CHAPTER-2

Chapter-2

THE DISCRETE FOURIER TRANSFORM

Introductin:

In this chapter we develop a special case of the continuous F.T. which is amenable to machine computation. The approach will be to develop the discrete F.T. from a graphical derivation based on continuous F.T. theory followed by theoretical development.

(2.1) A Graphical development

For the given function f(t) and its F.T. F(w) the modified F.T. pair will be in such a manner that the pair is ameanable to digital computer computation is called the discrete F.T.

To determine the F.T. of f(t) by means of digital analysis techniques, it is necessary to sample f(t). If it is sampled at a frequency of at least twice the largest frequency component of f(t), there is no loss of information as a result of sampling. If $F(w) \neq 0$ for some $|w| > w_k$ then sampling will produce overlapping. To avoid this error T should be small.

For machine computation it is necessary to truncate the sampled function f(t) so that only finite number of points, say N are considered. Also both the time and frequency domains are represented by discrete values. These N samples define the discrete F.T. pair and approximate the original F.T. pair. Hence, the discrete F.T. requires that both the original time and frequency functions be modified such that they become periodic functions and N time samples and N frequency values represent one period of the time and frequency domain-waveforms.

(2.2) Theoretical Development

The graphical development as explained above modify the continuous Fourier transform such that it is acceptable for machine computation. Now it is necessary to derive the mathematical relationships giving the modifications.

Let us consider f(t) and F(w), the F.T. pair. First it is necessary to sample f(t).

Let

$$\Delta_o(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$$ be the sampling function with

Sampling interval T. Therefore the sampled function can be written as

$$f(t)\Delta_o(t) = f(t)\sum_{k=-\infty}^{\infty} \delta(t-kT)$$

$$= \sum_{k=-\infty}^{\infty} f(kT)\delta(t-kT) \qquad \text{-------(2.2.1)}$$

If due to choice of T the aliasing occurs then we use the truncation function (rectangular function)

$$X(t) = 1 \text{ , } -T/2 < t < T_o - T/2$$

$$= 0 \quad \text{otherwise} \qquad \text{-------(2.2.2)}$$

where T_o is the duration of the truncation function. Especially it is designed to avoid the time domain aliasing. Thus the truncation gives.

$$[f(t)\Delta_o(t)]x(t) = [\sum_{k=-\infty}^{\infty} f(kT)\ \delta(t-kT)]x(t)$$

$$= \sum_{k=0}^{N-1} f(kT)\delta(t-kT) \qquad \text{-------(2.2.3)}$$

Where it has been assumed that there are N equidistant impulse functions lying within the truncation function i.e. $N = T_o/T$ (T_o is the length of rectangular function). This truncation in time domain may results in rippling in the frequency domain. Therefore it is necessary to sample the F.T. of equation (2.2.3). In time domain it is equivalent to the convolution of equation (2.2.3) with sampling function $\Delta_1(t)$ where

$$\Delta_1(t) = T_o \sum_{r=-\infty}^{\infty} \delta(t-rTo) \qquad \text{-------(2.2.4)}$$

Convolution gives,

$$[f(t).\Delta_o(t).x(t)] * \Delta_1(t) =$$

$$= [\sum_{k=o}^{N-1} f(kT)\delta(t-kT)] * [To\sum_{r=-\infty}^{\infty}\delta(t-rT_o)]$$

$$= . . . . . +T_o\sum_{k=o}^{N-1} f(kT)\delta(t+To-kT) \quad +T_o\sum_{k=o}^{N-1} f(kT)\delta(t-kT)+$$

$$+. . . \qquad \text{-------}(2.2.5)$$

$$f^\wedge(t) = T_0\sum_{r=-\infty}^{\infty}\sum_{k=o}^{N-1} f(kT)\delta(t-T_o-kT) \qquad \text{-------}(2.2.6)$$

Here f^(t) is the approximation of f(t) from equation (2.2.6) it is clear that if the end points of the truncation function coincides with the sample values, the convolution in (2.2.6) would result in time domain aliasing, that is, N^{th} point of one period would coincide with the first point of the next period. Hence the end points of the truncation function should lie at the mid-point of two adjacent sample values.

To develop the F.T. of equation (2.2.6) recall that the F.T. of periodic function is a equidistant impulses as

$$F^\wedge(n/T_o) = \sum_{n=-\infty}^{\infty} A_n\ \delta(w-nw_o), \qquad W_o=1/T_o \qquad \text{-------}(2.2.7)$$

Where $A_n = \frac{1}{T_o} \int_{-T/2}^{To-T/2} \hat{f}(t)\ e^{-in\pi t/To}\, dt,$

$$n=0,\pm1,\pm2,\pm3,... \qquad \text{-------}(2.2.8)$$

Using (2.2.6) in (2.2.8), we get

$$A_n = \frac{1}{T_o} \int_{-T/2}^{To-T/2} T_o \sum_{r=-\infty}^{\infty} \sum_{k=0}^{N-1} f(kT)\delta(t-kT-rT_o)e^{-in\pi t/To}\, dt,$$

Since integration is over only one period,

$$A_n = \int_{-T/2}^{To-T/2} \sum_{k=0}^{N-1} f(kT)\delta(t-kT)e^{-in\pi t/To}\, dt$$

$$= \sum_{k=0}^{N-1} f(kT) \int_{-T/2}^{To-T/2} e^{-in\pi t/To}\,\delta(t-kT)\,dt$$

$$= \sum_{k=0}^{N-1} f(kT)e^{-in\pi kT/To} \qquad \text{-------}(2.2.9)$$

But $T_o = NT$

$$\therefore\ A_n = \sum_{k=0}^{N-1} f(kT)e^{-in\pi k/N}\ ,\ n = 0,\pm1,\pm2,\pm3,... \qquad \text{------}(2.2.10)$$

Therefore eq.(2.2.7) gives,

$$\hat{F}(n/NT\ ) = \sum_{n=-\infty}^{\infty} \sum_{K=0}^{N-1} f(kT)\ e^{-in\pi k/N} \qquad \text{------}(2.2.11)$$

The waveform given by (2.2.11) is not necessarily periodic. But it gives N distinct complex values for n = r, any arbitrary integer.

$$\hat{F}(r/NT) = \sum_{k=0}^{N-1} f(kT)\ e^{-i\pi kr/N} \qquad \text{------}(2.2.12)$$

and if n=r+2N then

$$e^{-i\pi k(r+2N)/N} = e^{-i\pi kr/N}\, e^{-i2\pi k}$$

$$= e^{-i\pi kr/N}$$

$$\therefore\ F^{\wedge}\left[\frac{r+2N}{NT}\right] = \sum_{k=0}^{N-1} f(kT)\, e^{-i\pi k(r+2N)/N}$$

$$= \sum_{k=0}^{N-1} f(kT)\, e^{-i\pi kr/N} = F^{\wedge}(r/NT)$$

Therefore there are only N distinct values for which equation (2.2.12) can be evaluated and $\hat{F}(n/NT)$ is periodic with period of N samples.

Therefore F.T. of (2.2.6) equivalently expressed as

$$F^{\wedge}(n/NT) = \sum_{k=0}^{N-1} f(kT)\, e^{-in\pi k/N} \qquad \text{------(2.2.13)}$$

$$n=0,1,2,\ldots(N-1)$$

Normally it is written as

$$G(n/NT) = \sum_{K=0}^{N-1} g(kT)\, e^{-in\pi k/N} \qquad \text{------(2.2.14)}$$

$$n= 0,1,2,3 \ldots (N-1)$$

Thus continuous F.T. is approximated by discrete F.T. taking only N samples.

Discrete Inverse Fourier Transform :

The discrete inverse F.T. is defined by

$$g(kT) = (1/N) \sum_{n=0}^{N-1} G(n/NT\ ) e^{in\pi k/N}$$

$k=0,1,2,\ldots N-1$ ------(2.2.15)

Equation (2.2.15) shows periodicity in the same manner as the discrete transform. The period is defined by N samples of g(kT). Hence g(kT) is actually defined on the complete set of integers $k=0,\pm1,\pm2,\ldots$ It is also note that both time and frequency domain functions require to be periodic.

$$G(n/NT) = G\left[\frac{(rN+n)}{NT}\right] , \quad r=0,\pm1,\pm2,\ldots\ldots$$

$$g(KT) = g\left[(rN+k)\,T\right] , \quad r=0,\pm1,\pm2,\ldots\ldots .$$

(2.3) Band-Limited Periodic Waveforms

i) Truncation Interval Equal to Period :-

Consider any function f(t) and its F.T. In this class the waveforms in continuous and discrete F.T. are exactly the same within a period. For the equivalence the two waveforms requires

1) the time function f(t) must be periodic.

2) f(t) must be band-limited.

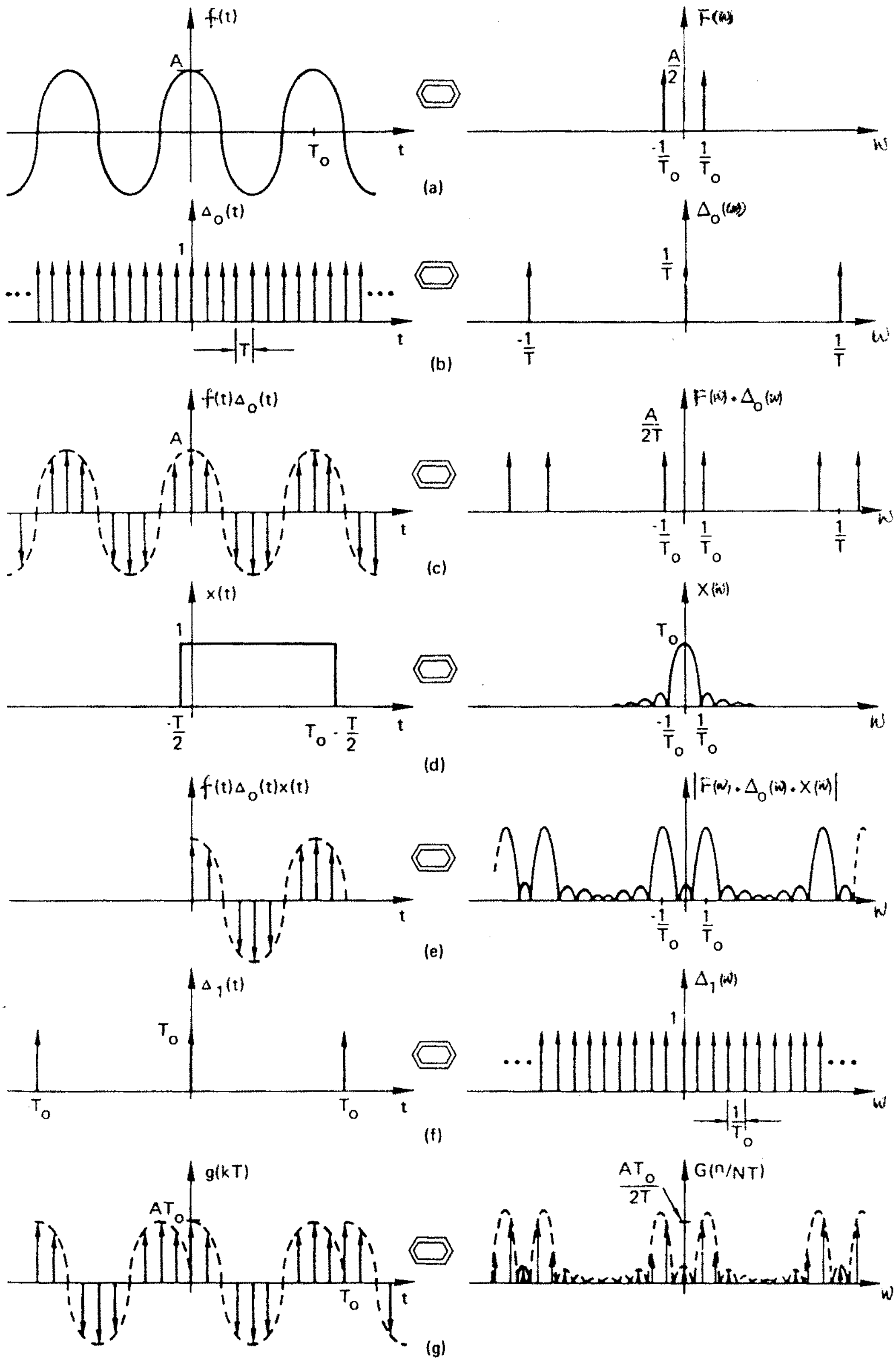
3) The sampling rate must be at least two times the largest frequency component of f(t).

4) The truncation function x(t) must be non-zero over exactly one period (or integer multiple period) of f(t).

In the sequence of sampling, truncating, convolution our point of discussion is truncation. The sampled waveform is truncated by multiplication with the rectangular function so that N sample values remaining after truncation equate to one period of the original waveform f(t). The F.T. of truncated function Significantly distorts with respect to the original transform F(w). However, when this function is sampled by the frequency sampling function the distortion is eliminated. This follows due to equidistant impulses of frequency sampling function.

II) **Truncation interval not Equal to Period**

If a periodic band-limited function is sampled and truncated to consist of other than an integer multiple of the period, the resulting discrete and continuous F.T. will differ considerably. Consider the following figure.


f(t)
A
T_o
t
(a)
F(w)
$\frac{A}{2}$
$-\frac{1}{T_o}$
$\frac{1}{T_o}$
w
$\Delta_o(t)$
1
T
(b)
$\Delta_o(w)$
$\frac{1}{T}$
$-\frac{1}{T}$
$\frac{1}{T}$
$f(t)\Delta_o(t)$
(c)
$F(w) * \Delta_o(w)$
$\frac{A}{2T}$
x(t)
$-\frac{T}{2}$
$T_o - \frac{T}{2}$
(d)
X(w)
T_o
$f(t)\Delta_o(t)x(t)$
(e)
$|F(w) * \Delta_o(w) * X(w)|$
$\Delta_1(t)$
T_o
(f)
$\Delta_1(w)$
$\frac{1}{T_o}$
g(kT)
AT_o
(g)
$\frac{AT_o}{2T}$
G(n/NT)

(figure 2.3)

Suppose that the function f(t) is sampled and truncated. Note that the sampled, truncated function is not an integer multiple of the period of f(t). After convolution this gives the periodic function but it is not the exact copy of original function f(t). Therefore it is necessary to examine the same result in frequency domain.

Fourier transform of the sampled truncated waveform of fig (2.3)(e) is obtained by convolving the frequency domain impulse functions of fig (2.3)(c) and fig (2.3)(d). Sampling of the resulting convolution at frequency intervals of $1/T_0$ yields the impulses as given in fig (2.3)(g). These sample values represent the F.T. of the periodic time waveform of fig (2.3)(g) There is an impulse at zero frequency. The average value is not expected to be zero because the truncated wave do not contain even number of cycles.

This discrepancy between the continuous and discrete F.T. is probably often encountered. Thus effect of truncation at other than a multiple of the period is to create a periodic function with sharp discontinuities. This results in additional frequency components in the frequency domain.

Discrete Fourier Transform properties :

Since discrete Fourier transform is simply a special case of the Fourier transform, and has similar properties as follows.

1) <u>Linearity</u> :-

If F(n/NT)and G(n/NT) are the Discrete Fourier transform of f(kT) and g(kT) respectively. then Discrete Fourier Transform of [f(kT) + g(kT)] is [F(n/NT) + G (n/NT)]

2) <u>Symmetry</u> :-If f(kT)and F(n/NT) are a discrete Fourier transform pair

Then f(-n)and(1/N)F(kT) are the Discrete Fourier transform pair

$$f(-n) = (1/N) \sum_{n=0}^{N-1} F(KT)e^{-i2\pi nk/N}$$

3) <u>Time shifting</u> :- If f(kT) is shifted by the integer 'r' its Discrete Fourier transform is obtained by multiplying F(n/NT) by the factor $e^{-i2\pi nr/N}$ i.e. $e^{-i2\pi nr/N}$ F(n/NT)

4) <u>Frequency shifting</u> :- If F(n/NT) is shifted by the integer 'r' then its inverse discrete Fourier transform is multiplied by $e^{i2\pi rk/N}$

We have

$$F(n/NT) = \sum_{k=0}^{N-1} f(KT)e^{-i2\pi kn/N}$$

$$F[(n-r)/NT] = \sum_{k=0}^{N-1} f(kT)e^{-i2\pi(n-r)k/N}$$

$$= \sum_{k=0}^{N-1}\left[f(kT)e^{i2\pi rk/N}\right] e^{-i2\pi nk/N}$$

Alternate Inversion Formula :

The discrete inversion formula may also written as -

$$f(kT) = \frac{1}{N} \sum_{k=0}^{N-1} [F^*(n/NT)e^{-i2\pi nk/N}]^*$$

Where * indicates complex conjugate

Consider $F(n/NT)=R(n/NT)+iI(n/NT)$

$$f(kT)=(1/N) \sum_{n=0}^{N-1} [R(n/NT)-iI(n/NT)] \; e^{-i2\pi nk/N}$$

Also it can be shown that

$$f(kT) = (1/N)\sum_{n=0}^{N-1} F(n/NT) \; e^{i2\pi nk/N}$$

This formula can be used to compute both F.T. and its inverse F.T.

- **Even Functions** :- If f(kT) is an even function then discrete Fourier transform of f(kT) is an even function and is real because,

$$F(n/NT) = \sum_{n=0}^{N-1} f(kT) e^{-i2\pi kn/N}$$

$$= \sum_{k=0}^{N-1} f(kT) \mathrm{Cos}(2\pi nk/N) - i \sum_{k=0}^{N-1} f(kT) \ \mathrm{Sin}(2\pi nk/N)$$

$$= \sum_{K=0}^{N-1} f(kT) \ \mathrm{Cos}(2\pi nk/N)$$

$$= R(n/NT)$$

Also, for the given real and even function F(n/NT), its inverse discrete Fourier transform f(kT) is an even function.

ODD FUNCTION :-

If f(kT) is odd function then its discrete Fourier transform is an odd and imaginary function given by

$$F(n/NT) = \sum_{n=0}^{N-1} f(kT) \ e^{-i2\pi nk/N}$$

$$= \sum_{n=0}^{N-1} f(kT) \ [\mathrm{Cos}(2\pi nk/N) - i \ \sin(2\pi nk/N)$$

$$= -i \sum_{n=0}^{N-1} f(kT) \ \sin \ (2\pi nk/N)$$

$$= iI(n/NT)$$

Similarly, if F(n/NT) is odd and imaginary function then its inverse discrete transform is odd function.

(2.4) Discrete Convolution and Correlation:-

These are the most important properties of discrete Fourier Transform. It is possible to obtain the relation between the continuous and discrete convolution.

Discrete Convolution:

If f(kT) and g(kT) are periodic functions with period N then Discrete convolution is defined by

$$Y(kT) = \sum_{i=0}^{N-1} f(iT)g[(k-i)T]$$

It is denoted by $y(kT) = f(kT) * g(kT)$

Here g[(k-i)T] is the image of g(iT) shifted by the amount kT.

Graphical Discrete Convolution:-

Graphical computation of discrete convolution is similar tc that of continuous convolution only differ from integration in continuous convolution is replaced by the summation. For discrete convolution the steps are

i) Folding ii) Shifting iii) Multiplication and

iv) Summation

The functions f(kT) and g(kT) are periodic functions, that is,

$$f(kT) = f[(k + rN)T]$$

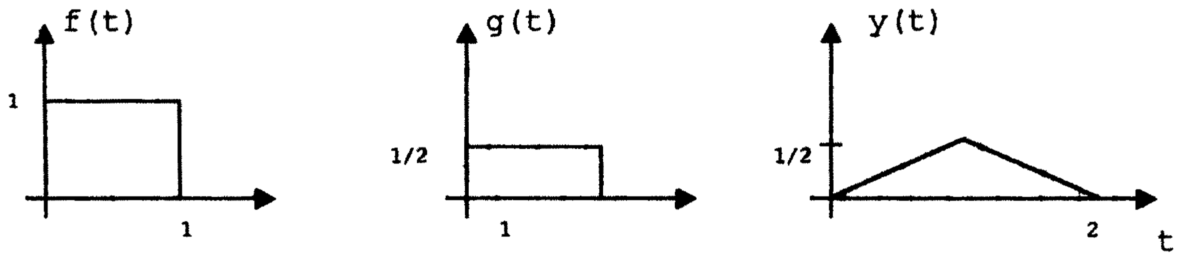
$$g(kT) = f[(k + rN)T], \quad r = 0, \pm 1, \pm 2, \ldots$$

hence the result of convolution are repeated after the period N.

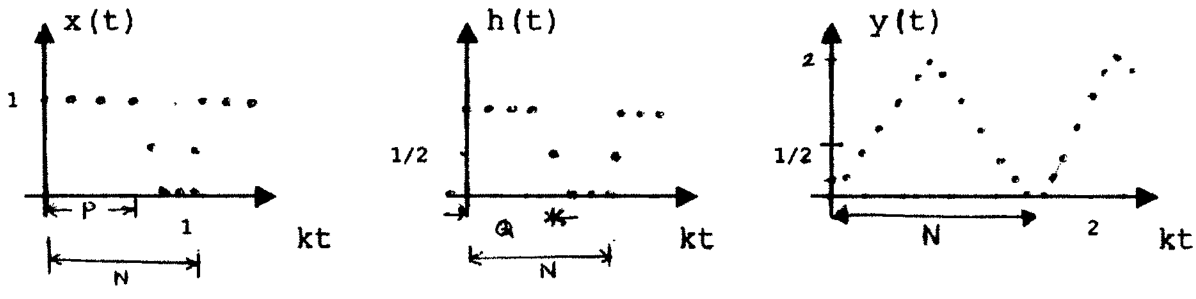
Now we will explain in detail the convolution of finite duration and infinite duration waveform.

Discrete Convolution of Finite Duration Waveforms

Consider the functions f(t) and g(t) as illustrated below. To evaluate the discrete convolution, we sample f(t) and g(t) with sample interval T and we assume that both sample functions are periodic with period N.



fig(a)



fig(b)

Figure (2.4)(b) shows that the discrete convolution y(kT) is very poor that is overlapping takes place due to lack of sufficiently large period.

If P ≡ No. of samples of f(t).

Q ≡ No. of samples of g(t) then the choice of N = P + Q - 1, results a function described by N = P+Q - 1, that is, there is no overlapping. Thus for this value of N the discrete convolution results in a periodic function where each period approximates the continuous convolution results.

From figure(2.4)(b) it is observed that there is difference in the scale T. Hence the modified result is

$$Y(kT) = T\sum_{i=0}^{N-1} f(iT)g[(k-i)T]$$

which is identical with the continuous convolution. The error introduced by the constant T is made negligible by making T very small.

Discrete Convolution of an Infinite Waveform and Finite Waveform:-

Let f(t) be infinite waveform and g(t) be finite but both the function are periodic. Since f(kT) is infinite duration waveform the convolution result is a funciton of f(kT) at both ends of the period, such a condition has no meaningful interpretation in terms of the desired continuous convolution. Similar values are obtained for each shift value until the Q points of g(kT) are shifted by Q-1, that is, the end effect exists until shift k = Q-1. But the end effect does not occur at the right end of the N sample values. If the sample interval T is chosen sufficiently small, then the discrete convolution closely approximation the continuous convolution except the end effect.

(2.5) Applying The Discrete Fourier Transform

We can obtain the approximation to the F.T. by means of discrete Fourier transform. The first step in applying the discrete F.T. is to choose the number of samples N and the sample interval T. Then using the formula

$$F(n/NT) = T \sum_{k=0}^{N-1} f(kT)\, e^{-i2\pi nk/N} \qquad , n = 0,\ 1,\ \ldots\ N-1 \qquad \text{-------}(2.5.1)$$

where T is introduced to produce equivalence between the continuous and discrete transforms, we can find the discrete transform of f(t).

- **<u>Inverse Fourier Transform</u>** :-

If a continuous real and imaginary frequency functions are given, by using the inverse discrete Fourier transforms,

$$f(kT) = \Delta w \sum_{n=0}^{N-1} [\ R(n\Delta w) + i\ I(n\Delta w)\ ]\ e^{i2\pi nk/N}$$

$$\text{for } k = 0,\ 1,\ \ldots\ N-1 \qquad \text{-------}(2.5.2)$$

where Δw is the sample interval in frequency, the corresponding time function can be determined.

We know that in equation (2.5.2) R(w) is even function so it is possible to fold R(w) about the sample point n = N/2. Also sample R(w) upto only n = N/2 and fold it to get the required samples.

Similarly, since I(w) in (2.5.2) is odd frequency function. it is require to fold it about $n = N/2$ and flip the result. In some of the cases it is require to set the sample value at some points to preserve the symmetry.

If sampled frequency function is specified correctly, the inverse discrete Fourier transform approximates the continuous results.

FREQUENCY CONVOLUTION THEOREM :-

If F(n/NT) & G(n/NT) are the discrete Fourier transform of f(kT) & g(kT) then discrete Fourier transform of the product f(kT).g(kT) is the convolution of the F(n/NT) and G(n/NT) By definition, convolution Y(n/NT) is given by

$$Y(n/NT) = \sum_{r=0}^{N-1} F(r/NT)\; G[(n/NT)-r]$$

$$= \sum_{r=0}^{N-1} \left\{ \left[\sum_{m=0}^{N-1} f(mT)\, e^{-i2\pi mr/N} \right] \left[\sum_{k=0}^{N-1} g(KT)\, e^{-i2\pi k(n-r)/N} \right] \right\}$$

$$= \sum_{m=0}^{N-1} \sum_{k=0}^{N-1} f(kT)\; g(kT)\, e^{-i2\pi kn/N} \left\{ \sum_{r=0}^{N-1} e^{-i2\pi mr/N}\, e^{i2\pi kr/N} \right\}$$

if $m = k$ bracketed term becomes N

$$Y(n/NT) = N \sum_{k=0}^{N-1} f(kT)\; g(kT)\, e^{-i2\pi kn/N}$$

$$\sum_{r=0}^{N-1} F(n/NT)\; G(n/NT - r) = N \sum_{k=0}^{N-1} f(kT)\; g(kT)\, e^{-i2\pi kn/N}$$

Discrete Correlation Theorem

If F(n/NT) and G(n/NT) are the discrete Fourier transform of f(kT) and g(kT) respectively then discrete Fourier transform of the correlation

$$\sum_{r=0}^{N-1} f(rT)\ g(k + rT) \quad \text{is } F^*(n/NT)\ G(n/NT)$$

where "*" indicate the complex conjugate of $F^*(n/NT)$

Parseval's Theorem:- The theorem states that

$$\sum_{k=0}^{N-1} f^2(kT) = (1/N) \sum_{k=0}^{N-1} |F(n/NT)|^2$$

Consider y(kT) = f(kT).f(kT) then by convolution theorem, the discrete Fourier transform of y(kT) is convolution of the discrete F.T. of f(kT), that is,

$$\sum_{k=0}^{N-1} f^2(kT) e^{-i2\pi nk/N} = (1/N) \sum_{r=0}^{N-1} F(r/NT)\ F(n - r/NT)$$

if n = 0, then we get,

$$\sum_{k=0}^{N-1} f^2(kT) = (1/N) \sum_{r=0}^{N-1} |F(r/NT)|^2$$

This proves the parseval's theorem,