CHAPTER-0

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PRELIMINARIES AND NOTATIONS

(0.1) Definitions and Theorems:

* **Periodic Function** :- A function f(x), which satisfies the relation f(x+T) = f(x) for all x is said to be a periodic function.

The smallest positive number T, for which this relation holds, is called the period of f(x).

* **Dirichlet's Conditions** :- A function f(x) defined in the interval $C_1 \leq x \leq C_2$ can be expressed as Fourier series if in the interval.

i) f(x) and its integrals are finite and single valued.

ii) f(x) has finite number of discontinuities.

iii) F(x) has finite number of maximum and minimum.

These conditions are known as Dirichlet's conditions.

* **Fourier Integral** :- If f(x) satisfies Dirichlet's conditions in each finite interval $-l \leq x \leq l$ and if f(x) is integrable in $(-\infty,\infty)$ then Fourier integral theorem states that

$$f(x) = (1/\pi)\int_0^{\infty} \int_{-\infty}^{\infty} f(s)\ \cos(w(s-x))\ dw\ ds \quad \text{-------}(0.1.1)$$

* **Fourier Sine and Cosine integrals** :- The above integrals can be written as

$$f(x) = (1/\pi)\int_{0}^{\infty}\int_{-\infty}^{\infty} f(s)[\cos ws \ \cos wx]\, dw\, ds$$

$$+(1/\pi)\int_{0}^{\infty}\int_{-\infty}^{\infty} f(s)\sin ws \ \sin wx\, dw\, ds$$

-------(0.1.2)

$$f(x) = (1/\pi)\int_{0}^{\infty} \cos wx \int_{-\infty}^{\infty} f(s) \cos ws\, dw\, ds$$

$$+(1/\pi)\int_{0}^{\infty} \sin wx \int_{-\infty}^{\infty} f(s) \sin ws\, dw\, ds$$

-------(0.1.3)

* **Fourier Cosine Integral** :- When f(x) is even function second integral in (0.1.3) will be Zero and we will get,

$$f(x) = (2/\pi)\int_{0}^{\infty} \cos ws \int_{0}^{\infty} f(s) \cos ws\, dw\, ds$$

-------(0.1.4)

Fourier Sine Integral :- When f(x) is odd function the first integral will be zero and we get,

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} \sin wx \int_{0}^{\infty} f(s) \sin ws\, dw\, ds$$

-------(0.1.5)

* **Complex form of Fourier Integral :-**

$$f(x) = (1/2\pi)\int_{-\infty}^{\infty} e^{iwx}\, dw \int_{-\infty}^{\infty} f(s)\, e^{-iws}\, ds \qquad \text{-------}(0.1.6)$$

is called the complex form of the Fourier integral. Then the expression defined by

$$\int_{-\infty}^{\infty} e^{-iws}\, f(s)\, ds \text{ or } \int_{-\infty}^{\infty} e^{-iwt}\, f(t)\, dt \qquad \text{-------}(0.1.7)$$

is called F.T. of f(t) and denoted by F(w). Typically f(t) is termed as a function of the variable time and F(w) is termed as a function of the variable frequency.

* **Inverse Fourier Transform :-** If F(w) is the F.T. of f(t) and if F(t) satisfies Dirichlet's conditions in every finite interval(-l,l)

and if $\int_{-\infty}^{\infty} |f(t)|\, dt$ is convergent then at every point of continuity

$$f(t) = (1/2\pi) \int_{-\infty}^{\infty} F(W)\, e^{iwt}\, dw \qquad \text{-------}(0.1.8)$$

f(t) is called the inverse F.T. of F(w).

Thus inversion transformation (0.1.8) allows the determination of a function of time from its F.T.

* **Some Important Properties** :-In dealing with F.T. there are a few properties which are basic to a thorough understanding

1) **Linearity** :- If F(w) and G(w) be the F.T. of f(t) & g(t) resp., then.

F[af(t) + b g(t)] = a F(w) + bG(w),

Where a,b are constants.

Proof :- We have

$$F(w) = \int_{-\infty}^{\infty} e^{-iwt} f(t)\, dt \;\&\; G(w) = \int_{-\infty}^{\infty} e^{-iwt} g(t)\, dt$$

Therefore,

$$F[a f(t) + b g(t)] = \int_{-\infty}^{\infty} e^{-iwt} [\, a f(t) + b g(t)\,]dt$$

$$= a\int_{-\infty}^{\infty} e^{-iwt} f(t)\, dt + b \int_{-\infty}^{\infty} e^{-iwt} g(t)\, dt$$

$$= a F(w) + b G(w)$$

2) **Change of Scale property (Time scaling)**:-

If F(w) is the F.T. of f(t) then 1/a F(w/a) is the F.T. of f(at) where a is real constant greater than 0.

Proof : $F(w) = \int_{-\infty}^{\infty} e^{-iwt} f(t)\, dt$

$$\therefore \int_{-\infty}^{\infty} e^{-iwt} f(at)\, dt = \int_{-\infty}^{\infty} e^{-iw(y/a)} f(y)\, dy/a \text{ (putting at = y)}$$

$$= 1/a \int_{-\infty}^{\infty} e^{-i(w/a)y} f(y)\, dy$$

$$= 1/a\ F(w/a)$$

this means that as the time scale expands, the frequency scale not only contracts but the amplitude increases vertically to keep the area constant

3) **Shifting property :-** (Time Shifting)

If $F(w)$ is the F.T.of $f(t)$then e^{-iaw} $F(w)$ is the Fourier transform of $f(t-a)$.

Proof :

$$F[f(t-a)] = \int_{-\infty}^{\infty} e^{-iwt} f(t-a)\ dt$$

Putting $t-a = x$

$$F[f(t-a)] = \int_{-\infty}^{\infty} e^{-iw(a+x)} f(x)\ dx$$

$$= e^{-iaw} \int_{-\infty}^{\infty} e^{-iwx} f(x)\ dx$$

$$= e^{-iaw} F(w)$$

Time shifting results in a change in the phase angle θ. It does not alter the magnitude of the F.T.

4) **Symmetry :-** If $F(w)$ is the F.T. of $f(t)$ then F.T.of $F(t)$ is $f(-w)$.

Proof : The inverse F.T. is defined as

$$\therefore\ f(t) = \int_{-\infty}^{\infty} F(w)\ e^{iwt}\ dw$$

$$\therefore \quad f(-t) = \int_{-\infty}^{\infty} F(w)\ e^{-iwt}\ dw$$

Now, interchanging the parameters f & w

$$f(-w) = \int_{-\infty}^{\infty} F(t)\ e^{-iwt}\ dt = F[F(t)]$$

4) **Change of Scale Property (frequency Scaling) :-** If the inverse F.T. of F(w) is f(t), the inverse F.T. of F(aw), a is real constant is given by 1/a f(t/a)

Proof :- The inverse F.T. is defined as

$$f(t) = \int_{-\infty}^{\infty} F(w)\ e^{+iwt}\ dw$$

$$\text{Now, } \int_{-\infty}^{\infty} F(aw)\ e^{iwt}\ dw = \int_{-\infty}^{\infty} F(w')\, e^{i(w'/a)t}\ 1/a\ dw \quad \text{(putting } aw=w'\text{)}$$

$$= 1/a \int_{-\infty}^{\infty} F\ (w')\ e^{iw'(t/a)}\ dw'$$

$$= 1/a\ f(t/a)$$

This means that as the frequency scale expands, the amplitude of the time function increases.

6) **Shifting property :-** (Frequency shifting) :-

If F(w) is shifted by a constant w_o, its inverse transform is multiplied by e^{iwot}

Proof :- The inverse F.T. is defined as

$$f(t) = \int_{-\infty}^{\infty} e^{iwt} \; F(w) \; dw$$

$$\therefore \int_{-\infty}^{\infty} e^{iwt} \; F(w-w_o) \; dw = \int_{-\infty}^{\infty} e^{i(x+wo)t} \; F(x) \; dx$$

$$= e^{iwot} \int_{-\infty}^{\infty} e^{ixt} \; F(x) \; dx$$

$$= e^{iwot} f(t)$$

- **Even Function:**-If f(t) is even function i.e. $f(t)=f(-t)$ then the F.T. of f(t) is an even function and is real.

We have

$$F(w) = \int_{-\infty}^{\infty} f(t) \; e^{-iwt} \; dt$$

$$= \int_{-\infty}^{\infty} f(t) \; \text{Cos}wt \; dt - i \int_{-\infty}^{\infty} f(t) \; \text{Sin}wt \; dt$$

$$= \int_{-\infty}^{\infty} f(t) \; \text{Cos}wt \; dt = \text{Re}(w)$$

The imaginary term is Zero since the integrand is an odd function.

Similarly, if F(w) is a real and even frequency function the inversion formula

$$f(t) = \int_{-\infty}^{\infty} F(w) \; e^{iwt} \; dw \quad \text{becomes}$$

$$f(t) = \int_{-\infty}^{\infty} F(w) \; \text{Cos}wt \; dw + i \int_{-\infty}^{\infty} F(w) \; \text{Sin}wt \; dw$$

$$= \int_{-\infty}^{\infty} F(w)\ \text{Cos}wt\ dw$$

$$= \text{Re}(t)$$

- **Odd Function**:-If f(t) is odd function that is f(-t)= -f(t) then F.T. of f(t)is an odd imaginary function.

  We have

$$F(w) = \int_{-\infty}^{\infty} e^{-iwt}\ f(t)$$

$$= \int_{-\infty}^{\infty} f(t)\ \text{Cos}wt\ dt - i\int_{-\infty}^{\infty} f(t)\ \text{Sin}wt\ dt$$

$$= -i\int_{-\infty}^{\infty} f(t)\ \text{Sin}wt\ dt$$

$$= i\ I_o(w)$$

real integral is Zero since the integral is odd function.

Similiarly, if F(w) is a imaginary and odd function the inversion formula

$$f(t) = \int_{-\infty}^{\infty} F(w)\ e^{iwt}\ dw \quad \text{becomes}$$

$$f(t) = \int_{-\infty}^{\infty} F(w)\ \text{Cos}wt\ dw + i\int_{-\infty}^{\infty} F(w)\ \text{Sin}wt\ dw$$

$$= i\int_{-\infty}^{\infty} F(w)\ \text{Sin}wt\ dw$$

$$= f_o(t)$$

An arbitrary function can always be decomposed or separated into the sum of an even and odd function.

$$f(t) = \frac{f(t)}{2} + \frac{f(t)}{2}$$

$$= \left[\frac{f(t)}{2} + \frac{f(-t)}{2}\right] + \left[\frac{f(t)}{2} - \frac{f(-t)}{2}\right]$$

$$= f_e(t) + f_o(t)$$

Such a type of decomposition is important to increase the speed of computation of the discrete fourier transform

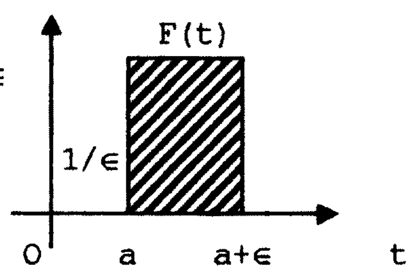
* **Impulse function :-**

Consider the function F(t) defined by

$$F(t) = \left.\begin{array}{ll} 0, & t < a \\ 1/\epsilon, & a \le t \le a + \epsilon \\ 0, & t > a + \epsilon \end{array}\right\} \text{-------} (0.1.9)$$

The function is represented by the adjoining figure integrating F(t), we get,

$$\int_{-\infty}^{\infty} F(t)\, dt = \int_{a}^{a+\epsilon} 1/\epsilon\, dt = 1 \text{ for all } \epsilon$$



As $\epsilon \to 0$, the function F(t) tends to infinity at a and is Zero every where else. But the integral of F(t) is unity. Hence, the limiting form of F(t) (as $\epsilon \to 0$) is known as unit impulse function and is denoted by $\delta(t-a)$.

$$\therefore \delta(t-a) = \lim_{\epsilon \to o} F(t)$$

When a = o, the unit function at t = 0 is

$$\therefore \delta(t) = \lim_{\epsilon \to o} F(t)$$

Also it is defined as

$$\delta(t-a) = 0, \ t \neq to \text{ and}$$

$$\int_{-\infty}^{\infty} \delta \ (t-to) \ dt = 1, \qquad \text{------(0.1.10)}$$

That is, we define the δ-function as having undefined magnitude at the time of occurrence and zero else where with the additional property that area under the function is unity.

- **Properties of impulse function :-**

For any arbitrary function Φ(t) the impulse function satisfies

$$\int_{-\infty}^{\infty} \delta \ (t) \ \Phi(t)dt = \Phi(0), \qquad \text{------(0.1.11)}$$

it gives some useful properties :

(i) Shifting property :-

The function δ(t-to) is defined by

$$\int_{-\infty}^{\infty} \delta(t-t_o) \ \Phi(t)dt = \Phi(t_o), \text{ otherwise} \qquad \text{------(0.1.12)}$$

that is, the function Φ(t) is shifted if the value of t_o varies continuously. This is the most important property of the δ-function.

(ii) Scaling property :- The distribution δ(at) is defined by

$$\int_{-\infty}^{\infty} \delta(at)\ \Phi(t)dt = \frac{1}{|a|} \int_{-\infty}^{\infty} \delta(t)\ \Phi(t/a)dt \qquad \text{------}(0.1.13)$$

from the equality, we mean that

$$\delta(at) = (1/|a|)\delta(t) \qquad \text{------}(0.1.14)$$

Multiplication of δ-function with an ordinary function

The product of a δ-function by an ordinary function f(t) is defined by

$$\int_{-\infty}^{\infty} \left[\delta(t)f(t)\right]\Phi(t)dt = \int_{-\infty}^{\infty}\delta(t)\left[f(t)\Phi(t)\right]dt \qquad \text{------}(0.1.15)$$

(iv)Convolution property :-

The convolution of two impulse functions is given by

$$\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty}\delta_1(y)\delta_2(t-y)\,dy\right]\Phi(t)\,dt = \int_{-\infty}^{\infty}\delta_1(y)\left[\int_{-\infty}^{\infty}\delta_2(t-y)\Phi(t)\,dt\right]dy$$

Hence

$$\delta_1(t-t_1) * \delta_2(t-t_2) = \delta\left[t-(t_1+t_2)\right]$$