. e. $\sup_i (q_{ij} * a_i) \le r_j, j = 1, 2, 3, \ldots, m$

Hence, $A = (a_1, a_2, \ldots, a_n)$ is a solution of $p * Q = r$

Theorem 4.2.6 [L]: If $A = (a_1, a_2, \ldots, a_n)$ is a solution of equation $p * Q = r$, then $D(A) = (d(A)_{ij})_{n \times m}$, is a submatrix of D and $\sup_i d_{ij}^* > 0$ for $r_j \neq 0, j = 1, 2, 3, \ldots, m$, where

$$d(A)_{ij} = \begin{cases} d_{ij}, & q_{ij} * a_i = r_j \\ 0, & \text{otherwise} \end{cases}$$

Proof: Since $t = (t_1, t_2, \ldots, t_n)$ is the maximum solution, $a_i \le t_i, i = 1, 2, 3, \ldots, n$

Hence, $q_{ij} * t_i \ge q_{ij} * a_i$

i. e. $q_{ij} * a_i \ge r_j$

Now $q_{ij} * t_i = q_{ij} * (\inf_k (r_{ik} \,\bar{*}\, r_k))$

$\le q_{ij} * (q_{ij} \,\bar{*}\, r_j)$

$\leq \; r_j$

Thus, $q_{ij} * t_i = r_j$

Hence, $q_{ij} * t_i = r_j$

Therefore, D(A) is a submatrix of D

If $A = (a_1, a_2, \ldots, a_n)$ is a solution of $p * Q = r$, then $\sup_i (r_{ij} * a_i) = r_j$, $j = 1, 2, 3, \ldots, m$

For $r_j \neq 0$, there exists i_o such that, $r_{i_{oj}} * a_{i_{oj}} = r_j$

Therefore, $d(A)_{i_{oj}} \geq 0$

Hence, $\sup_i d(A)_{ij} > 0$, $j = 1, 2, 3, \ldots, m$

Theorem 4.2.7 [L]: $\sup_j d(A)_{ij} \leq a_i$

Proof:

$$d_{ij} = \begin{cases} q_{ij} \,\bar{\bullet}\, r_j, & \text{if } q_{ij} * t_i = r_j \\ 0, & \text{otherwise} \end{cases}$$

Also

$$d(A)_{ij} = \begin{cases} d_{ij}, & q_{ij} * a_i = r_j \\ 0, & \text{othewise} \end{cases}$$

Therefore, $d(A)_{ij} = q_{ij} \,\bar{\bullet}\, r_j$

Since $A = (a_1, a_2, \ldots, a_n)$ is a solution, $d_{ij}^* = d_{ij}$ or 0 with $a_i = \sup_i d_{ij}^*$

Therefore, $d(A)_{ij} = q_{ij} \,\bar{\bullet}\, r_j = q_{ij} \,\bar{\bullet}\, (q_{ij} * a_i) \leq a_i$, for $r_j \neq 0$

Hence, $\sup_i d(A)_{ij} \leq a_i$, for $r = 1, 2, 3, \ldots, n$

Method for calculating all minimal solutions [L]

Step1: Write the formula P of the matrix by

$$P = \Pi \left(\sum_{i=1}^{n} d_{ij}\right), \text{ product over } j,\ r_j \neq 0.$$

(If $d_{ij} = 0$, then omit it)

Here, 'Σ' indicates the logical 'or' and 'Π' indicates the logical 'and'

Then calculate P in Σ form according to polynomial multiplication as:

$P_1 = \Sigma\, d_{k1}l_1\, d_{k2}l_2 \ldots d_{kr}l_r$

Step 2: 'Multiplicate' for all terms

$$dpq \,.\, drs = \begin{cases} \max(dpq, drs) & r = p \\ \text{unchanged} & \text{otherwise} \end{cases},$$

until the term $d_{k1l1}\, d_{k2l2} \ldots d_{krlr}$ satisfies $ki \neq kj$ for $i \neq j$

Step 3: 'Sum' among terms

$d_1r_1\, d_2r_2 \ldots d_nr_n - d_1s_1\, d_2s_2 \ldots\ldots d_ns_n$

$$= \begin{cases} d_1r_1\, d_2r_2 \ldots d_nr_n,\ d_kr_k \leq d_ks_k,\ k = 1, \ldots, n \\ \text{unchanged}, \quad \text{otherwise} \end{cases}$$

All the minimal solutions are given by $t^* = (t_1^*, t_2^*, \ldots, t_n^*)$, where $t_k^* = d_kr_k$,

$k = 1, 2, 3, \ldots, n$

4.3 FUZZY RELATION EQUATIONS WITH INTERSECTION PRESERVING OPERATOR

Definition 4.3.1: Let P(X, Y) and Q(Y, Z) be fuzzy relations. Let $\bullet$ be an intersection preserving operator. Then $\bullet$ - inf composition of P and Q is a fuzzy relation, R(X, Z), defined as follows:

$$P \bullet Q\,(x, z) = \inf_{y \in Y} \{Q(x, y) \bullet P(z, y)\}$$

Definition 4.3.2: Let P(X, Y), Q(Y, Z) and R(X, Z) be fuzzy relations. Then the equation P • Q = R is called • - inf fuzzy relation equation or fuzzy relation equation with intersection preserving operator •.

This problem, P • Q = R, can be partitioned into a set of simpler problems $p_i \bullet Q = r_i$, ∀ i.

If $p = (p_1, p_2, \ldots, p_n)$, $Q = (q_{ij})_{n \times m}$ and $r = (r_1, r_2, \ldots, r_m)$. Let • be an intersection preserving operator. Then $r_j = \inf_{i} (q_{ij} \bullet p_i)$, for j = 1, 2, …, m

We will discuss above equation when Q and r are given and give the method for obtaining maximal solutions.

The proofs of the following theorems 4.3.3 to 4.3.7 can be obtained dually from Theorem 4.2.3 to Theorem 4.2.7 respectively.

Theorem 4.3.3 [L]: There exists a solution to fuzzy relation equation p • Q = r if and only if $t \bullet Q \leq r$, where $t = (t_1, t_2, \ldots, t_n)$ and $t_i = \sup_{j}\{q_{ij} * r_j\}$, * is inverse operator of •.

Theorem 4.3.4 [L]: Given Q and r if we define $t_i = \sup_{j} (q_{ij} * r_j)$, ∀ i = 1, 2, … n, then $t = (t_1, t_2, \ldots, t_n)$ is the minimum solution of the equation p • Q = r.

Let us assume that there exists solutions to the equation p • Q = r so that $T = (t_1, t_2, \ldots, t_n\}$ with $t_i = \sup_{j} (q_{ij} * r_j)$, ∀ i = 1, 2, … n, is the minimum solution . To obtain maximal solutions of p • Q = r for p, we shall introduce a matrix $D = (d_{ij})_{n \times m}$ as follows:

$$d_{ij} = \begin{cases} q_{ij} \bar{*} r_j, & \text{if } q_{ij} \bullet t_i = r_j \\ 1, & \text{otherwise} \end{cases}$$

Here $a \bar{*} b$ is the minimum solution of $a \bullet x = b$.

We assume that the maximum solution of $a \bullet x = b$ exists.

Theorem 4.3.5 [L]: If $D^* = (d_{ij}^*)_{n \times m}$ is a submatrix of matrix D, where $d_{ij}^* = d_{ij}$ or 1, such that $\inf_i d^*_{ij} < 1$, for $r_j \neq 1$; $j = 1, 2, ..., m$. Then $A = (a_1, a_2, \ldots, a_n)$ with $a_i = \inf_j d^*_{ij}$, $i = 1, 2, 3, \ldots, n$ is a solution of the equation $p \bullet Q = r$.

Theorem 4.3.6 [L]: If $A = (a_1, a_2, \ldots, a_n)$ is a solution of equation $p \bullet Q = r$, then $D(A) = (d(A)_{ij})_{n \times m}$, is a submatrix of D and $\inf_i d^*_{ij} < 1$, for $r_j \neq 1, j = 1, 2, 3, ..., m$, where

$$d(A)_{ij} = \begin{cases} d_{ij}, & q_{ij} \bullet a_i = r_j \\ 1, & \text{othewise} \end{cases}$$

Theorem 4.3.7 [L]: $\inf_i d(A)_{ij} \geq a_i$.

Method for calculating all maximal solutions [L]

Step 1: Write down the formula P of a matrix D as

$$P = \Pi \left(\sum_{i=1}^{n} d_{ij}\right), \text{ product over } j, r_j \neq 0.$$

(If $d_{ij} = 1$ then omit it).

Calculate P in Σ from according to polynomial multiplication :

$$P = \Sigma\, dk_{1l1}\, dk_{2l2} \ldots\ldots dk_{rlr}.$$

Step 2: 'Multiplicate' for all terms.

$$\text{dpq drs} = \begin{cases} \min(\text{dp}, \text{drs}) & r = p \\ \text{unchanged} & r \neq p \end{cases}$$

until the term $dk_1l_1\ dk_2l_2 \ldots\ldots dk_rl_r$. satisfies $k_i \neq k_j,\ i \neq j$.

Step 3: 'Sum' among terms

$d_1r_1\ d_2r_2\ d_nr_n + d_1s_1\ d_2s_2\ d_ns_n$

$$= \begin{cases} d_1r_1\ d_2r_2 \ldots d_nr_n ; d_kr_k > d_ks_k\ ;\ k = 1; n \\ \text{unchanged} \qquad\qquad \text{otherwise} \end{cases}$$

All maximal solutions are

$T^* = (t_1^*, t_2^*, \ldots\ldots, t_n^*)$

where $t_k^* = d_{k\,rk}^*$; $k = 1,2,\ldots.,n$.